Math 8680: Cluster Algebras and Quiver Representations

Homework 1 (Due Monday September 30, 2016)

I encourage collaboration on the homework, as long as each person understands the solutions, writes them up in their own words, and indicates on the homework page their collaborators.

Please do at least **five** of the following nine problems. (Front and Back)

1) Let Q be the quiver $1 \longrightarrow^{\alpha} 2$. (Here α is meant to be the name or value of this arrow.)

   (a) Show that the only indecomposable representations are

$$S_1 = K \longrightarrow 0, \qquad S_2 = 0 \longrightarrow K, \quad \text{and} \quad P_1 = K \longrightarrow^{1} K.$$

   (b) Suppose that M is a representation of Q with vector spaces $M(1) = K^n$ and $M(2) = K^m$ and linear map $M(\alpha) = A : K^n \longrightarrow K^m$.

   Show that $M \cong S_1^{d_1} \oplus S_2^{d_2} \oplus P_1^r$ where d_1 is the dimension of the kernel of A, d_2 is the dimension of the cokernel of A, and r is the rank of A.

   (c) Show that the representation $P(1)$ is not simple (i.e. irreducible).

2) Problem 1.3 of Schiffler

3) Let Q be the quiver $1 \overset{\alpha}{\underset{\beta}{\rightleftarrows}} 2$ (i.e. with two arrows: α from 1 to 2 and β from 2 to 1). Show that the path algebra KQ is isomorphic to the quotient of the free associative algebra $K\langle \alpha, \beta \rangle / (\alpha^2, \beta^2)$.

4) Problem 1.6 of Schiffler

5) Problem 2.6 of Schiffler

6) Let Q_n be the unidirectional quiver of type A_n which consists of n vertices linearly ordered and labeled as $\{1, 2, \ldots, n\}$, and arrows a_i between vertex $\mathbf{i}$ and $(\mathbf{i+1})$. Let k be a field and kQ_n be the associated path algebra. Let $B_n(k)$ denote the k-algebra of lower-triangular n-by-n matrices over field k.

   a) Show that kQ_n and $B_n(k)$ are isomorphic as k-algebras.

   Hint: It suffices to exhibit a map between basis elements and then show that the same relations are satisfied in both k-algebras.

   b) Describe the projective and injective indecomposable representations of Q_n.

7) Problem 3.1 of Schiffler for the two quivers of type A_6, i.e. parts (1) and (2).

8) Problem 2.11 of Schiffler

9) Problem 2.12 of Schiffler