4/22/15

Lecture 25: Pyramid Partition Functions and Dimer shuffling Combinatorial Model

We begin with Section 2 of Mozgovoy-Reineke

Let $A = \mathbb{C}Q/I = P_1 \oplus P_2 \oplus \cdots P_n$ $(n = |Q_0|)$
where $P_i = Ae_i$ = projective A-module

P_i consists of paths starting at vertex i

Any (left) A-module M decomposes as

$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ where $M_i = e_i M$ (note that e_i is now on the other side)

Def: For $i \in Q_0$, an i-cyclic A-module is a pair (M, m) where M = fin. dim. A-module and $m \in M_i$ with $Am = M$ (i.e. m generates M)

In general, i-cyclic modules are fin. dim. quotients of P_i.

Hilbert scheme := moduli space of isom classes of i-cyclic modules

Euler characteristic with compact support

Partion function (for $i \in Q_0$) $\quad Z_i(A) = \sum_{\alpha \in \mathbb{N}^{Q_0}} \chi_c(\mathrm{Hilb}_i^{\alpha}(A))\, x^{\alpha}$

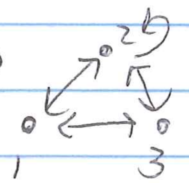
$\mathrm{Hilb}_i^{\alpha}(A)$ = moduli space of i-cyclic modules with dim vector α
i.e. $\dim M_j = \alpha_j$ for $j \in Q_0$.

4/22/15 ② We wish to describe today how to compute $Z_i(A)$'s.

We first define a weighting on $\mathbb{Z}^{Q_1}$.

Define $\Lambda = \mathbb{Z}^{Q_1} / d_e^{-1}(0)$

e.g.'s  and , $\Lambda = \mathbb{Z}^{Q_1}$

SPP , $\Lambda = \mathbb{Z}^{Q_1} / \langle A+C-B,\ D+F-E-G \rangle$

Define $wt: Q_1 \to \Lambda$ by projecting $a \in Q_1$ to its image in Λ.

<u>Lemma 3.3 [MR]</u> If dimer model (Q, W) satisfies
- there is at least one perfect matching OR
- all faces in Q_2 contain the same number of arrows

Then Λ is a free group, i.e. no torsion.

Recall a dimer model is nondegenerate as long as all of its edges belong to some perfect matching.

<u>Lemma 3.5 [MR]</u> If dimer model is
- nondegenerate OR
- all faces in Q_2 contain the same number of arrows

Then any nontrivial element $\sum_{a \in Q_1} c_a a \in \mathbb{N}^{Q_1}$ projects to a nonzero element in Λ.

4/22/15 ③ Thus in this case, the arrows of Q generate a <u>strongly convex cone in $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$.</u>

<u>Lemma 4.11 [MR]</u> A dimer model on a torus is <u>consistent</u>

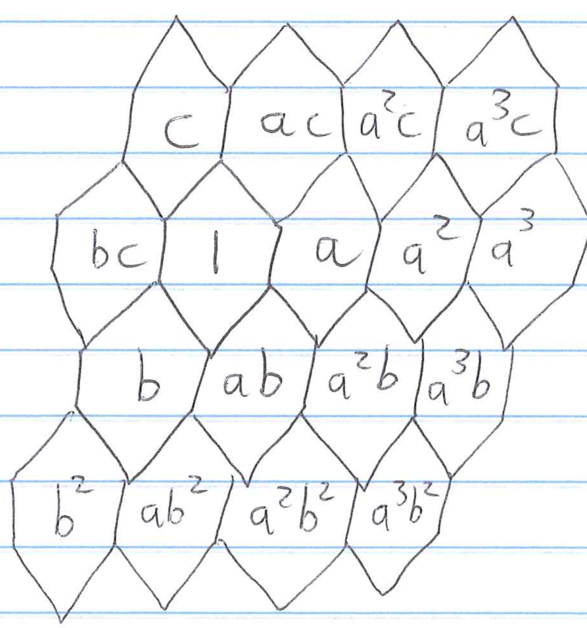
$\Leftrightarrow$ two paths in $\mathbb{C}Q$ with the same starting point are equivalent in $A = \mathbb{C}Q/\partial W$ if and only if their <u>weights</u> in Λ are equal.

$\Leftrightarrow$ two paths in $\mathbb{C}Q$ " " in $A = \mathbb{C}Q/\partial W$ if and only if these two paths are <u>weakly equivalent</u>.

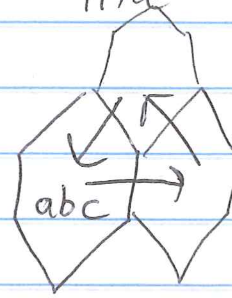
allowing both a & a^{-1} for every arrow

<u>Examples:</u> , $\Lambda = \mathbb{Z}^{Q_1}$

Two paths are equivalent in $A = \mathbb{C}Q/\partial W \cong \mathbb{C}[a,b,c]$ $\Leftrightarrow$ they consistent of the same number of a's, b's, & c's.



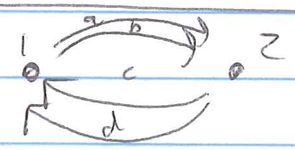
as well as paths like

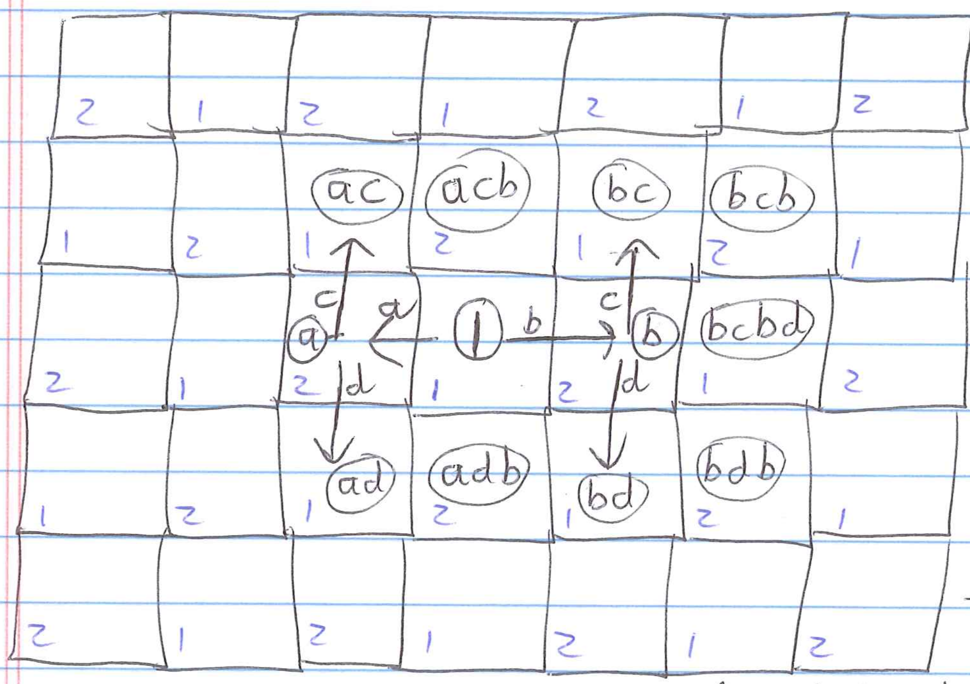


4/22/15 (4) Monomials in $\mathbb{C}[a,b,c]$ in bijection with

$(abc)^k \alpha^m \beta^n$ where $(\alpha,\beta) = (a,b), (a,c),$ or (b,c)

$(\circlearrowleft)^k \longrightarrow^m \nearrow^n$

Next example  $\Lambda = \mathbb{Z}^{Q_1}$



Also

wt$(\circlearrowright) = a^1 b^1 c^1 d^1$

only paths containing cycles will contain all four arrows

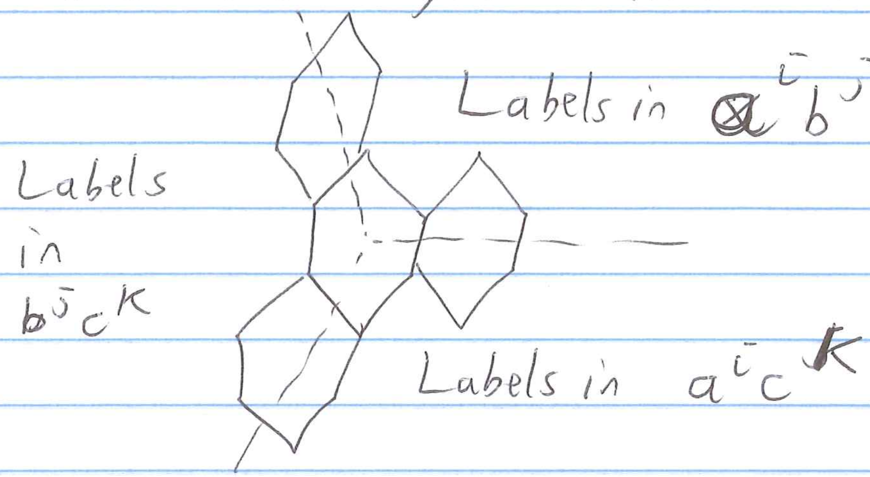
same Λ-weights too

in $\mathbb{C}Q/\partial W$, $acb \sim bca$, $bcbd \sim bdbc$

starting from Face 1 (as opposed to Face 2) paths must begin with arrow a or b.

Moral: every square of checkerboard could be labeled with $a^i b^j c^k d^l$ signifying the wt of paths from "origin" to that square (assuming no backtracking or cycles).

4/22/15 (5) Just like hexagonal lattice divided into



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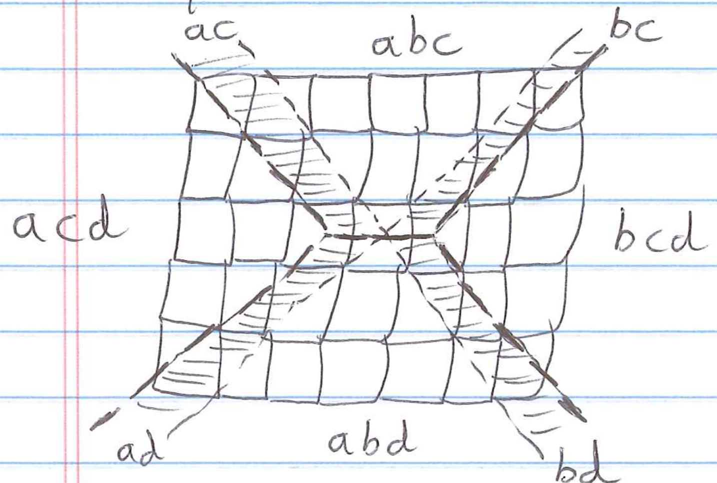
square lattice divides into



as possible weights for paths to squares (without cycles or backtracking)

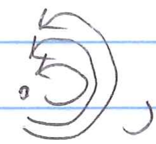
In general, define $\Delta_i = \{\text{paths in } \mathbb{C}Q \text{ starting at vertex } i \text{ (i.e. certain face in dimer model)}\} \big/ p \sim q \text{ if } wt(p) = wt(q)$

We turn Δ_i into a poset by $p \leq q$ if $\exists$ path r s.t. $rp \sim q$.

Lemma 2.5 [MR] Bijection between isom. classes of Λ-graded i-cyclic A-modules and finite order ideals of Δ_i.

$$(M, m) \longmapsto \{p \in \Delta_i \mid M_{wt(p)} \neq 0\}$$

4/22/15 (6) In the above two examples, elements of Δ_i in bijection with (Faces in dimer model, $\mathbb{N}$) where the second parameter measures how many times to add a cycle (e.g. abc or $abcd$). Like adding a height dimension to the bipartite tiling.

For , ison classes of Λ-graded i-cyclic A-modules in bijection with

- monomials $a^i b^j c^k$ (equivalently)
- paths decomposed as $(abc)^I a^J b^K$ or $(abc)^I a^J c^L$ or $(abc)^I b^K c^L$ (equivalently)
- 3-dim Young Tableaux, i.e. plane partitions.

Cor: $Z_1(A) = \dfrac{1}{\prod_{n \geq 1} (1-q^n)^n}$ MacMahon Formula

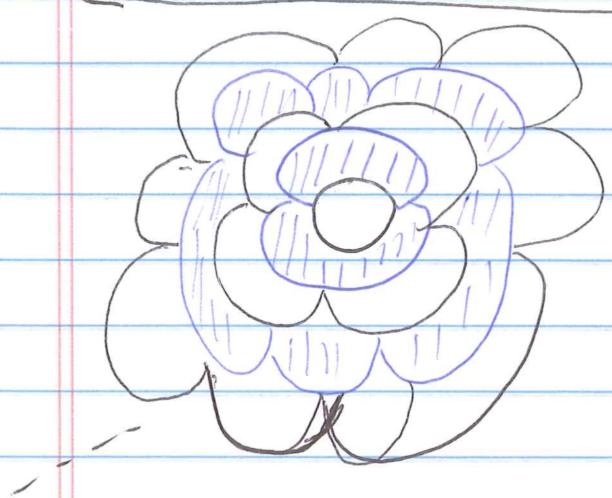
In general, Prop 4.14 [MR] For a consistent non-degenerate dimer model,

$$Z_i(A) = \sum_{\substack{\Omega \subset \Delta_i \\ \text{Finite order ideal}}} \ \prod_{\substack{\text{path } p \in \Omega \\ i \rightsquigarrow t(p)}} x_{t(p)} .$$

4/22/15 (7) For  ^{2} $$Z_1(A) = \sum_{\pi \text{ pyramid partition}} x_1^{wt_1(\pi)} (-x_2)^{wt_2(\pi)}$$

where a pyramid partition is a collection of stones pulled off the top of

$\boxed{\text{Thm 2.7.1 of Szendrői}}$



odd layers have m^2 stones

even layers have $m(m+1)$ stones

wt_1 and wt_2 measure pulling off odd/even layer, resp.

Instead of working with pyramids directly, we instead consider <u>dimer coverings</u> instead

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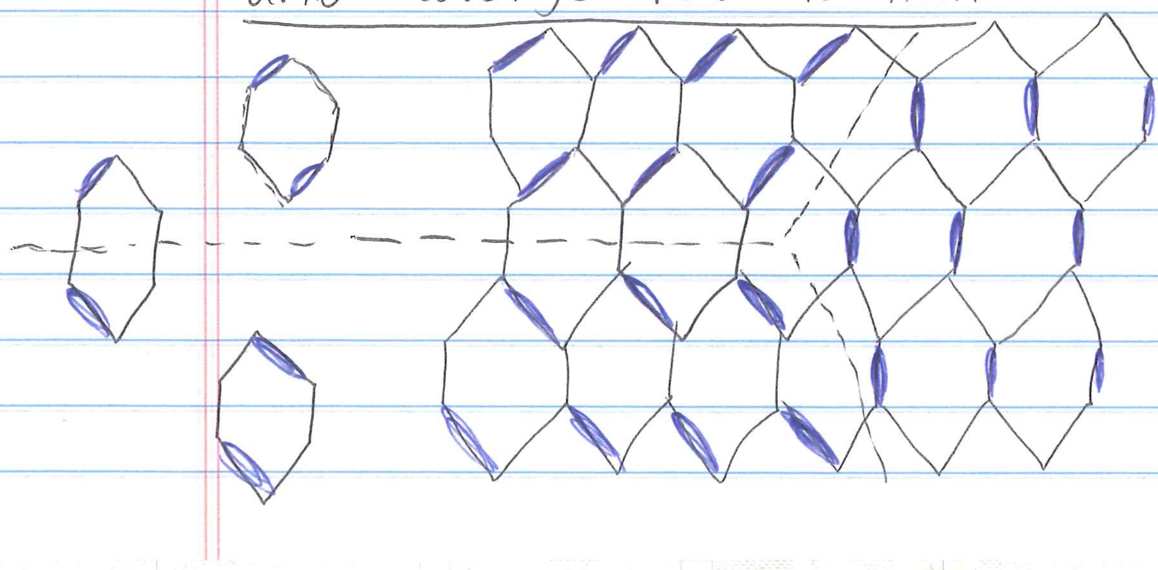
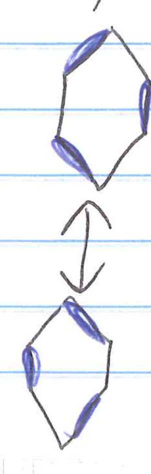
Firstly,  Plane partitions in bijection with

<u>dimer coverings reachable from</u>



by



4/22/15 8 First, can "twist" origin face

Second, can twist origin's neighbours

And can iterate to twist any hexagon in this lattice by a path of twisting.

Furthermore, can only twist a face a second time if all three of origin's neighbours twisted $\longleftrightarrow abc$ then can twist origin a second time and will allow other faces to be twisted a second time henceforth.

- Monomials in $a^i b^j c^k \longleftrightarrow$ record which hexagons twisted

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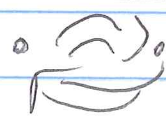
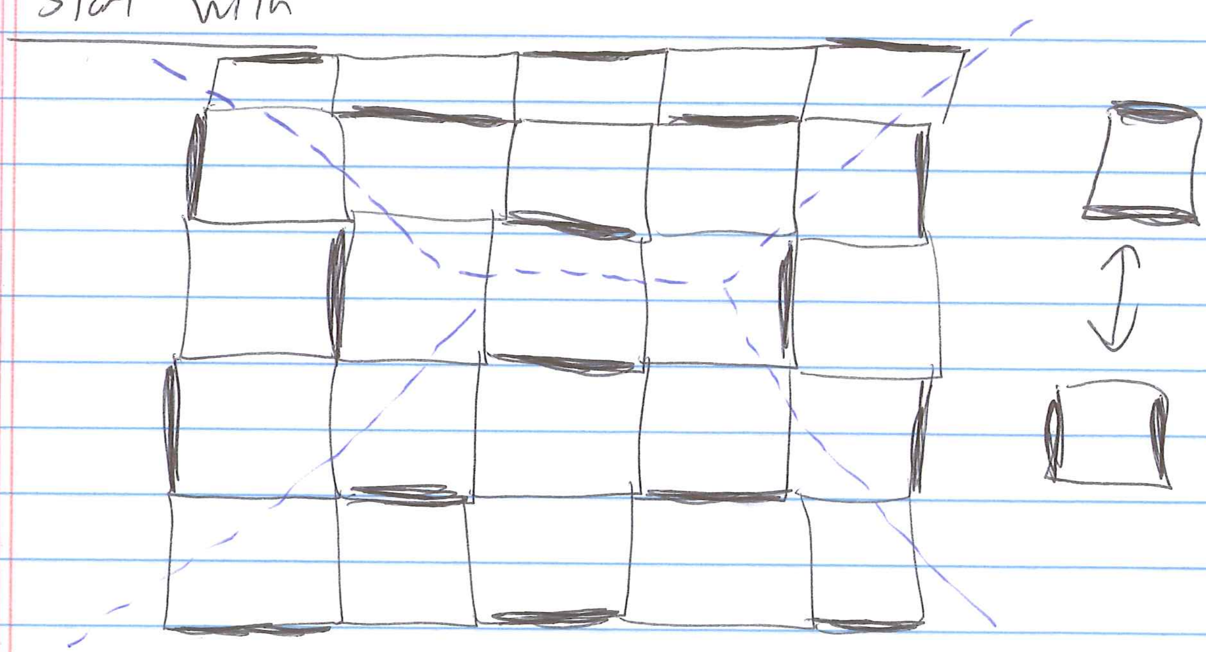
We can model  pyramid partitions the same way

Start with



4/22/15 9 Benjamin Young uses this dimer covering model to obtain a closed formula for corresp. $Z_1(A)$

"Computing a Pyramid Partition Generating Function with Dimer Shuffling" (arXiv: 0709.3079)

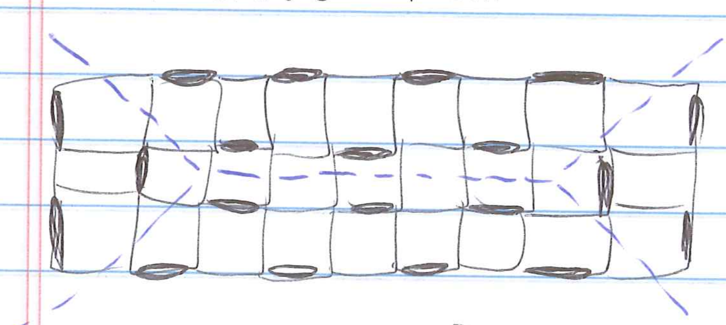
Conjecture of Szendrői

Thm (Young) $Z_1(A) = M(-q_0q_1)^2 \prod_{k\geq 1} (1+q_0^k(-q_1)^{k-1})^k \cdot (1+q_0^k(-q_1)^{k+1})^k$

where $M(x)$ is the MacMahon func.

$$M(x) = \prod_{k\geq 1} \frac{1}{(1-x^k)^k}$$

In fact Ben Young proves more generally, the partition function $Z^{(n)}(A)$ of dimer coverings reachable from



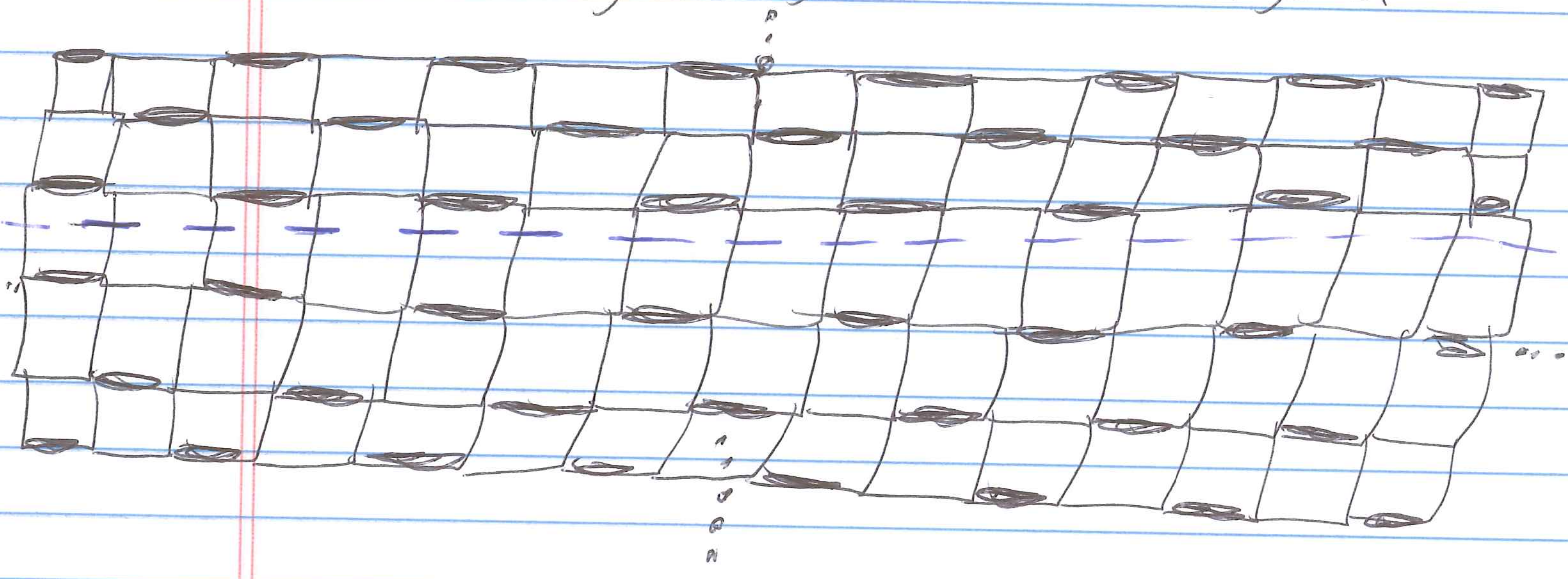
e.g. $n=3$

case above is $n=1$ (i.e. $Z_1(A) = Z^{(1)}(A)$).

$$Z^{(n)}(A) = \underbrace{\left(\prod_{k\geq 1} \frac{1}{(1-q_0^k(-q_1)^k)^k}\right)^2}_{= M(-q_0q_1)^2} \prod_{k\geq 1} (1+q_0^k(-q_1)^{k-1})^{k+n-1} \cdot \prod_{k\geq 1}(1+q_0^k(-q_1)^{k+1})^{\max(k-n+1,0)}$$

4/22/15 (10) Young's method of proof

If $n \to \infty$, looking at dimer coverings of



i.e., only two regions rather than four

infinitely squares that can be twisted in first step.

Claim:

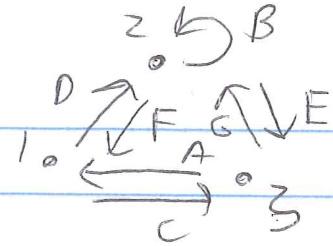
$$Z^{(\infty)}(A) = \frac{\left(\prod_{k \geq 1} \frac{1}{\left(1 - q_0^k (-q_1)^k\right)^k} \right)^2}{\prod_{k \geq 1} \frac{1}{\left(1 - \frac{q_0^k (-q_1)^k}{q_1}\right)^k}}$$

Mimics Seiberg duality

- Uses this to give a weighting to edges of entire lattice.

- Then describes a "dimer shuffling" rule to get from dimer covering in $Z^{(n)}$ to one in $Z^{(n+1)}$.

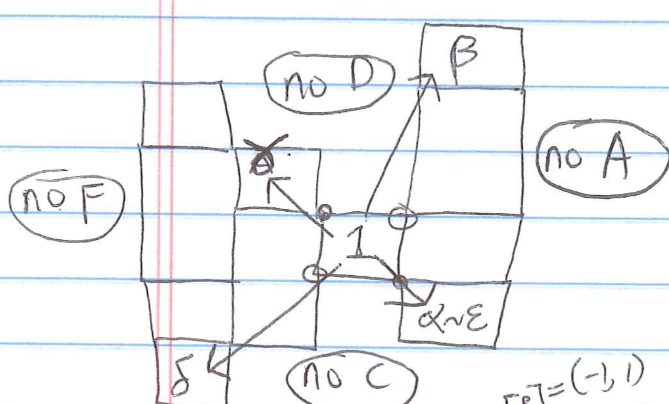
4/22/15 11 SPP e.g.



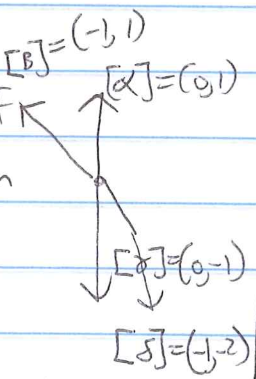
$W = ADCF - BFD + BEG - ACGE$

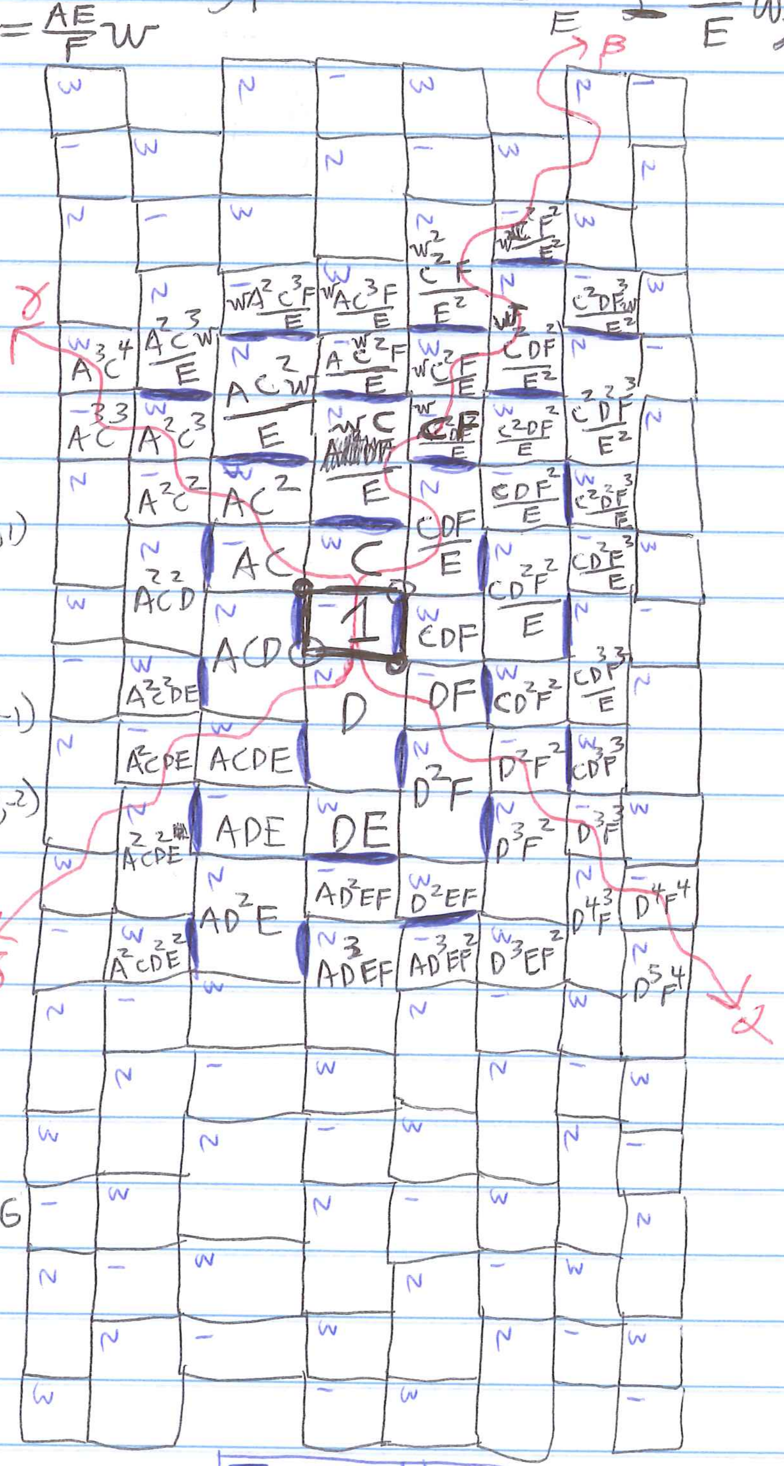
$\Lambda = \langle A, C, D, E, F \rangle$ with $B \equiv A + C$, $G \equiv D + F - E$

Zig-zag paths $\alpha = DF \sim \varepsilon = EG$, $\beta = BCFG \sim \frac{AC^2DF^2}{E} = \frac{CF}{E}w$, $\gamma = AC$, $\delta = ABDE \sim A^2CDE = \frac{AE}{F}w$

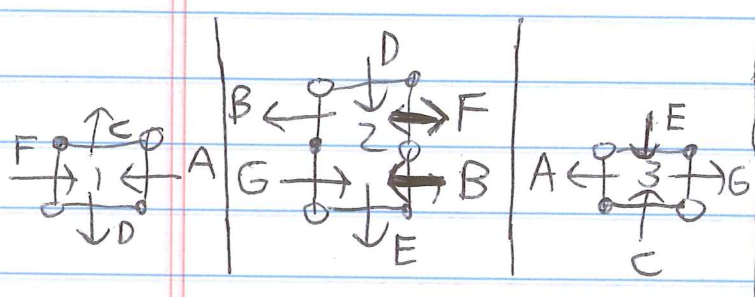


rotated version of Global Zig Zag Fan





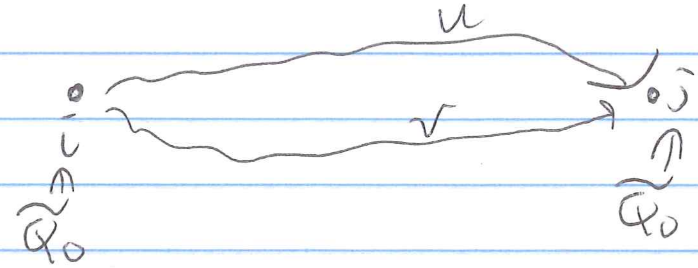
Cycle w has weight $\boxed{ACDF}$



$\boxed{\text{write values so } w\text{'s factored out}}$

$\boxed{I_0 \text{ in blue}}$

4/22/15 (12) Remark 4.10 [MR] Let $\widetilde{Q}_0$ denote the set of vertices in the unfolded quiver Q (using potential W) and consider two weak paths u and v between points i & j.



Then there exists $k \in \mathbb{Z}$ s.t. $v \sim u w^k$ where w is a $\overset{\text{fundamental small}}{\text{cycle}}$ in $(\widetilde{Q}_0, \widetilde{Q}_1)$.

In consistent case

Hence why we can label each face $\overset{\text{dual to}}{\text{}}$ $\widetilde{Q}_0$ as a weight in Λ which corresponds to the weight of a shortest path from i_0 (fixed) to $j \in \widetilde{Q}_0$.

All weights are $wt(v_{i_0 j}) + k\,wt(w)$ for shortest path $v_{i_0 j}$ and some integer k,

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Leads to the definition in Prop 5.2

$I_0 := \{ a : i \to j \mid a v_{i_0 i} > v_{i_0 j} \}$ which is proven to be dual to a perfect matching of the bipartite tiling.

Cor 5.7 [MR] Partition function $Z^{i_0}(A)$ equals

$$\sum_{\substack{I \text{ is a perfect} \\ \text{function reachable} \\ \text{from } I_0}} \prod_{i \in \widetilde{Q}_0} x_{\pi(i)}^{h_I(i)}$$

$\boxed{h_I(i) = \text{height} = \#\text{ times face } \boxed{i} \text{ has to be twisted in a seq from } I_0 \text{ to } I}$

$\pi(i)$: projection of vertex label in Q_0

4/22/15 13 In the conifold case

Claim: $Z^{(n)}(A)\, Z^{(n-2)}(A) = \left(1 + q_0^{n-2}(-q_1)^{n-3}\right) Z^{(n-1)}(A)$

Mimics Cluster Mutation of the quiver

See "Wall Crossing of BPS States on the Conifold from Seiberg Duality and Pyramid Partitions" by Wu-yen Chuang and Daniel Louis Jafferis (arXiv: 0810.5072) for related discussion.

Question: How are these infinite pyramid partition functions related to

$$f_n f_{n-2} = f_{n-1}^2 + f_{n-1}^2 \quad \text{where}$$

f_n counts # perf. matching in an Aztec Diamond

Question: For more complicated (Q,W)'s is there a more combinatorial/direct way to build canonical matchings I_0?

"Colored BPS Pyramid Partition Functions, Quivers, and Cluster Transformations"

Rem: Eager-Franco also explore finite & infinite pyramids for more complicated cases.