4/27/15 Lecture 26: Speyer's Crosses and Wrenches Graphs & the Octahedron Recurrence

"Perfect matchings and the octahedron recurrence" by David Speyer (JACO 2007)

Set-up: $\mathcal{L} = \{(n,i,j) : i+j+n \equiv 0 \bmod 2\} \subset \mathbb{Z}^3$

$h : \mathbb{Z}^2 \to \mathbb{Z}$ (Speyer "height function")

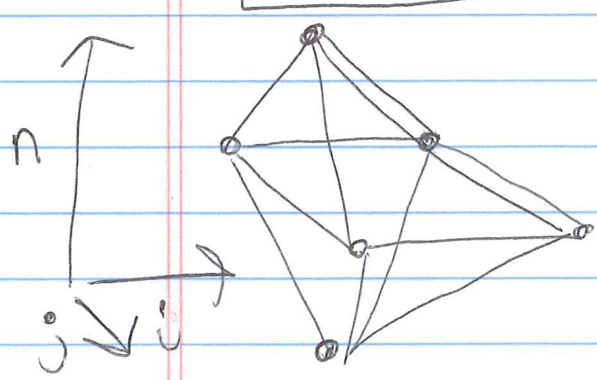
initial conditions $\mathcal{I} \subset \mathcal{L}$ defined as

$\mathcal{I} = \{(h(i,j), i, j)\}$ "~~two~~-dimensional slice" in $\mathcal{L}$

Define rational functions $f(n,i,j)$ for $(n,i,j) \in \mathcal{L}$

$$f(n,i,j) = X_{ij} \text{ for } (n,i,j) \in \mathcal{I}$$

Otherwise
$$\boxed{f(n,i,j)f(n-2,i,j) = f(n-1,i,j+1)f(n-1,i,j-1) + f(n-1,i+1,j)f(n-1,i-1,j)}$$

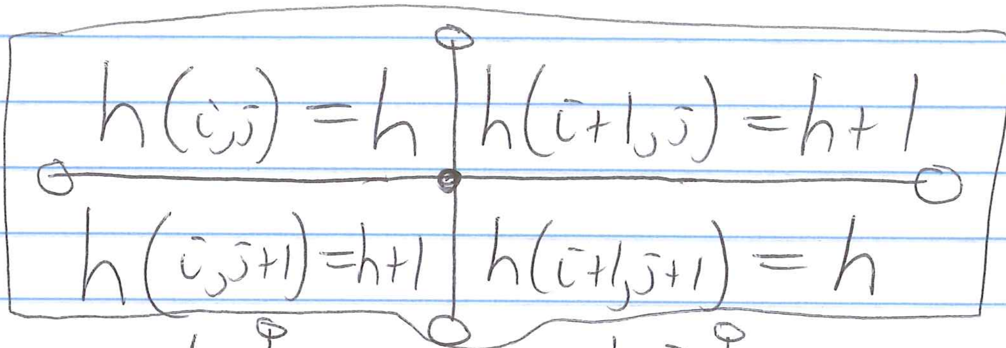


Speyer even allows coefficients on this recurrence which we suppress for now for convenience.

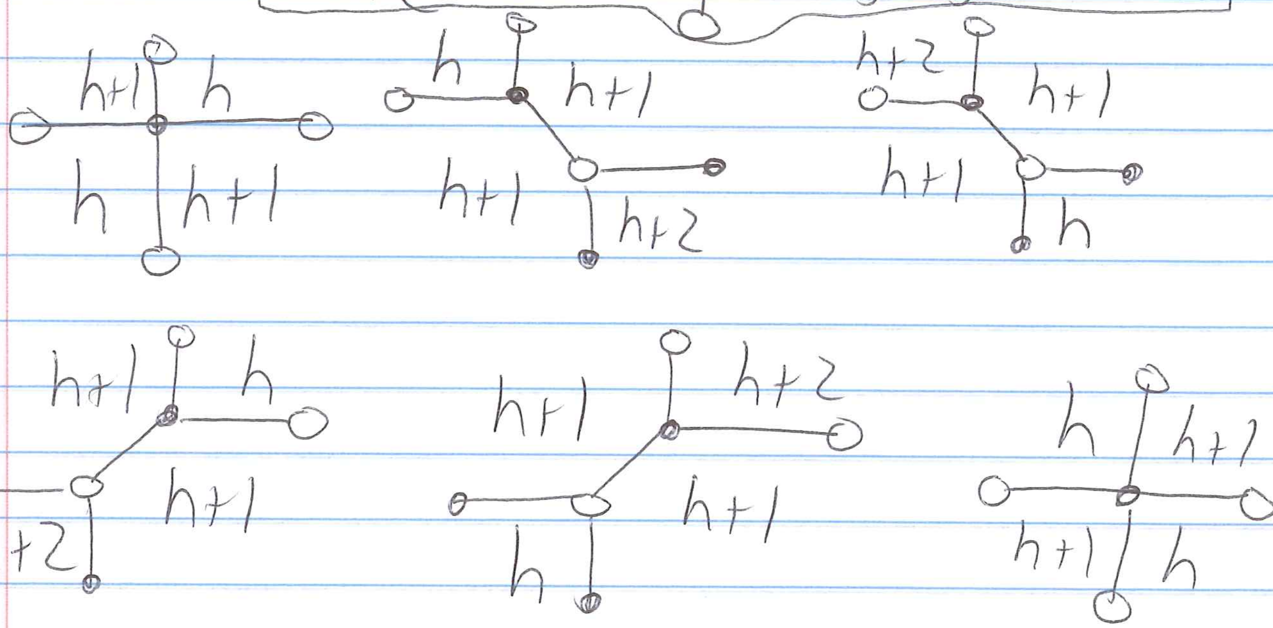
Method: Speyer uses $h(i,j)$'s to define an infinite graph $\mathcal{G}_h = \mathcal{G}$.

Called an "infinite graph with open faces".

Locally



or



Rem: $h : \mathbb{Z}^2 \to \mathbb{Z}$ defined so that

$$h(i,j) = h(i \pm 1, j) \pm 1,$$

$$h(i,j) = h(i, j \pm 1) \pm 1,$$

$$h(i,j) + i + j \equiv 0 \mod 2, \quad \text{and}$$

$$\lim_{|i|+|j| \to \infty} h(i,j) + |i| + |j| = \infty$$

4/27/15

3

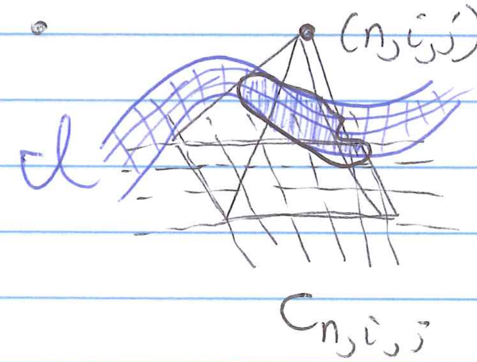
Method continued: Given a choice of $(n,i,j) \in \mathcal{L}$,

$F(n,i,j)$ is a positive Laurent polynomial with the following combinatorial interpretation:

— For $(n,i,j) \in \mathcal{L}$, we build a cone $\mathring{C}_{n,i,j}$

— We build finite subgraph $G_{n,i,j} \subset \mathcal{G}_h$ by using the faces (k,l) of $\mathcal{G}_h$ such that

$$\big(h(k,l), k, l\big) \in \mathcal{L} \cap \mathring{C}_{n,i,j}.$$



$$F(n,i,j) = \sum_{\substack{M = \text{perfect} \\ \text{matching of} \\ G_{n,i,j} \subset \mathcal{G}_h}} w(M)$$

where $w(M)$ is a weight (Laurent monomial of x_{kl}'s) based on the number of edges in M bordering face (k,l) in $\overline{G_{n,i,j}} = G_{n,i,j} \cup \partial G_{n,i,j}$

Warning: Speyer's weight function $w(M)$ depends on whether (k,l) is a face of $G_{n,i,j}$, i.e. a "closed face" or a face on the boundary of $G_{n,i,j}$, an "open face".

$$\mathring{C}_{n,i,j} = \left\{ (n',i',j') \in \mathcal{L} : n' < n - |i'-i| - |j'-j| \right\}$$

4/27/15

(4) More specifically, the weight is defined as

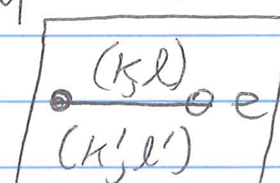
$$w(M) = \left(\prod_{\substack{\text{faces } (k,l) \\ \text{in } G_{n,i,j}}} x_{kl}^{d-m} \right) \Big/ \prod_{\substack{\text{closed faces} \\ (k,l) \text{ in } G_{n,i,j}}} x_{kl}^{1}$$

where closed face (k,l) is a $2d$-gon with exactly m of its edges in M

or open face (k,l) has $2d$ or $(2d-1)$ edges bordering $G_{n,i,j}$ and m of them in M

Rem: $F(n,i,j)$ can be rewritten as

$$F(n,i,j) = cm(G_{n,i,j}) \sum_{\substack{M \text{ a perf.} \\ \text{matching of } G_{n,i,j}}} \prod_{\text{edge } e \in M} \frac{1}{x_{kl}\, x_{k'l'}}$$

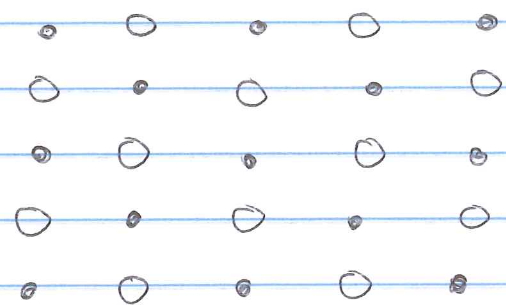


where cm = covering monomial
= monomial recording which closed & open faces in $G_{n,i,j}$ where hexagons double-counted, octagons triple counted, etc. (as closed faces) and related rules for open faces

4/27/15 (5)

Examples:

If the initial conditions are $\mathcal{C} = \{(\underset{\uparrow \atop \{0,-1\}}{i+j \bmod 2}, i, j)\}$



i.e. $h(i,j) = -1$ if •

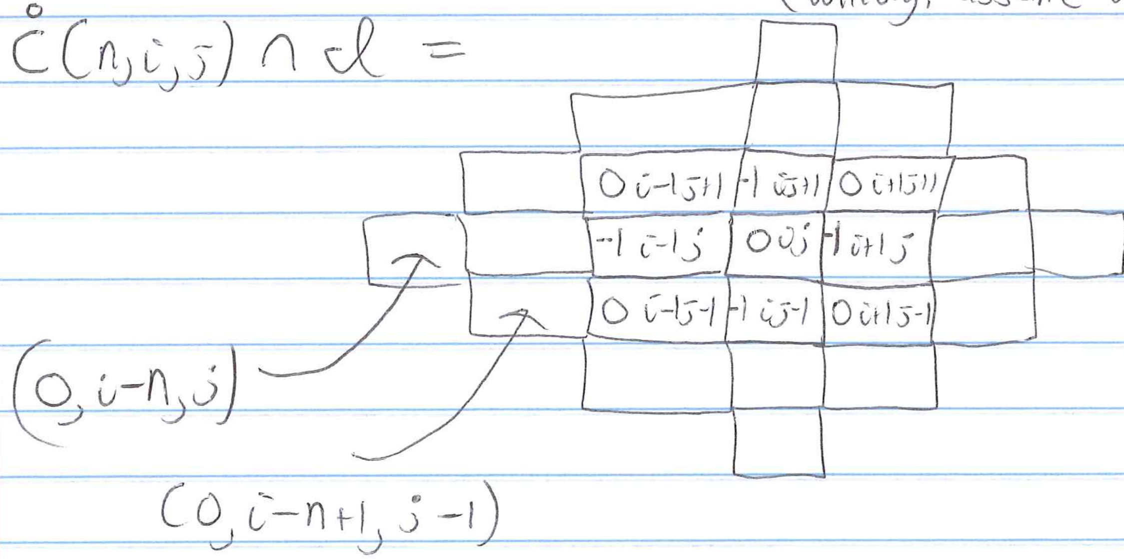
$h(i,j) = 0$ if o

All the local configurations look like $\begin{array}{c|c} -1 & 0 \\ \hline 0 & -1 \end{array}$ or $\begin{array}{c|c} 0 & -1 \\ \hline -1 & 0 \end{array}$

so infinite graph $\cancel{\mathcal{G}}_h$ = checkerboard/square lattice

Given a specific (n,i,j) s.t. $n \geq$ [redacted] & $i+j+n \equiv 0 \bmod 2$
(w.l.o.g. assume $i+j \equiv 0 \bmod 2$)

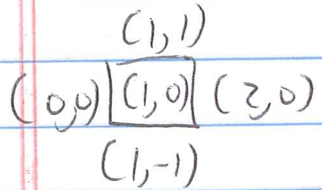
$\mathring{C}(n,i,j) \cap \mathcal{C} =$

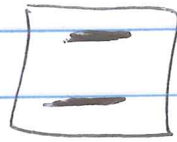


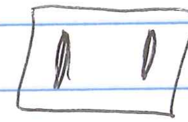
$\boxed{\text{i.e. } G_{n,i,j} = \text{Aztec Diamond of order } n \text{ centered at } (i,j)}$

4/27/15

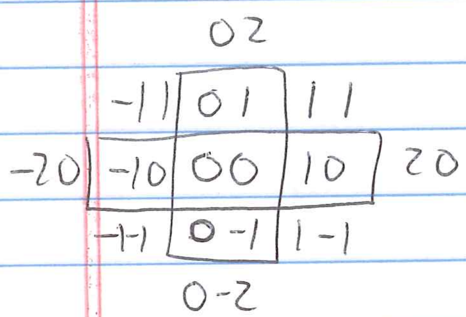
6. $F(1,1,0) = \dfrac{x_{00}\,x_{20}}{x_{10}} + \dfrac{x_{11}\,x_{1-1}}{x_{10}}$

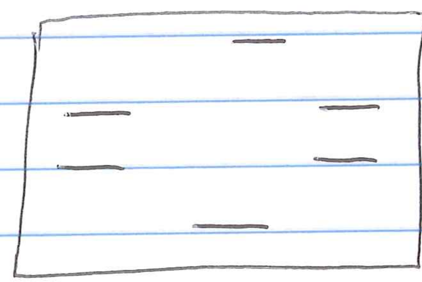


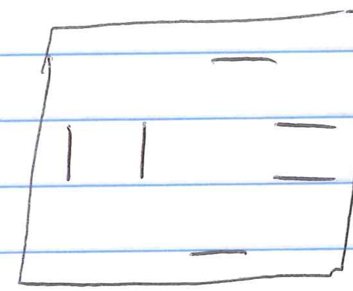




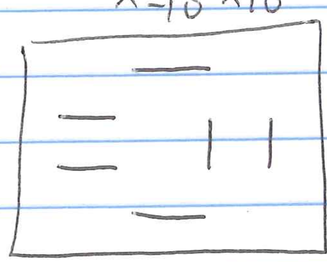
$F(2,0,0) = \dfrac{X_{-20}\,X_{00}\,X_{20}}{X_{-10}\,X_{10}} + \dfrac{X_{-11}\,X_{1-1}\,X_{20}}{X_{-10}\,X_{10}}$

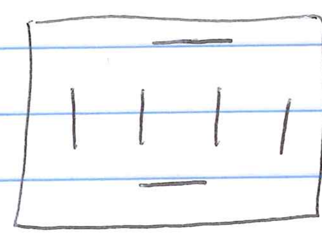


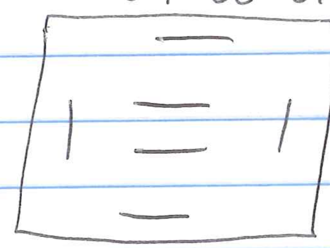




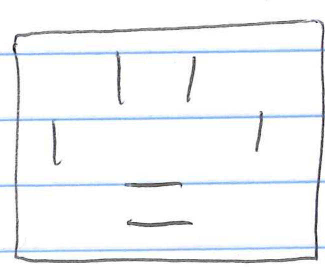
$+ \dfrac{X_{-20}X_{11}\,X_{1-1}}{X_{-10}\,X_{10}} + \dfrac{X_{-11}\,X_{11}\,X_{1-1}\,X_{-1-1}}{X_{-10}\,X_{00}\,X_{10}} + \dfrac{X_{-11}X_{11}X_{1-1}\,X_{-1-1}}{X_{0-1}X_{00}X_{01}}$

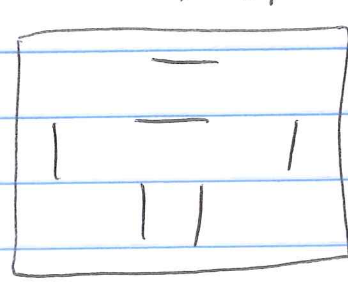


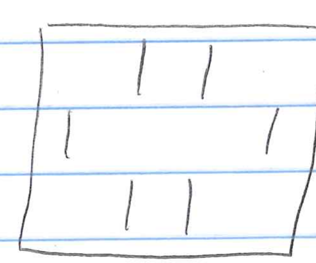




$+ \dfrac{X_{02}\,X_{-1-1}\,X_{1-1}}{X_{0-1}\,X_{01}} + \dfrac{X_{-11}\,X_{11}\,X_{0-2}}{X_{0-1}\,X_{01}} + \dfrac{X_{02}\,X_{00}\,X_{0-2}}{X_{0-1}\,X_{01}}$

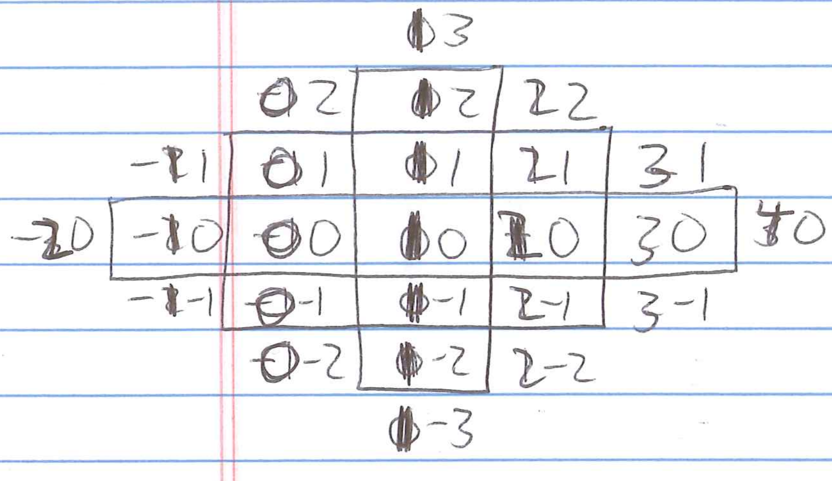


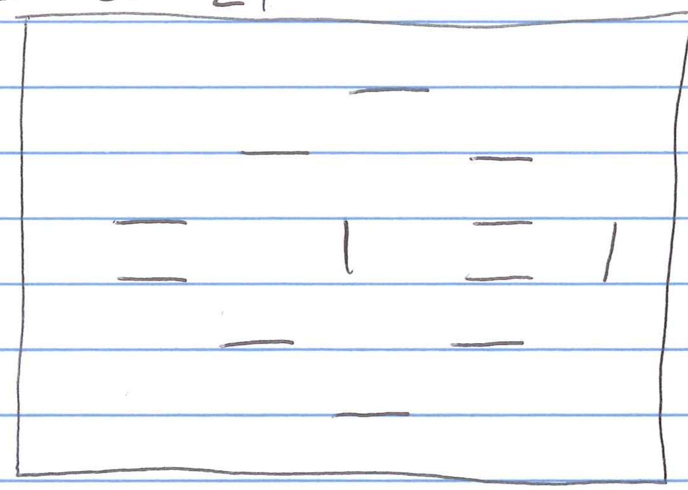




4/27/15

7 $F(3,1,0) = \dfrac{x_{-20}\,x_{11}\,x_{1-1}\,x_{01}\,x_{0-1}}{x_{-10}\,x_{21}\,x_{20}\,x_{2-1}} + \dots$



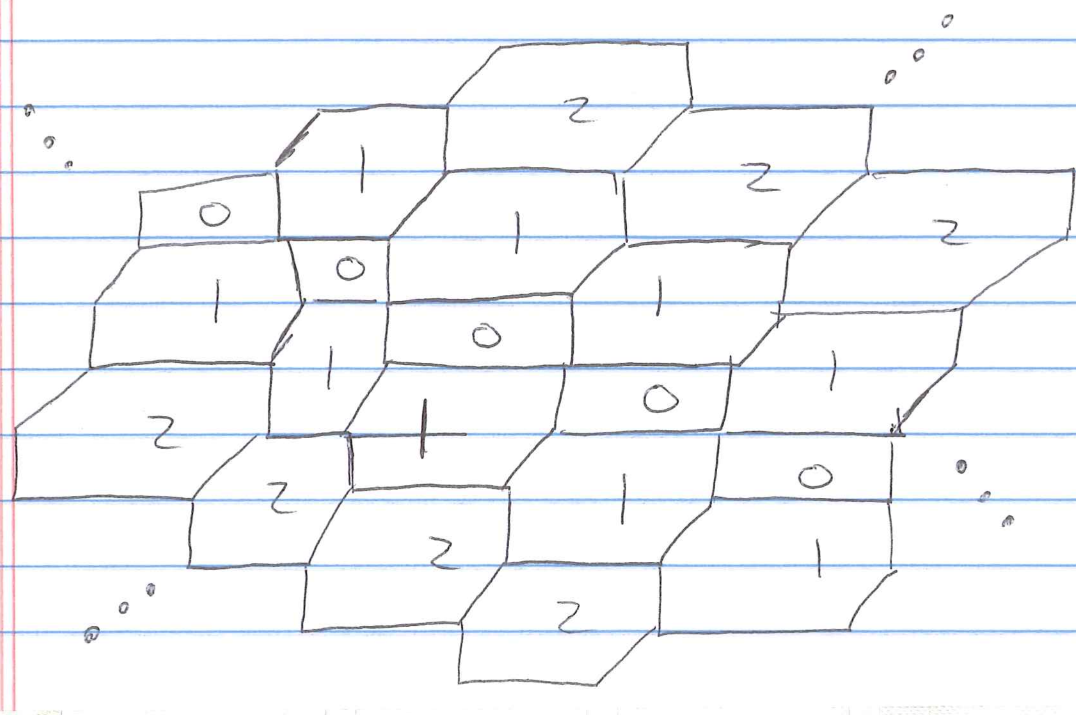


Next example: $h(i,j) = |i+j|$

0	1	2	3	4	
1	0	1	2	3	...
2	1	0	1	2	
3	2	1	0	1	
4	3	2	1	0	

$d = \{(0,0,0), (0,1,-1), (0,2,-2), (0,-1,1), (1,1,0), (2,2,0), \dots\}$

$\mathcal{G}_h$



4/27/15

8

Rem: Crossed out faces on $\partial \mathcal{C}_{(3,1,0)}$

open faces always on subset of $\partial C \cap d$ but not nec. all of $\partial C \cap d$

$G_{3,1,0}$ in $\mathcal{D}_h$

$$\text{Cone } \mathring{C}_{(3,1,0)} = \Big\{ \overbrace{(1,1,0), \cancel{(3,1,0)}}^{\text{distance } 0}, \overbrace{\cancel{(2,0,0)},}^{\text{distance } 1}$$

$$\overbrace{(0,0,0), \cancel{(2,2,0)}, (0,2,0), \cancel{(2,1,1)}, (0,1,1),}^{\text{distance } 1}$$

$$\overbrace{\cancel{(2,1,-1)}, (0,1,-1)}^{\text{distance } 1}, \overbrace{\cancel{(1,3,0)}, \cancel{(1,2,1)}, \cancel{(1,2,-1)},}^{\text{distance } 2}$$

$$\overbrace{\cancel{(1,1,2)}, \cancel{(1,1,-2)}, \cancel{(1,0,1)}, \cancel{(1,0,-1)}, \cancel{(1,-1,0)},}^{\text{distance } 2}$$

$$\overbrace{\cancel{(0,4,0)}, \cancel{(0,3,1)}, \cancel{(0,3,-1)}, \cancel{(0,2,2)}, \cancel{(0,2,-2)},}^{\text{distance } 3}$$

$$\cancel{(0,1,3)}, \cancel{(0,1,-3)}, \cancel{(0,0,2)}, \cancel{(0,0,-2)}, \cancel{(0,-1,1)}$$

$$\cancel{(0,-1,-1)}, \cancel{(0,-2,0)}, \ \ldots$$

$$\mathring{C}_{(3,1,0)} \cap d = \Big\{ \overbrace{(1,1,0)}^{\text{distance } 0}, \overbrace{(0,0,0), \cancel{(2,2,0)},}^{\text{distance } 1}$$

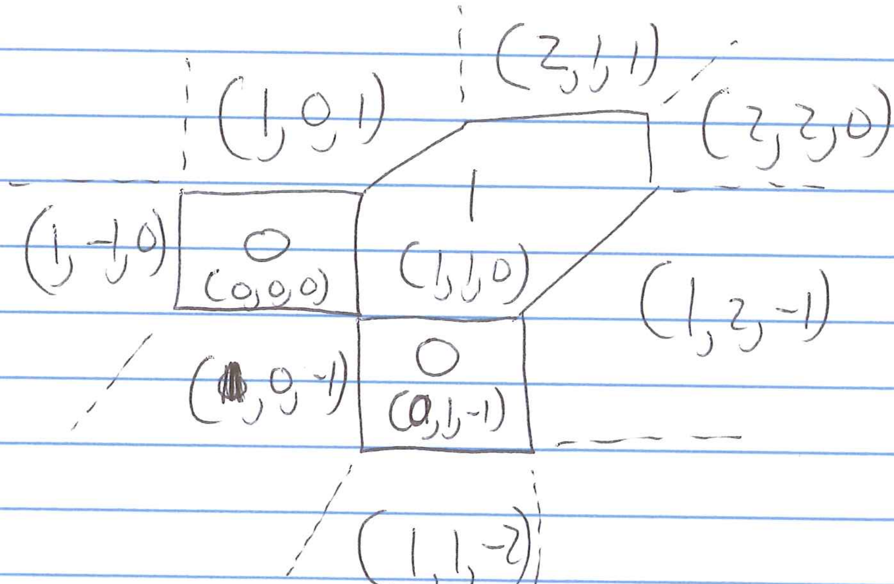
$$(0,1,-1), \cancel{(2,1,1)},$$

$$\overbrace{\cancel{(1,-1,0)}, \cancel{(1,2,-1)}, \cancel{(1,0,1)}, \cancel{(1,1,-2)}, \cancel{(1,0,-1)}}^{\text{distance } 2} \Big\}$$

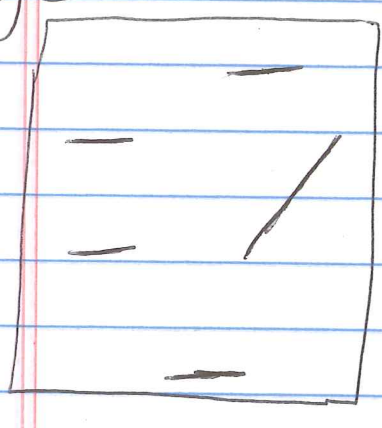
4/27/15

(9) Remark: Since $h(i,j)$ increasing as $|i|$ & $|j|$ increase, only part of distance 2 cone and none of distance 3 part of cone intersect $\mathcal{d}$.

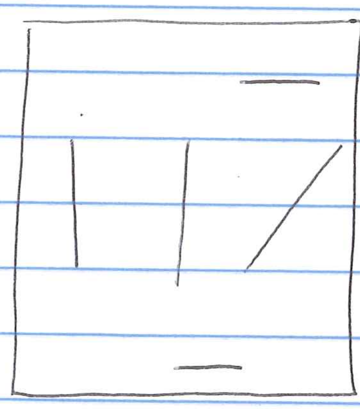
$G_{3,1,0} =$



$F(3,1,0) = \dfrac{x_{10}\,x_{20}}{x_{00}}$

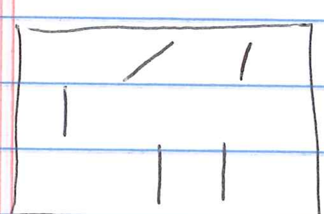


$+ \dfrac{x_{01}\,x_{20}\,x_{0-1}}{x_{00}\,x_{10}}$



$+ \dfrac{x_{11}\,x_{2-1}\,x_{0-1}}{x_{10}\,x_{1-1}}$

$+$



$\dfrac{x_{11}\,x_{1-2}}{x_{1-1}}$.

See Section 3, Fig 1 of [Speyer]

4/27/15

(10) Gale-Robinson Sequence as a special case (Sections 1.3 and 4.3 of [Speyer])

and Propp also cited in "Laurent Phenomenon" by Fomin-Zelevinsky for writing GR sequence as an example of the Octahedron Recurrence

$$\boxed{g(n)g(n-k) = g(n-r)g(n-k+r) + g(n-s)g(n-k+s)}$$

Then $f(n,i,j) = g\left(\frac{kn+(2r-k)i+(2s-k)j}{2}\right)$ satisfy octahedron recurrence

Speyer's definition: Let $h(i,j) = h$ be the unique integer satisfying $\boxed{\bullet\ h+i+j \equiv 0 \bmod 2}$

and $\boxed{\bullet\ -k < \frac{kh+(2r-k)i+(2s-k)j}{2} \leq 0}$

Claim: $\boxed{h(i,j) = i+j-2\left\lceil \frac{ri+sj}{k} \right\rceil}$

Pf: Suppose $\frac{ri+sj}{k} = m + \frac{d}{k}$ with $0 \leq d < k$

Then $h(i,j) = \begin{cases} i+j-2m & \text{if } d=0 \\ i+j-2m-2 & \text{if } d>0 \end{cases}$

$$\Rightarrow \frac{kh+(2r-k)i+(2s-k)j}{2} = \begin{cases} \frac{-2km+2(ri+sj)}{2} \\ \frac{-2km-2k+2(ri+sj)}{2} \end{cases}$$

$$= \begin{cases} 0 & \text{if } d=0 \\ -k+d & \text{if } d>0. \end{cases}$$

4/27/15

(11) Example of Somos-4 $(r=1, s=2, k=4)$

$$h(i,j) = i + j - 2\left\lceil \frac{i+2j}{4} \right\rceil$$

$$\boxed{\begin{array}{l} j \text{ even} \Rightarrow \ h(i,j) = i - 2\left\lceil \frac{i}{4} \right\rceil \\ j \text{ odd} \Rightarrow \ h(i,j) = i+1 - 2\left\lceil \frac{i+2}{4} \right\rceil \end{array}}$$

Thus $\mathcal{A}$ and infinite graph $\mathcal{G}_h$ look like

(-2,-1,3)	(-1,0,3)	(0,1,3)	(1,2,3)	(0,3,3)	(1,4,3)	(2,5,3)	(3,6,3)
(-1,-1,2)	(0,0,2)	(-1,1,2)	(0,2,2)	(1,3,2)	(2,4,2)	(1,5,2)	(2,6,2)
(-2,-1,1)	(-1,0,1)	(0,1,1)	(1,2,1)	(0,3,1)	(1,4,1)	(2,5,1)	(3,6,1)
(-1,-1,0)	(0,0,0)	(-1,1,0)	(0,2,0)	(1,3,0)	(2,4,0)	(1,5,0)	(2,6,0)

i.e. $h(i,j)$ proceeds in horizontal strips of height 2 as

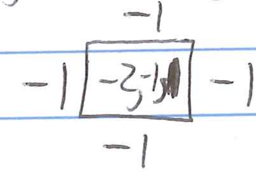
...	-3 -4	-3 -2	-1 -2	-1 0	10	1 2	3 2	3 4	5 4	5 6	...
	-4 -3	-2 -3	-2 -1	$\boxed{0}$ -1	01	2 1	2 3	4 3	4 5	6 5	

$(i,j) = (0,0)$

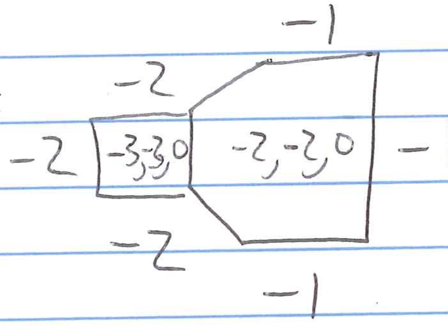
4/27/15

(12) Consider $F(n,i,j) = g(2n-i)$

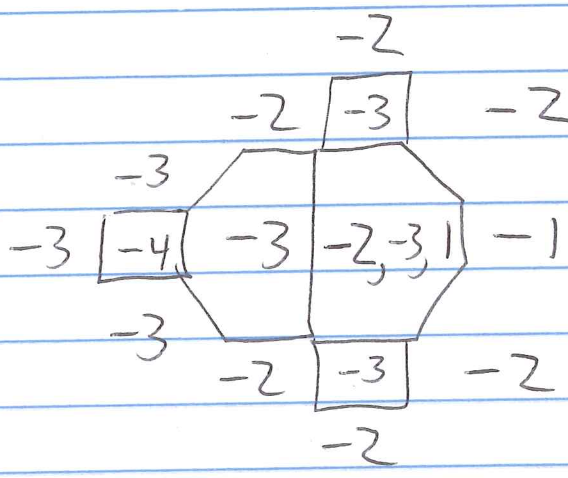
$\boxed{G_{0,-1,1}}$ =



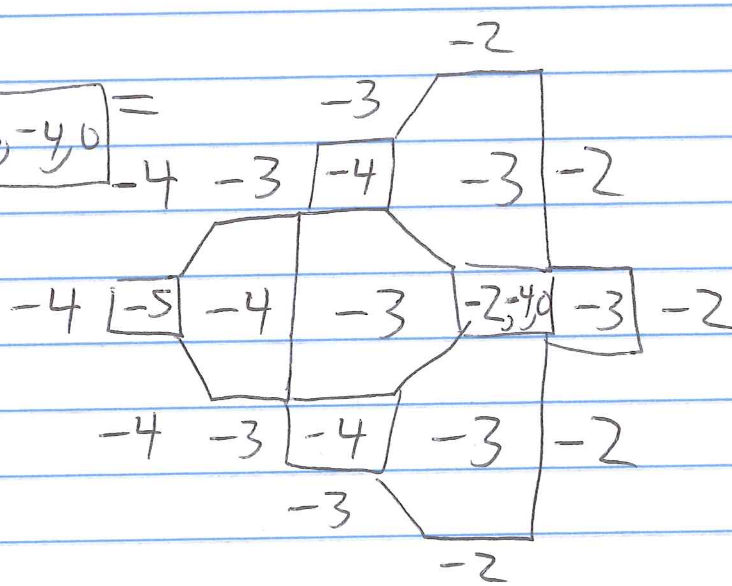
$\boxed{G_{0,-2,0}}$ =



$\boxed{G_{0,-3,1}}$ =



$\boxed{G_{0,-4,0}}$ =



See Figure 11 of [Speyer]

Alternate
$F(0,-2m+1,1)$,
$F(0,-2m,0)$

4/27/18 13

Example of Somos-5 $(r=2, s=1, k=5)$

heights in $\mathcal{L}$ look like

6	5	6	5	6	7	6	7	6	7	8	7	8	7	8	9	8	9	8	9
5	4	5	6	5	6	5	6	7	6	7	6	7	8	7	8	7	8	9	8
4	5	4	5	4	5	6	5	6	5	6	7	6	7	6	7	8	7	8	7
3	4	3	4	5	4	5	4	5	6	5	6	5	6	7	6	7	6	7	8
2	3	4	3	4	3	4	5	4	5	4	5	6	5	6	5	6	7	6	7
3	2	3	2	3	4	3	4	3	4	5	4	5	4	5	6	5	6	5	6
2	1	2	3	2	3	2	3	4	3	4	3	4	5	4	5	4	5	6	5
1	2	1	2	1	2	3	2	3	2	3	4	3	4	3	4	5	4	5	4
0	1	0	1	2	1	2	1	2	3	2	3	2	3	4	3	4	3	4	5
-1	0	1	0	1	0	1	2	1	2	1	2	3	2	3	2	3	4	3	4
$\boxed{0}$	-1	0	-1	0	1	0	1	0	1	2	1	2	1	2	3	2	3	2	3

$(i,j) = (0,0)$

periodicities (up to addition)

add 3 $\uparrow$ 5 steps | add 1 $\longrightarrow$ 5 steps | add 1 $\to\to\uparrow$

$$f(n,i,j) = g\left(\frac{5n + i - 3j}{2}\right)$$ in this case

Thus, we can consider $G_{0,-2m,0}$'s for $g(m)$.

Next time: - Speyer's proof by Kuo Condensation

- Speyer's proof by Urban Renewal

- Di Francesco's related T-system proof by matrices and networks

Later: Relationship between Octahedron Recurrence and Quivers/Brane Tilings/Cluster Variables

4/27/15 14 In Gale-Robinson case, $\mathcal{G}_h$ is the bipartite tiling dual to Gale-Robinson quiver after it is unfolded.

To recover (folded) Gale-Robinson quiver,

we let Face (h, i, j) be labelled by

$$\frac{kh+(2r-k)i+(2s-k)j}{2}$$ which is a

value in $\{-k+1, -k+2, \ldots, -2, -1, 0\}$

or easier label it as $\frac{k(h+1)+(2r-k)i+(2s-k)j}{2}$

in $\{1, 2, \ldots, k-2, k-1, k\}$

and/or could alter definition of height $h(i,j)$ by adding constant one everywhere.

As in our proof on page 10, equivalent to labelling Faces as $\boxed{d \equiv ri+sj \bmod k \text{ in } \{1,2,\ldots,k\}}$.

Rem: This is discussed in Section 6 of "Non-commutative geometry and brane tilings" by Richard Eager (arXiv:1003.2862)

Eager goes further and describes every arrow in Gale-Robinson quiver as an irred. morphism or product of two irred. morphisms

4/27/15 15

Table 2 of Eager		change in (h,i,j) coordinates
$\circ\rightarrow\circ$	a	$(1,1,0)$
$\circ\leftarrow\circ$	b	$(1,-1,0)$
$\uparrow$	c	$(-1,0,1)$
$\downarrow$	d	$(-1,0,-1)$
$\nwarrow$	bd	$(0,-1,1)$
$\swarrow$	bc	$(0,-1,-1)$
$\nearrow$	ac	$(0,1,1)$
$\searrow$	ad	$(0,1,-1)$

Seiberg duality (i.e. quiver mutation) locally changes

$$\begin{tikzcd}
& (h-1,i,j+1) & \\
(h-1,i-1,j) \arrow[r] & (h,i,j) \arrow[u] \arrow[d] & (h-1,i+1,j) \arrow[l] \\
& (h-1,i,j-1) &
\end{tikzcd}$$

to

$$\begin{tikzcd}
& (h-1,i,j+1) \arrow[d] & \\
(h-1,i-1,j) & \boxed{(h-2,i,j)} \arrow[l] \arrow[r] & (h-1,i+1,j) \\
& (h-1,i,j-1) \arrow[u] &
\end{tikzcd}$$