9/26/18 Lecture Math 8680: Poisson brackets compatible with cluster algebra structures

**Def:** A Poisson algebra is a commutative associative algebra $g$ equipped w/ a Poisson bracket

$\{\cdot,\cdot\} : g \times g \to g$, which is a skew-symmetric bilinear map

$$\left( \begin{array}{l} \{ax+by, cw+dz\} = ac\{x,w\} + ad\{x,z\} + bc\{y,w\} + bd\{y,z\} \\ \text{and} \quad \{y,x\} = -\{x,y\} \end{array} \right)$$

satisfying the Leibniz identity

$$\{F_1F_2, F_3\} = F_1\{F_2,F_3\} + \{F_1,F_3\}F_2$$

and the Jacobi identity

$$\{F_1,\{F_2,F_3\}\} + \{F_2,\{F_3,F_1\}\} + \{F_3,\{F_1,F_2\}\} = 0.$$

**Rem:** If we let $g = \mathcal{C}^\infty(M)$, the algebra of smooth functions on a symplectic manifold, the usual product & chain rules of differentiation imply these identities.

Rem: Given an associative algebra $A$, let $\{x, y\}$ be defined as the commutator

$[x,y] = xy - yx$. This yields a Lie algebra with Lie bracket $[\circ,\circ]$ and together with the multiplicative structure of $A$ this is a Poisson algebra.

Rem: The tensor algebra of a Lie algebra is a Poisson algebra.

Def: Given a cluster algebra $\mathcal{A}$, we say a Poisson bracket $\{\circ,\circ\}$ on $\mathcal{A}$ is compatible with the cluster algebra structure if every cluster $\mathbb{X}$ of $\mathcal{A}$ is log-canonical w.r.t. $\{\circ,\circ\}$, i.e.

$\exists$ skew-symmetric matrix $\Omega_{\mathbb{X}}$ w/ entries $[\Omega_{ij}]$ s.t. $\{x_i, x_j\} = \Omega_{ij}\, x_i x_j \quad \forall\, x_i, x_j \in \mathbb{X}$.

Equivalently, $\{\log x_i, \log x_j\} = \Omega_{ij}$.

Lemma: Let $\widetilde{B}$ be the extended exchange matrix for a cluster algebra, such that $\widetilde{B}$ is $(m+n)$-by-$n$ and rank $n$.

Let $\Omega_{\mathbb{X}}$ be the $(m+n)$-by-$(m+n)$ matrix assoc. to a compatible Poisson bracket.

Then $\widetilde{B}^T \Omega_{\mathbb{X}} = [D\ 0]$ where $D = n$-by-$n$ diagonal w/ nonzero integer entries.

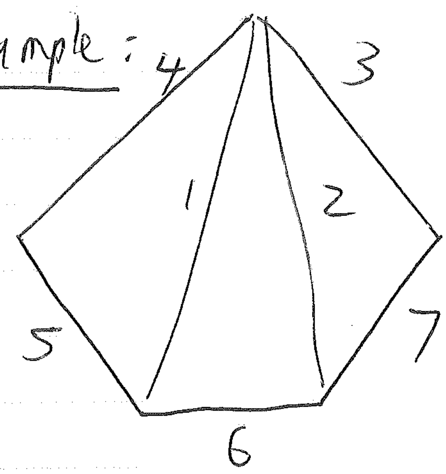
Sec 6.2

(3) <u>Construction</u> [Muller '16] For a cluster algebra from an unpunctured surface $\mathcal{A}$ such that $\widetilde{B}$ has full rank, can build a compatible Poisson structure via

$$\Omega_{ij} := \left(\#\text{ endpts of } \tau_i \,\&\, \tau_j \text{ s.t. } \tau_j \text{ clockwise from } \tau_i \left[\begin{matrix}\text{not nec.}\\ \text{immediately}\end{matrix}\right]\right)$$

$$- \left(\#\text{ endpts of } \tau_i \,\&\, \tau_j \text{ s.t. } \tau_j \text{ counter-clockwise from } \tau_i\right).$$

<u>Example:</u>




yields $\widetilde{B} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ \hline 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$

and $\Omega_{\mathbb{X}} = \begin{bmatrix} 0 & -1 & -1 & 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & -1 \\ -1 & -1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & -1 & 0 \end{bmatrix}.$

<u>Note:</u>

$\widetilde{B}^T \Omega_{\mathbb{X}} = \left[\begin{array}{cc|ccccc} 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$

(4) Can find other compatible Poisson structures as well, an entire subspace of choices.

Another one appears in Example 4.1 of [GSV10]:

$$\Omega_{\mathbb{X}_2} \quad \begin{bmatrix} 0 & 1 & -1 & -1 & 1 & 1 & 0 \\ -1 & 0 & -1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 & 2 & 2 & 1 \\ 1 & 1 & -1 & 0 & 1 & 2 & 0 \\ -1 & 0 & -2 & -1 & 0 & 1 & 0 \\ -1 & -1 & -2 & -2 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

satisfying

$$\tilde{B}^T \Omega_{\mathbb{X}_2} = \left[\begin{matrix} -2 & 0 \\ 0 & -2 \end{matrix} \middle| \; 0 \right].$$

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In particular, $x_1' = \dfrac{x_2 x_5 + x_4 x_6}{x_1}$ & we wish to show $\{\cdot,\cdot\}$ also log-canonical on the neighboring cluster $\mathbb{X}' = \{x_1', x_2, x_3, \dots, x_7\}$, i.e. $\exists\, \lambda \in \mathbb{R}$ s.t.

$\boxed{\text{Wish to show}}$

$$\{x_1', x_i\} \overset{?}{=} \lambda x_1' x_i = \lambda \frac{x_2 x_5 x_i}{x_1} + \frac{\lambda x_4 x_6 x_i}{x_1}$$

$\forall\, i = 2, 3, \dots, 7$, using $\Omega_{\mathbb{X}}$ or $\Omega_{\mathbb{X}_2}$.

First, more generally observe that $\boxed{\left\{\frac{1}{a}, b\right\} = \frac{-1}{a^2}\{a, b\}}$

Essentially from $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \dfrac{-1}{x^2}$ and chain rule, but we will use Leibniz identity

(5) Pf: Note that $\{1, b\} = 0 \ \forall b$ since $1$ is a constant function.

Thus $0 = \{\frac{1}{a} \cdot a, b\} = \frac{1}{a}\{a,b\} + a\{\frac{1}{a}, b\}$

$\Rightarrow a\{\frac{1}{a}, b\} = -\frac{1}{a}\{a,b\} \Rightarrow \{\frac{1}{a}, b\} = -\frac{1}{a^2}\{a,b\}$. ☒

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In the special case $\{a,b\} = \lambda ab$ (log-canonical), then $\boxed{\{\frac{1}{a}, b\} = -\frac{\lambda b}{a}}$.

(Letting, for $j,k \in \{1,2,\dots,7\}$, $\{x_j, x_k\} = \lambda_{jk} x_j x_k$, $w[\lambda_{jk}] = \Omega$)

We now compute $\{x_1', x_i\}$:

$$\{x_2 \cdot \frac{x_5}{x_1}, x_i\} + \{x_4 \cdot \frac{x_6}{x_1}, x_i\} =$$

$$x_2\{x_5 \cdot \frac{1}{x_1}, x_i\} + \frac{x_5}{x_1}\{x_2, x_i\} + x_4\{x_6 \cdot \frac{1}{x_1}, x_i\} + \frac{x_6}{x_1}\{x_4, x_i\}$$

$$= x_2 x_5\{\frac{1}{x_1}, x_i\} + \frac{x_2}{x_1}\{x_5, x_i\} + \frac{x_5}{x_1}\{x_2, x_i\}$$
$$+ x_4 x_6\{\frac{1}{x_1}, x_i\} + \frac{x_4}{x_1}\{x_6, x_i\} + \frac{x_6}{x_1}\{x_4, x_i\}$$

$$= -\lambda_{1i}\frac{x_2 x_5 x_i}{x_1} + \lambda_{5i}\frac{x_2 x_5 x_i}{x_1} + \lambda_{2i}\frac{x_2 x_5 x_i}{x_1}$$
$$+ \frac{x_4 x_6 x_i}{x_1}(-\lambda_{1i} + \lambda_{6i} + \lambda_{4i})$$

Hence

$$\{x_1', x_i\} = \lambda_{1'i}\, x_1' x_i \quad \text{if}$$

$$-\lambda_{1i} + \lambda_{5i} + \lambda_{2i} = -\lambda_{1i} + \lambda_{6i} + \lambda_{4i} = \lambda_{1'i} \quad \forall i$$

In fact, looking at both of these $\Omega$-matrices, we see that

sum of 2nd & 5th rows = sum of 4th & 6th rows (except in first column)

This calculation also helps us define how $\Omega$-matrix changes under mutation:

replace entries of first row & column $\lambda_{1i} \longmapsto \lambda_{1'i}$, the common value defined above

and leave $\lambda_{jk}$ the same otherwise (for $j, k \neq 1$)

(7)

Rem: Having a full rank $\tilde{B}$ is Key, See Example 4.2 of [GSV10].

If $\tilde{B} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ (Which has rank 2 not 3)

$x_1' = \dfrac{x_2 + x_3}{x_1}$ and a Poisson compatible structure would require

$$\{x_1', x_2\} = \left\{\frac{x_2}{x_1}, x_2\right\} + \left\{\frac{x_3}{x_1}, x_2\right\}$$

$$= \frac{1}{x_1}\{x_2, x_2\} + x_2\left\{\frac{1}{x_1}, x_2\right\} + \frac{1}{x_1}\{x_3, x_2\} + x_3\left\{\frac{1}{x_1}, x_2\right\}$$

$$= 0 - \lambda_{12}\frac{x_2^2}{x_1} - \lambda_{23}\frac{x_2 x_3}{x_1} - \lambda_{12}\frac{x_2 x_3}{x_1}$$

$$= \lambda_{1'2}\, x_1' x_2 = \lambda_{1'2}\frac{x_2^2}{x_1} + \lambda_{1'2}\frac{x_2 x_3}{x_1}$$

$$\Rightarrow -\lambda_{12} = \lambda_{1'2} = -\lambda_{12} - \lambda_{23} \Rightarrow \boxed{\lambda_{23} = 0}$$

we can similarly use log-compatibility of $\{x_2', x_3\}$ & $\{x_3', x_1\}$ to conclude

we'd need $\lambda_{13}$ & $\lambda_{12} = 0$ and thus only Poisson compatible structure is trivial in this case.