Math 8680: Cluster Algebras and Quiver Representations

Homework 3 (Due Monday December 12, 2016)

I encourage collaboration on the homework, as long as each person understands the solutions, writes them up in their own words, and indicates on the homework page their collaborators. You may use computer algebra packages for calculations but should also briefly describe your calculations in words in this case.

Please do **either** Problems 1 & 2 **or** two out of the four Problems 3, 4, 5 & 6. (**Note: Problem 6 added Friday Dec. 2nd**)

1) Let $Q_{\tilde{A}_2}$ be the quiver of (affine) type $\widetilde{A}_2$ shown below.

$$\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[rr] & & 3 \end{tikzcd}$$

a) Describe a marked surface and triangulation whose cluster algebra model has $Q_{\tilde{A}_2}$ as its corresponding quiver.

b) Compute the dimension vectors of $\tau^{-1}P(i)$ for $i = 1, 2, 3$ and draw the beginning of the pre-projective component of the Auslander-Reiten quiver, showing the six vertices $P(i)$ and $\tau^{-1}P(i)$.

c) Compute the dimension vectors of $\tau I(i)$ for $i = 1, 2, 3$ and draw the ending of the pre-injective component of the Auslander-Reiten quiver, showing the six vertices $I(i)$ and $\tau I(i)$.

d) Compute the τ-orbit of $S(2)$ and construct the dimension vector of at least six indecomposable modules in its component in the Auslander-Reiten quiver. Show that this component contains two distinct indecomposable modules that both have the dimension vector $(1, 1, 1)$. (This component is called a *tube* of rank two is a *regular* component.)

e) Which arcs on the surface (from part (a)) correspond to indecomposables in pre-projective component? pre-injective component? regular component?

2) a) Pick several arcs γ on the surface and compute their Laurent expansion (either by the snake graph formula or by explicit mutation sequences).

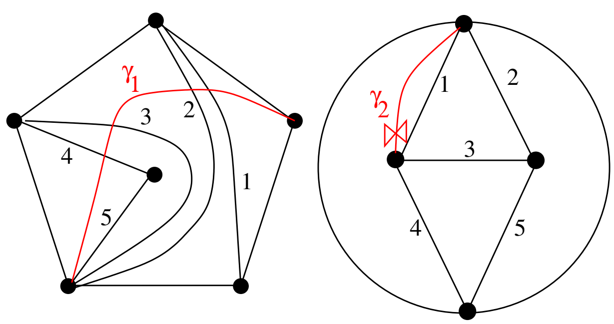
b) Writing your Laurent polynomial in reduced terms (so that there is no common factor in the numerator and denominator), how do the denominator vectors of your Laurent expressions in x_γ compare with how γ crosses the triangulation T? Make a conjecture.

Hint: Do enough examples until you see that crossings do not exactly correspond to elements of the denominator.

c) Consider the indecomposable modules from parts (c), (d), and (e). Use the Caldero-Chapoton formula to compute the corresponding cluster variables for several examples.

d) How do the dimension vectors of the indecomposables compare with the denominators of the corresponding cluster variables? Do you notice any other quantities that dimension vectors match up more closely with? Make a conjecture.

3) Let (S_1, M_1) and (S_2, M_2) be the punctured surfaces on the left and right sides of the figure below, respectively.



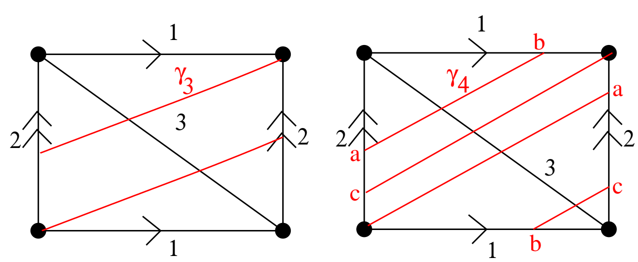
a) What finite type or affine type cluster algebras do (S_1, M_1) and (S_2, M_2) correspond to?

b) Let γ_1 and γ_2 be the following (tagged) arcs in marked surfaces (S_1, M_1) and (S_2, M_2) shown above. Find mutation sequences to reach clusters that contain γ_1 and γ_2.

c) Using snake graph formulas for cluster variables from surfaces, compute the Laurent expansions of x_{γ_1} and x_{γ_2}. (At least draw the appropriate graphs and a few terms of the Laurent expansions.)

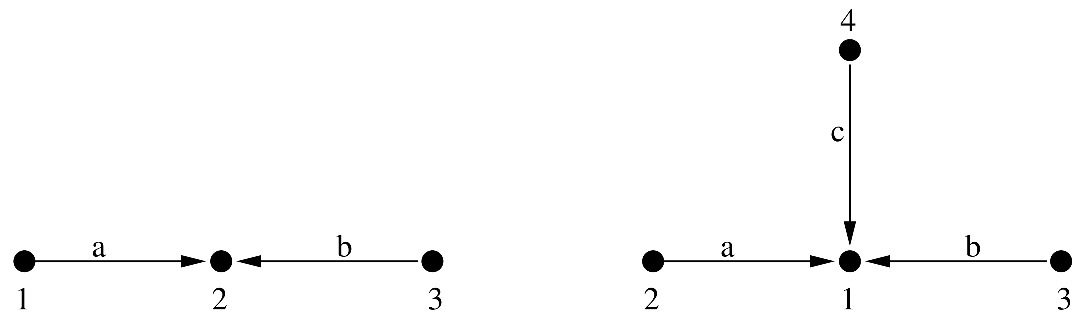
4) a) In the once-punctured torus below, find mutation sequences to reach clusters that contain γ_3 and γ_4.

b) Compute the Laurent expansions of x_{γ_3} and x_{γ_4}.



5) Let Q_{A_3} be the quiver shown below on the left and Q_{D_4} be the quiver on the right.

a) For each indecomposable representation V of Q_{A_3}, compute the corresponding unique (non-initial) cluster variable X_V according to the Caldero-Chapoton formula.



b) Do the same for each indecomposable representation V of Q_{D_4}.

6) Consider the once-punctured bi-gon with boundary segments b_1 and b_2 triangulated with two radii, r and s. The corresponding cluster algebra from this surface is of type $D_2 = A_1 \times A_1$ and has four cluster variables.

a) Use cluster mutation to write the two non-initial cluster variables in terms of b_1, b_2, x_r and x_s. (Here x_r and x_s are the cluster variables corresponding to arcs r and s, and by abuse of notation we leave the frozen variables labeled as b_1 and b_2.)

b) Let ℓ_1 and ℓ_2 be the loops in this triangulated bigons that you get by flipping the radii r and s. Use the skein relation to expand $x_{\ell_1} x_{\ell_2}$ as a sum of three non-zero terms. (**Note:** a contractible monogon corresponds to zero in the cluster algebra.)

c) Comparing your computation in part (b) to the product of the four cluster variables, from part (a), conclude that the value of a closed loop around a puncture is simply the number 2 in the cluster algebra.

Consider the Kronecker quiver and its cluster algebra, which also corresponds to a cluster algebra from an annulus with a single marked point on each boundary. If the initial cluster is $\{x_1, x_2\}$, let x_n for $n \in \mathbb{Z} - \{1, 2\}$ denote the non-initial cluster variables with the recurrence $x_n x_{n-2} = x_{n-1}^2 + 1$.

d) Solve for z in the equation

$$x_3 x_0 = x_1 x_2 + z$$

using the skein relation and the above recurrence. Write z as a Laurent polynomial in $\{x_1, x_2\}$. Conclude that this Laurent polynomial is the value of a closed loop around the inner boundary for this cluster algebra.

Let z_k denote a closed loop that winds around the inner boundary k times (so that it has $(k-1)$ self-crossings). Note that $z = z_1$.

e) Use the skein relation to prove that the Laurent polynomials corresponding to the z_k's satisfy

$$z_k = z \cdot z_{k-1} - z_{k-2}$$

for $k \geq 3$. Show that this formula still works for $k = 1, 2$ if you define z_{-1} and z_0 appropriately.