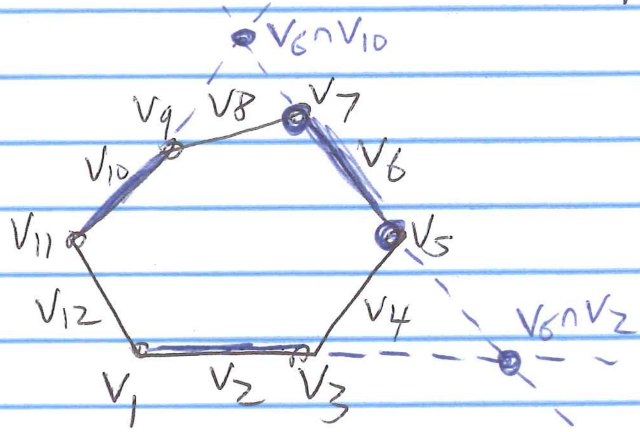
10/2/18 Last time we discussed <u>corner invariants</u> $X_i(P)$, $\in \mathbb{R}\cup\{\infty\}$
where $0 \le i \le n-1$ for convex n-gon P.

Today, we coordinatize instead w/ $2n$ real numbers for vertices & sides of P.

$$Y_i(P) := \begin{cases} \dfrac{-1}{\chi(\langle v_i, v_{i-4}\rangle, v_{i-1}, v_{i+1}, \langle v_i, v_{i+4}\rangle)} & \text{if } v_i \text{ is a vertex of } P \\[2ex] -\chi(v_i \cap v_{i-4}, v_{i-1}, v_{i+1}, v_i \cap v_{i+4}) & \text{if } v_i \text{ is a side of } P \end{cases}$$

e.g. P = hexagon

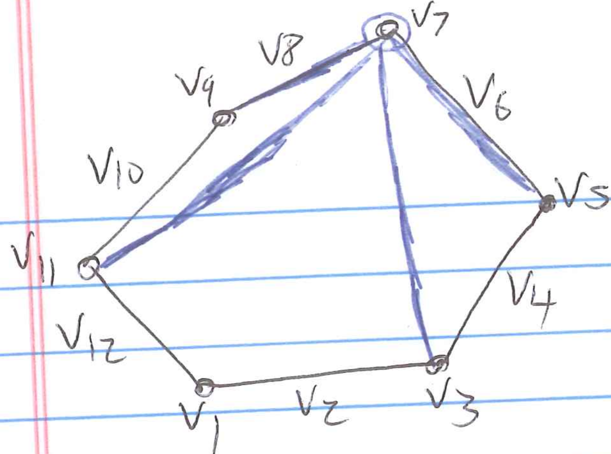


v_6 is a side $\Rightarrow$ $Y_6(P) = -\chi(v_6 \cap v_2, v_5, v_7, v_6 \cap v_{10})$

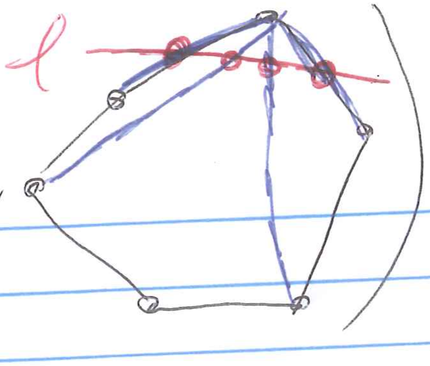
sides v_6 and v_2 intersect in a unique pt
and so do v_6 and v_{10}.

$v_6 \cap v_2$, v_5, v_7, $v_6 \cap v_{10}$ all collinear so we define cross-ratio accordingly.

(2)



OR equiv



V_7 is a vertex $\Rightarrow$ $Y_7(P) = -\dfrac{1}{\chi(\langle V_7, V_3\rangle, V_6, V_8, \langle V_7, V_{11}\rangle)}$

sides V_6 and V_8 meet at vertex V_7

and $\langle V_7, V_3\rangle$, $\langle V_7, V_{11}\rangle$ define two more lines meeting at V_7.

We thus define $\chi(\langle V_7, V_3\rangle, V_6, V_8, \langle V_7, V_{11}\rangle)$ using the slopes of these 4 lines which all meet at the same point.

OR equivalently, intersect w/ any ℓ and take cross ratio of the four collinear intersection points.

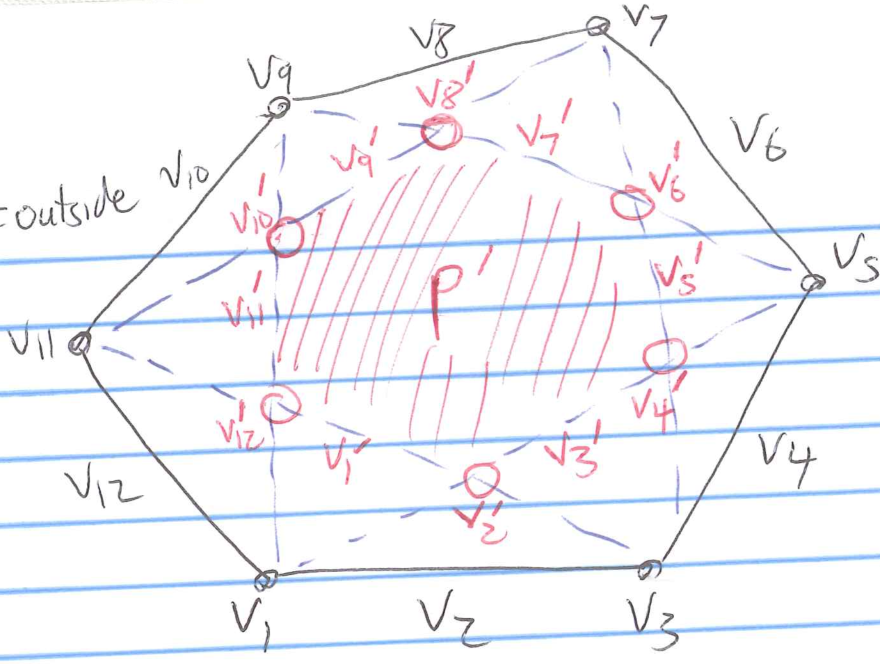
Call these the y-parameters of polygon P.

$\boxed{\text{subscripts taken modulo } 2n}$

Prop 6.6 of [GR18] Let $P' = T(P)$, the polygon after applying the pentagram map. For $i = 1, 2, \ldots, 2n$, let Y_i' denote the y-parameters of P'. Then

$$Y_i' = \begin{cases} Y_i^{-1} & \text{if } i \text{ is a side of } P' \text{ (a vertex of } P) \\ Y_i \dfrac{(1+Y_{i-1})(1+Y_{i+1})}{(1+Y_{i-3}^{-1})(1+Y_{i+3}^{-1})} & \text{if } i \text{ is a vertex of } P' \text{ (a side of } P) \end{cases}$$

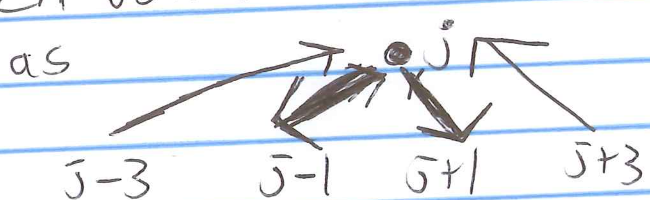
③ E.g. P = outside



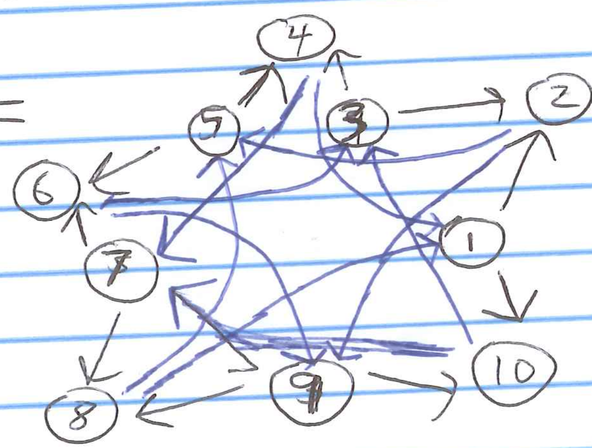
$P \mapsto T(P) = P'$ switches the roles of vertices & sides.

Def: Consider the bipartite quiver Q_n defined on $2n$ vertices with arrows around each vertex j when j is odd.

as



E.g. $Q_5 =$



Let $\mu_{odd} = \mu_{2n-1} \circ \mu_{2n-3} \circ \dots \circ \mu_3 \circ \mu_1$

$\mu_{even} = \mu_{2n} \circ \mu_{2n-2} \circ \dots \circ \mu_4 \circ \mu_2$

Just like in the case of Zamolodchikov periodicity, $\mu_{2i} \circ \mu_{2j}$ commute so can rearrange these seqs or think of as simultaneous mutation. (Same for constituents of μ_{odd}.)

(4) Thm 6.7 of [GR18] (originally from Glick 2010) Consider the

Y-seed $((y_1, y_2, \dots, y_{2n}), Q_n)$ and apply K

compound mutations $\underbrace{\mu_{even} \circ \mu_{odd} \circ \mu_{even} \circ \dots}_{K}$

alternating between the two.

The resulting Y-seed is $\left(\left(y_1^{(K)}, y_2^{(K)}, \dots, y_{2n}^{(K)}\right), (-1)^K Q_n\right)$

where if we let

original Q_n or its reverse

y_i's = y-parameters of convex n-gon P

then $y_i^{(K)}$'s = y-parameters of n-gon $T^K(P)$

where $T^K(P)$ is the result after applying the petagram map T to P K times.

Furthermore, just as cluster variables w/ principal coeffs (or in general setting w/ semifield $\mathbb{P}$) could be rewritten in terms of F-polynomials, we have the following from Cl. Alg IV

Prop 3.13 Y-seed components $Y_{j,t}$ for $j = 1, \dots, N$ (N = # vertices in Q) can be rewritten as

t = seed

$$Y_{j,t} = \vec{y}^{\,\vec{c}_{j,t}} \cdot \prod_{i=1}^{N} F_{i,t}^{(b_{ij})_t}(y_1, y_2, \dots, y_N) \quad \text{where}$$

$\vec{c}_{j,t}$ = c-vector, i.e. bottom of column j in $\tilde{B}_t = \begin{bmatrix} B_t \\ C_t \end{bmatrix}$ assuming $\tilde{B}_{t_0} = \begin{bmatrix} B_{t_0} \\ I \end{bmatrix}$.

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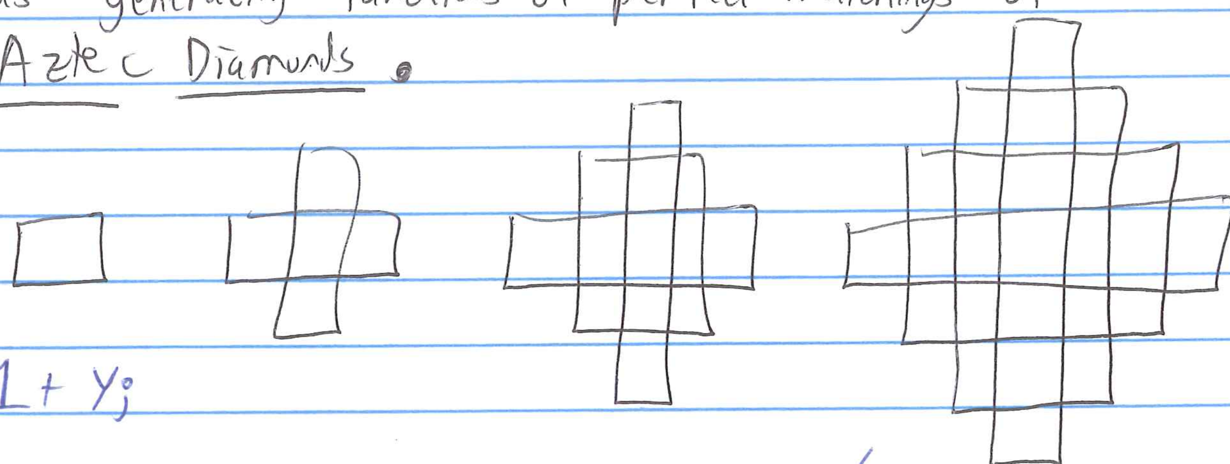
Thm 1.2 (Glick '10) Given convex n-gon P, the y-parameters of $T^k(P)$ are given by

$$y_j\left(T^k(P)\right) = \begin{cases} \left(\prod_{i=-k}^{k} y_{j+3i}\right) \dfrac{F_{j-1,k}\,F_{j+1,k}}{F_{j-3,k}\,F_{j+3,k}} & \text{when } j+k \text{ is even} \\[2ex] \left(\prod_{i=-k+1}^{k-1} y_{j+3i}\right) \dfrac{F_{j-3,k-1}\,F_{j+3,k-1}}{F_{j-1,k-1}\,F_{j+1,k-1}} & \text{when } j+k \text{ is odd} \end{cases}$$

taking subscripts of y_ℓ's modulo $2n$

and $$F_{j,k} = \sum_{\substack{\text{order ideals} \\ I \text{ in poset } P_k}} \prod_{(r,s,t)\in I} y_{3r+s+j}$$

Instead of giving Glick's definition of posets P_k, we will re-express the F-polynomials as generating functions of perfect matchings of Aztec Diamonds.



e.g. $F_{j,1} = 1 + y_j$

$$F_{j,2} = 1 + y_{j-3} + y_{j+3} + y_{j-3}y_{j+3} + y_{j-3}\,y_j\,y_{j+3}\left(1 + y_{j-1} + y_{j+1} + y_{j-1}y_{j+1}\right)$$

⑥ Based on Elkies-Kuperberg-Larsen-Propp, Max Glick defines <u>poset P_K</u> to have the vertices

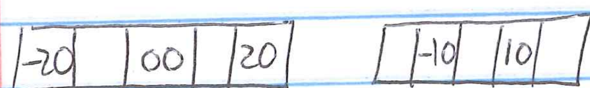
$$\left\{(r,s,t)\in\mathbb{Z}^3 \;\middle|\; \begin{array}{l} 2|s|-(K-2)\le t\le (K-2)-2|r| \\ \text{and } 2|s|-(K-2)\equiv t\equiv (K-2)-2|r| \bmod 4 \end{array}\right\}$$

$$\cup \left\{(r,s,t)\in\mathbb{Z}^3 \;\middle|\; \begin{array}{l} 2|s|-(K-1)\le t\le (K-1)-2|r| \\ \text{and } 2|s|-(K-1)\equiv t\equiv (K-1)-2|r| \bmod 4 \end{array}\right\}$$

with cover relations $(r',s',t')\gtrdot(r,s,t) \iff t'=t+1$ and $|r'-r|+|s'-s|=1$.

e.g. P_3 has five layers

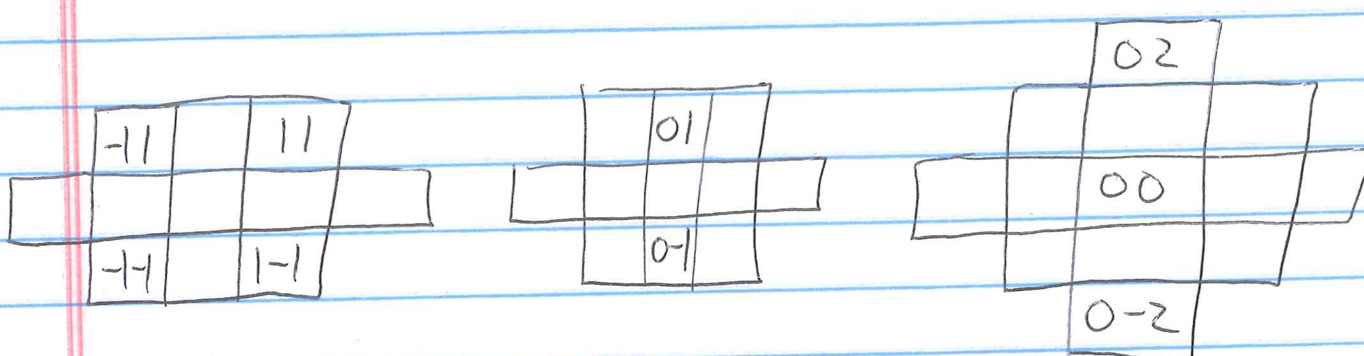
$t=-2$ $\qquad$ $t=-1$



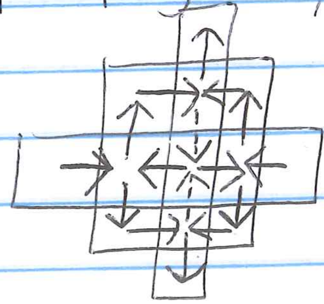
(r,s) values illustrated

Notice: $(0,0,-2)$ and $(0,0,2)$ both in P_3 but otherwise (r,s) determines (r,s,t) uniquely.

$t=0$ $\qquad$ $t=1$ $\qquad$ $t=2$

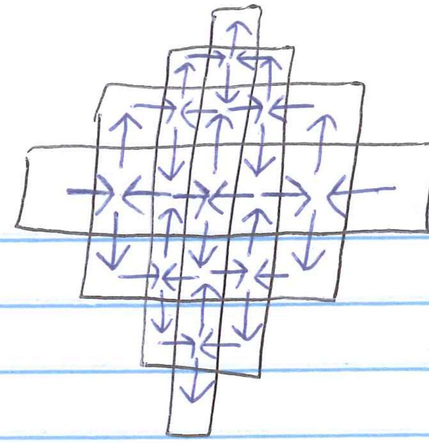


Superimposing all layers together, we see an Aztec Diamond

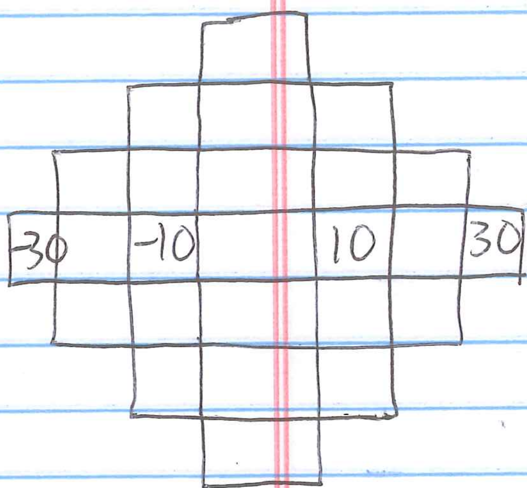


<u>cover relations illustrated</u>

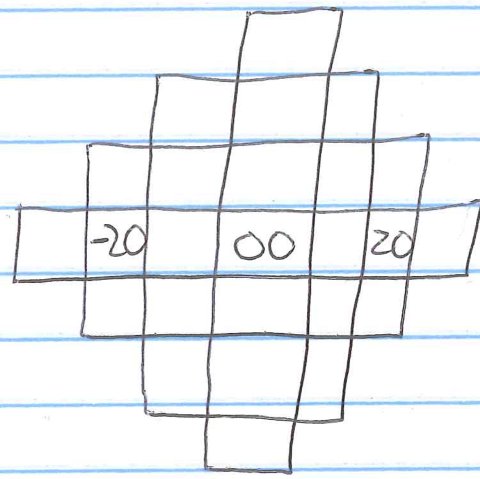
(7) e.g. P_4 has seven layers



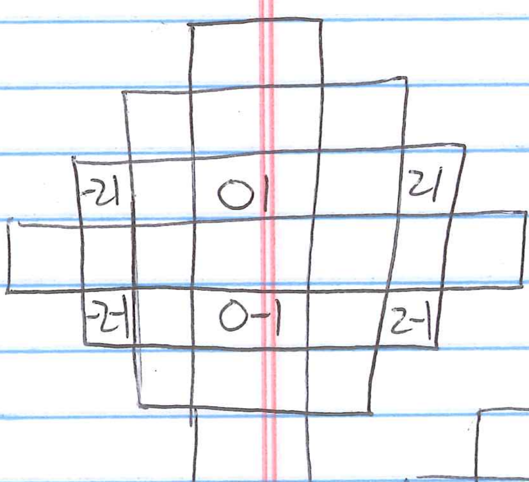
$t=-3$



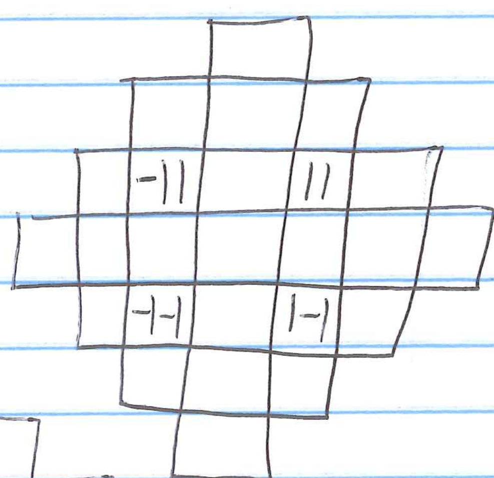
$t=-2$



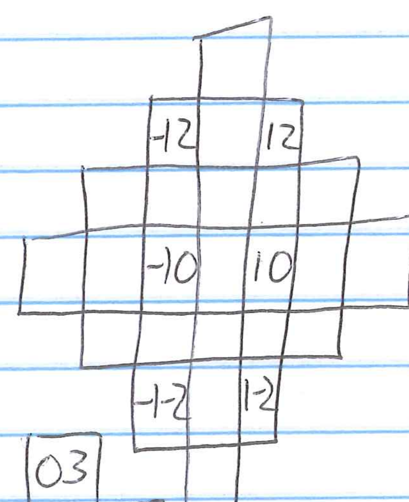
$t=-1$



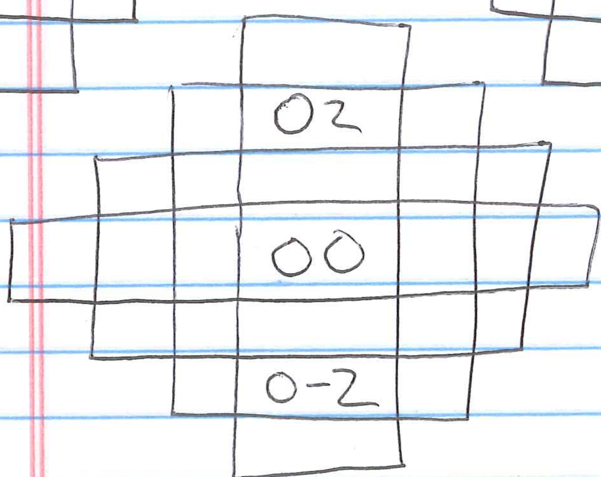
$t=0$



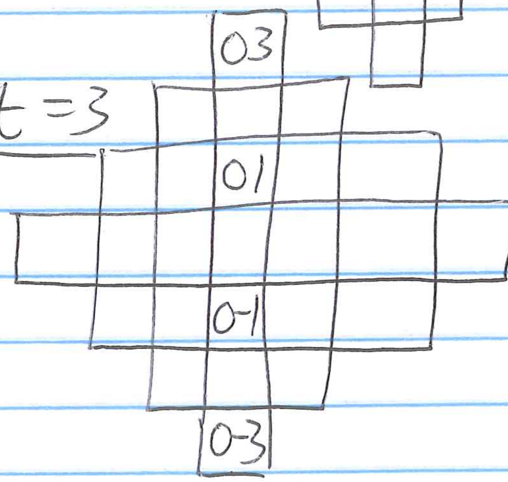
$t=1$



$t=2$



$t=3$

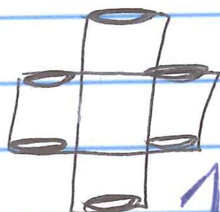


8

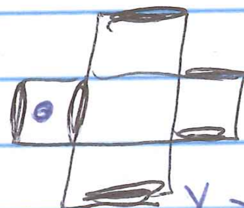
<u>order ideals</u> $\mathcal{I}$ in a poset are subsets that are downward-closed under the cover relation.

<u>Claim:</u> Order ideals $\mathcal{I}$ of $\mathcal{P}_K$ are in bijection with perfect matchings of the <u>Aztec Diamond</u> with height and width of $(2K-1)$ squares.

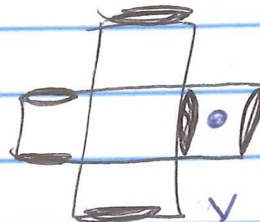
e.g. P_2) $\emptyset$ $\quad \{(-1,0,-1)\}$ $\quad \{(1,0,-1)\}$ $\quad \{(-1,0,-1),(1,0,-1)\}$



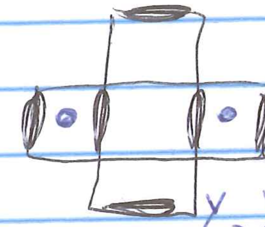
1



Y_{-3+j}



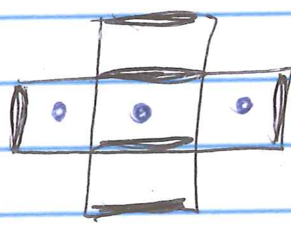
Y_{3+j}



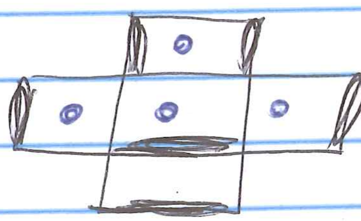
$Y_{-3+j}Y_{3+j}$

Summing these 8 summands together yields $\boxed{F_{j,2}}$ from the bottom of pg. 5

$\{(-1,0,-1),(1,0,-1),(0,0,0)\}$ $\quad \{(-1,0,-1),(1,0,-1),(0,0,0),(0,1,1)\}$



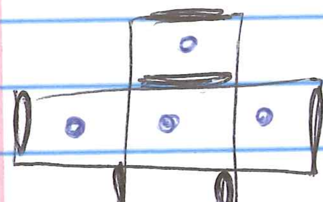
$Y_{-3+j}Y_jY_{3+j}$



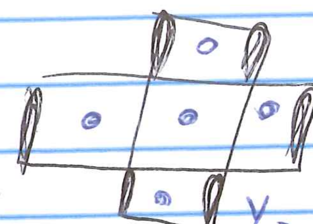
$Y_{-3+j}Y_jY_{1+j}Y_{3+j}$

$\{(-1,0,-1),(1,0,-1),(0,0,0),(0,-1,1)\}$ $\quad \{\text{entire } P_2\}$

These weights called <u>height funcs</u> of the perf. matchings



$Y_{-3+j}Y_{-1+j}Y_jY_{3+j}$

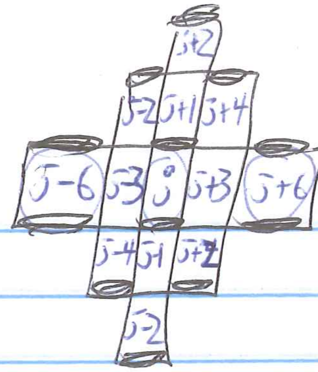


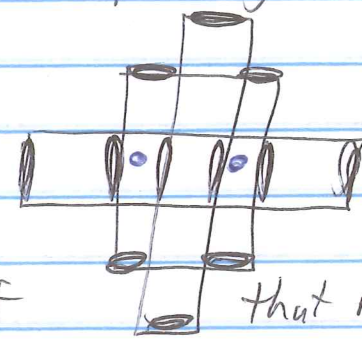
$Y_{-3+j}Y_{-1+j}Y_jY_{1+j}Y_{3+j}$

<u>Recall:</u> <u>weights</u> are $\prod_{(r,s,t)\in\mathcal{I}} Y_{3r+s+j}$ [equiv to recording flips:

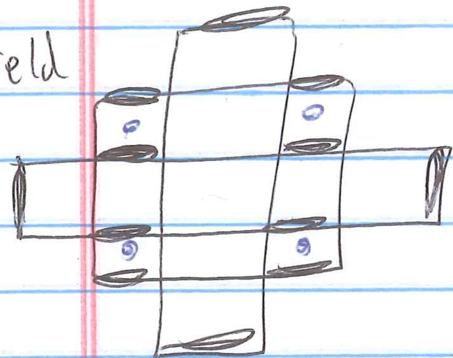
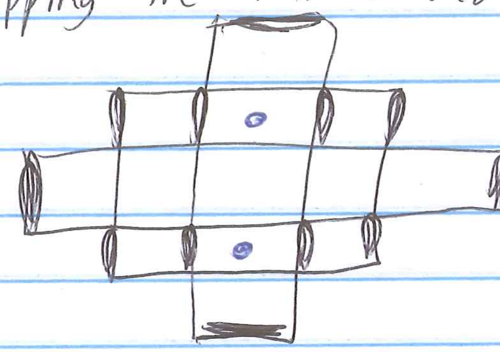
	$1+j$	
$-3+j$	j	$3+j$
	$-1+j$	

]

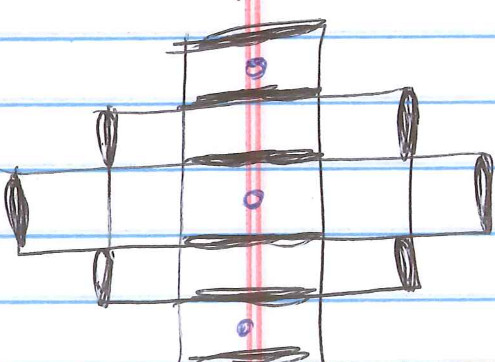
(9) For P_3, we see  for y-weights.

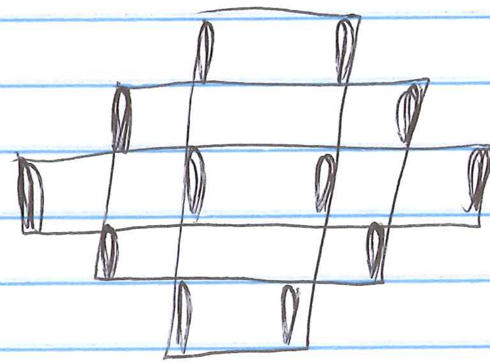
Flipping three squares in central row yields 

we then flip the remaining two squares of that row

to yield . The Flipping the four labeled squares yields 

Followed by Flipping two squares in the center column:



Flipping the rest of the central column concludes with .

Note that the central square is flipped twice, both in the first step and in the last step.

Coincides w/ $(0,0,-2)$ & $(0,0,2)$ both appearing in P_3.

Rem: Aztec Diamonds are subgraphs of infinite checkerboard w/ labels $y_{j+\ell}$ for $\ell =$

-4	-1	2	5	8
-5	-2	1	4	7
-6	-3	0	3	6
-7	-4	-1	2	5
-8	-5	-2	1	4