Math 8680: Cluster Algebras and Quiver Representations

Homework 2 (Due Monday November 14, 2016)

I encourage collaboration on the homework, as long as each person understands the solutions, writes them up in their own words, and indicates on the homework page their collaborators.

Please do at least **five** of the following nine problems.

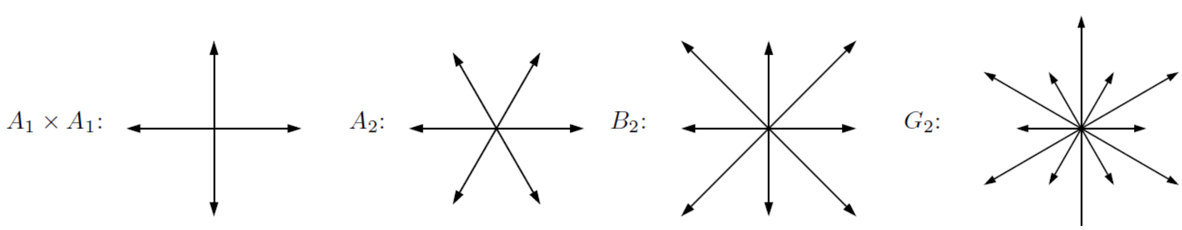
1) Consider the quiver of type E_6 with the given orientation:

$$\begin{array}{ccccccccc} & & & & \circ & & & & \\ & & & & \downarrow & & & & \\ \circ & \longrightarrow & \circ & \longrightarrow & \circ & \longleftarrow & \circ & \longleftarrow & \circ. \end{array}$$

Write down the dimension vectors of all indecomposable representations by applying powers of the coxeter element c.

2) Problem 7.1 of Schiffler (you may skip Parts (2) and (7)).

3) Consider the following families of vectors in two-dimensional space.



a) Verify that each of these collections satisfy the axioms of a root system: i.e. each set Φ is a finite non-empty collection of non-zero vectors from a vector space V such that

(i) Each one-dimensional subspace of V either contains no roots or contains the two roots $\{-\alpha, \alpha\}$.

(ii) For each $\alpha_i \in \Phi$, the reflection s_i permutes Φ.

b) Prove that these four root systems are the only root systems of rank 2, up to scaling and rotation. (By rank, we mean the dimension of the span of Φ as a subspace of V.)

Hint: Argue why you may assume that the unit vector e_1 is included in your root system.

c) Show that $\{e_j - e_i : i, j \in \{1, 2, \ldots, n+1\}, i \neq j\}$ is a rank n root system. (Called A_n)

d) Show that $\{\pm e_i : i \in \{1, 2, \ldots, n\}\} \cup \{\pm e_i \pm e_j : 1 \leq i < j \leq n\}$ is a rank n root system. (Called B_n)

e) Show that $\{\pm 2e_i : i \in \{1, 2, \ldots, n\}\} \cup \{\pm e_i \pm e_j : 1 \leq i < j \leq n\}$ is a rank n root system. (Called C_n)

f) Show that $\{\pm e_i \pm e_j : 1 \leq i < j \leq n\}$ is a rank n root system. (Called D_n)

Hint: You will use similar techiques for exercises (c)-(f) so feel free to reuse your work.

4) Suppose that Q is a quiver without oriented cycles with n vertices.

(a) Show that you can choose a bijection $\varphi : \{1, 2, \ldots, n\} \to Q_0$ such that for every arrow, we have

$$\varphi^{-1}(sa) > \varphi^{-1}(ta).$$

(b) Show that under this ordering, each vertex $(i+1)$ is a sink of the quiver $s_i s_{i-1} \cdots s_1 Q$.

(c) Define $F^+ = F_n^+ F_{n-1}^+ \cdots F_1^+$. Show that F^+ is independent of the choice of the bijection φ.

(d) Let c be the Coxeter element $s_n s_{n-1} \cdots s_1$. Show that for every two dimension vectors α and β, we have

$$\langle \alpha, c(\beta) \rangle = -\langle \beta, \alpha \rangle$$

where $\langle \cdot, \cdot \rangle$ is the standard bilinear form associated to Q.

(e) Show that if $\langle \cdot, \cdot \rangle$ is positive definite, and there exists $\alpha \in \mathbb{R}^n$ such that $c(\alpha) = \alpha$, then $\alpha = 0$.

(f) For $j \in \{1, 2, \ldots, n\}$, let $\mathbf{p(j)}$ be the dimension vector of the projective $P(j)$ and let $\mathbf{i(j)}$ be the dimension vector of the injective $I(j)$. Show that

$$c(\mathbf{p(j)}) = -\mathbf{i(j)}.$$

(g) Show that if V is indecomposable, then $F^+ V = 0$ if and only if V is projective.

(h) Show that if V is indecomposable, then $F^- V = 0$ if and only if V is injective.

5) In this problem, you will describe the connection between the original definition of cluster algebras and those of geometric type.

Let $\mathbb{P}$ be the semifield $\mathrm{Trop}(u_1, u_2, \ldots, u_m)$ defined with $\oplus$ as $u_1^{d_1} \cdots u_m^{d_m} \oplus u_1^{e_1} \cdots u_m^{e_m} = u_1^{\min(d_1, e_1)} \cdots u_m^{\min(d_m, e_m)}$. Let $\mathcal{A}$ be a rank n cluster algebra defined over the ground ring $\mathbb{ZP}$ with seed

$$(\mathbf{x} = \{x_1, x_2, \ldots, x_n\}, \quad \mathbf{y} = \{y_1, y_2, \ldots, y_n\}, \quad B).$$

We remind the reader that the set $\mathbf{x}$ is algebraically independent, the y_i's are in $\mathbb{P}$, and B is an n-by-n skew-symmetrizable matrix. Elements of $\mathbb{ZP}$ are Laurent polynomials in the u_i's.

Let $\tilde{\mathcal{A}}$ denote a cluster algebra of *geometric type* (over the ground ring $\mathbb{Z}[u_1^{\pm 1}, \ldots, u_m^{\pm 1}]$) with seed given by

$$\left(\tilde{\mathbf{x}} = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n, u_1, u_2, \ldots, u_m\}, \quad \tilde{B}\right),$$

where $\tilde{B}$ is an $(m+n)$-by-n matrix, whose top n-by-n submatrix is skew-symmetrizable.

a) Show that if $\tilde{B} = \left[\frac{B}{C}\right]$ where $C = [c_{ij}]$ is a general m-by-n integer matrix such, and we let $x_i = \tilde{x}_i$, $y_j = \prod_{i=1}^m u_i^{c_{ij}}$ for each column index $j \in \{1, 2, \ldots, n\}$, then $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are isomorphic as cluster algebras.

In particular, show that the mutation rules (1) and (2) agree, and that the matrix mutation rule for the matrix $\tilde{B}$ is equivalent to the matrix mutation rule for B plus the coefficient mutation rule (3).

$$x'_k = \frac{y_k \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}}{(y_k \oplus 1)x_k} \tag{1}$$

$$\tilde{x}'_k = \frac{\prod_{\substack{\tilde{b}_{n+i,k}>0 \\ 0\le i\le m}} u_i^{\tilde{b}_{n+i,k}} \prod_{\substack{\tilde{b}_{ik}>0 \\ 0\le i\le n}} \tilde{x}_i^{\tilde{b}_{ik}} + \prod_{\substack{\tilde{b}_{n+i,k}<0 \\ 0\le i\le m}} u_i^{-\tilde{b}_{n+i,k}} \prod_{\substack{\tilde{b}_{n+i,k}<0 \\ 0\le i\le n}} \tilde{x}_i^{-\tilde{b}_{ik}}}{\tilde{x}_k} \tag{2}$$

$$y'_j = \begin{cases} y_k^{-1} & \text{if } \ j = k, \text{ and} \\ y_j \frac{y_k^{\max(b_{kj},0)}}{(y_k \oplus 1)^{b_{kj}}} & \text{if } \ 1 \le j \le n, j \ne k \end{cases} \tag{3}$$

b) As an example, reformulate the cluster algebra of geometric type described below in Problem 8 as a cluster algebra with coefficients y_1, $y_2 \in \mathrm{Trop}(u_1, u_2, u_3)$. Compute $\mu_1(y_1), \mu_1(y_2), \mu_2\mu_1(y_1)$, and $\mu_2\mu_1(y_2)$.

6) (a) Which orientations of an n-cycle are mutation equivalent?

(b) Exercise 2.6.6 of Fomin-Williams-Zelevinsky (`https://arxiv.org/pdf/1608.05735.pdf`).

7) Consider the cluster algebra $\mathcal{A}(2,2)$ with exchange matrix $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$. Let z be the Laurent polynomial $z = \frac{x_1^2 + x_2^2 + 1}{x_1 x_2}$.

a) Show that for all $n \in \mathbb{Z}$, with respect to the cluster $\{x_n, x_{n+1}\}$ that $z = \frac{x_n^2 + x_{n+1}^2 + 1}{x_n x_{n+1}}$.

This is called a **conserved quantity**.

b) Show that the cluster algebra exchange relation linearizes as $x_{n+1} = z x_n - x_{n-1}$ in this case.

8) Consider the cluster algebra of geometric type defined by the initial labeled seed given by

$$\mathbf{x} = \{x_1, x_2, u_1, u_2, u_3\} \text{ and } B = \begin{bmatrix} 0 & 2 \\ -1 & 0 \\ -1 & 0 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

a) Compute all cluster variables generating this cluster algebra.

b) Try to find a geometric model for this cluster algebra.

9) a) Let $\{f_n\}$ be a sequence of rational functions defined by the following initial conditions and recurrence: $f_1 = x_1,\ f_2 = x_2,\ f_3 = x_3$, and for $n \geq 3$, $f_n f_{n-3} = f_{n-1} f_{n-2} + 1$. Use the Caterpillar Lemma or model this sequence by a cluster algebra to show that f_n is a Laurent polynomial in $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$ for all $n \geq 1$.

b) By a similar method, show that Somos-5, defined as $f_1 = x_1,\ f_2 = x_2,\ f_3 = x_3,\ f_4 = x_4,\ f_5 = x_5$, and for $n \geq 5$, $f_n f_{n-5} = f_{n-1} f_{n-4} + f_{n-2} f_{n-3}$ gives rise to Laurent polynomials in $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, x_4^{\pm 1}, x_5^{\pm 1}]$.