10/24/18

From Principal Coeffs to Y-systems

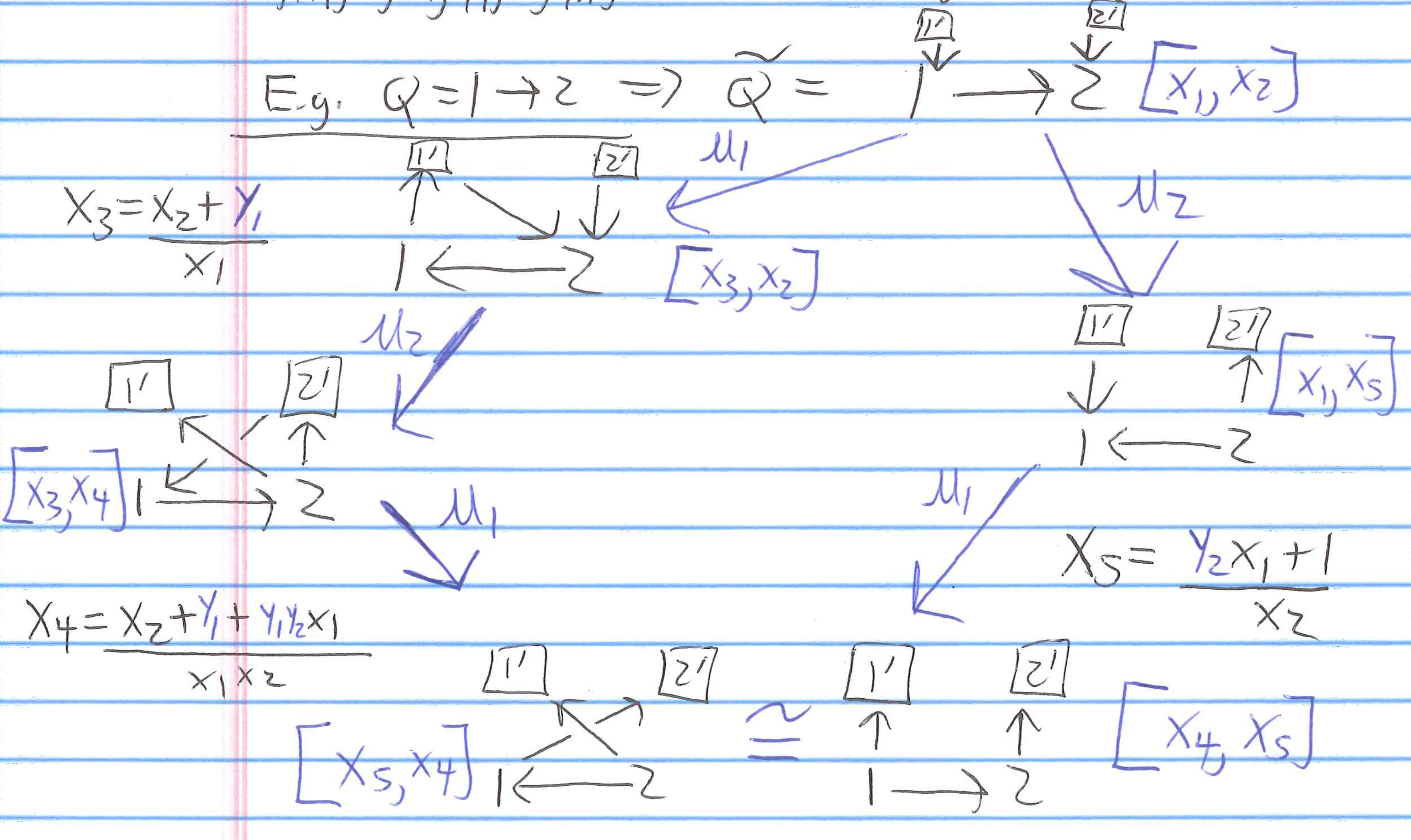
Def: Given $n \times n$ skew-symmetrizable B (or quiver Q in skew symmetric case) we define the associated cluster algebra with principal coefficients using extended exch. matrix/quiver

$$\tilde{B} := \left[\begin{array}{c} B \\ \hline I \end{array}\right] \quad 2n\text{-by-}n \quad \text{OR } \tilde{Q} \text{ w/ } n \text{ frozen vertices } \& \ \boxed{i'} \to i.$$

By convention, we let extended cluster $= \{x_1, \dots, x_n, y_1, \dots, y_n\}$ and not allowed to mutate $y_1, y_2, \dots, y_n$.

Hence, updated cluster variables are Laurent polys in $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ BUT ONLY x_i's in the denominator.

E.g. $Q = 1 \to 2 \Rightarrow \tilde{Q} =$



$$X_3 = \frac{X_2 + Y_1}{X_1}$$

$$X_5 = \frac{Y_2 X_1 + 1}{X_2}$$

$$X_4 = \frac{X_2 + Y_1 + Y_1 Y_2 x_1}{x_1 x_2}$$

②

Def (F-polynomials and g-vectors) Given a cluster algebra seed B or Q, and its associated seed $\widetilde{S}$ w/ principal coeffs $\widetilde{B}$ or $\widetilde{Q}$, and a mutation sequence

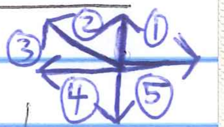
$$\mu_{i_1} \circ \mu_{i_2} \circ \dots \circ \mu_{i_k}$$

The F-polynomials associated to labeled cluster

$$\widetilde{S}' := \mu_{i_1} \circ \mu_{i_2} \circ \dots \circ \mu_{i_k} (\widetilde{S})$$

are the cluster vars w/ principal coeffs in $\widetilde{S}'$ where all y_i's set to 1.

The g-vectors are exponent vector (in x_i's) of Laurent monomial obtained from cluster var w/ principal coeffs w/ all y_i's set to 0.



E.g. continued

Labeled cluster	Cluster var w/ principal coeffs	F-polys	g-vectors
$[x_1, x_2]$	x_1, x_2	$1, 1$	$\begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix}$
$[x_3, x_2]$	$\frac{x_2+y_1}{x_1}, x_2$	$y_1+1, 1$	$\begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$
$[x_3, x_4]$	$\frac{x_2+y_1}{x_1}, \frac{x_2+y_1+y_1y_2x_1}{x_1x_2}$	$y_1+1, y_1y_2+y_1+1$	$\begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}-1\\0\end{bmatrix}$
$[x_5, x_4] \cong [x_4, x_5]$	$\frac{y_2x_1+1}{x_2}, \frac{x_2+y_1+y_1y_2x_1}{x_1x_2}$	$y_2+1, y_1y_2+y_1+1$	$\begin{bmatrix}0\\-1\end{bmatrix}, \begin{bmatrix}-1\\0\end{bmatrix}$
$[x_1, x_5]$	$x_1, \frac{y_2x_1+1}{x_2}$	$1, y_2+1$	$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\end{bmatrix}$

③ Cluster algebras defined with a general semifield $(\mathbb{P}, \oplus)$

Define a seed by a triple $(B, \mathbf{X}, \mathbf{Y})$

where $B = n \times n$ skew-symmetrizable matrix (as usual)

$\mathbf{X} = \{x_1, \dots, x_n\} \subset \mathbb{QP}(x_1, \dots, x_n)$ initial cluster

$\mathbf{Y} = \{y_1, \dots, y_n\} \subset \mathbb{P}$ initial coefficients

We mutate by $(B, \mathbf{X}, \mathbf{Y}) \xrightarrow{\mu_K} (B', \mathbf{X}', \mathbf{Y}')$ where

$B' = \mu_K(B)$ (usual mutation of exchange matrices)

$$\mathbf{X}' = \{x_1, \dots, x_K', \dots, x_n\} \text{ w/ } x_K' := \frac{y_K \prod_{b_{iK}>0} x_i^{b_{iK}} + \prod_{b_{iK}<0} x_i^{-b_{iK}}}{(y_K \oplus 1)\, x_K}$$

$$\mathbf{Y}' = \{y_1', \dots, y_n'\} \text{ with } y_j' = \begin{cases} y_K^{-1} & \text{if } j = K \\ y_j\, y_K^{[b_{Kj}]_+} (y_K \oplus 1)^{-b_{Kj}} & \text{if } j \neq K \end{cases}$$

Example ($\mathbb{P}$ is Tropical semifield)

Suppose $\mathbb{P} = \{\text{Laurent monomials in } u_1^{d_1} \cdots u_m^{d_m} : d_i \in \mathbb{Z}\}$

where mult, division defined as usual

and auxiliary addition $\oplus$ defined as

$$(u_1^{a_1} u_2^{a_2} \cdots u_m^{a_m} \oplus u_1^{b_1} u_2^{b_2} \cdots u_m^{b_m}) = u_1^{c_1} u_2^{c_2} \cdots u_m^{c_m}$$

where $c_i = \min(a_i, b_i) \ \forall\ 1 \leq i \leq n$.

$$\text{e.g. } u_1^{a_1} \cdots u_m^{a_m} \oplus \underset{\substack{\uparrow \\ \text{all exponents are zero}}}{1} = u_1^{-[-a_1]_+} u_2^{-[a_2]_+} \cdots u_m^{-[a_m]_+} \overset{\text{e.g.}}{=} 1 \text{ if all } a_i\text{'s} \geq 0$$

④ If we define $y_j = \prod_{i=1}^{m} u_i^{c_{ij}}$ for $j=1,2,\dots,n$ and consider the extended exchange matrix $\tilde{B} = \begin{array}{r} \\ m\{ \end{array}\!\!\overbrace{\begin{bmatrix} B \\ \hline C \end{bmatrix}}^{n}$

<u>Claim:</u> $(\tilde{B}, \tilde{X}) \xrightarrow{\mu_K} (\tilde{B}', \tilde{X}')$ is equivalent to

$$(B, \mathbb{X}, \{y_1, \dots, y_n\}) \xrightarrow{\mu_K} (B', \mathbb{X}', \{y_1', \dots, y_n'\})$$

when $(\mathbb{P}, \oplus)$ is the tropical semifield.

E.g. $\tilde{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ \hline p & r \\ -q & s \end{bmatrix}$ w/ $p, q, r, s \geq 0$

$\mu_1)$ $\tilde{X}_1' = \dfrac{u_1^p + u_2^q x_2}{x_1}$, $\tilde{B}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ \hline -p & r+p \\ q & s \end{bmatrix}$

vs. $X_1' = \dfrac{y_1 + x_2}{(y_1 \oplus 1)\, x_1} = \dfrac{u_1^p u_2^{-q} + x_2}{(u_2^{-q})\, x_1} = \dfrac{u_1^p + u_2^q x_2}{x_1}$

$B' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\mathbb{Y} = \{y_1', y_2'\}$ with $y_1' = \dfrac{1}{y_1} = u_1^{-p} u_2^{q}$

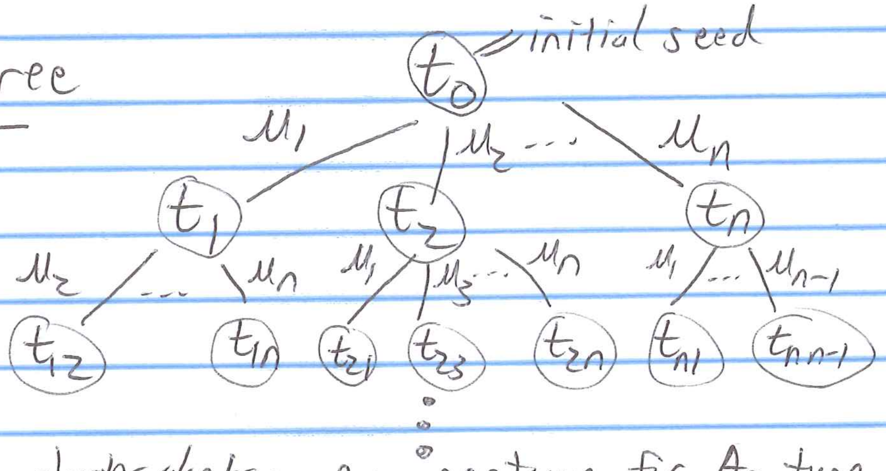
$y_2' = \dfrac{y_2 y_1}{(y_1 \oplus 1)} = \dfrac{(u_1^r u_2^s)(u_1^p u_2^{-q})}{(u_2^{-q})} = u_1^{r+p} u_2^{s}$.

Thm 3.7/Cor 6.3 of Cl. Alg. IV by Fomin–Zelevinsky

$\downarrow$

⑤ More generally, we get the Separation Formulas

Consider exchange tree



where we consider all mut. seq's w/o backtracking and abstractly thinking of infinite # distinct seeds w/o considering possible equalities for specific cluster algebras, e.g. pentagon for A_2 type

Let $\left(B_t, \underline{X}_t, \underline{Y}_t\right)$ be seed data for seed t.

$\underline{X}_t = \{X_{1,t}, \ldots, X_{n,t}\}$, $\underline{Y}_t = \{Y_{1,t}, \ldots, Y_{n,t}\}$

Define $\{F_{1,t}, \ldots, F_{n,t}\}$ and $\{\vec{g}_{1,t}, \ldots, \vec{g}_{n,t}\}$ as associated F-polynomials and g-vectors.

Separation Formulas: $$X_{j,t} = \frac{F_{j,t}(\hat{y}_1, \ldots, \hat{y}_n)\, x_1^{g_1} \cdots x_n^{g_n}}{F_{j,t}|_{\mathbb{P}}(y_1, \ldots, y_n)}$$

where $\hat{y}_j = y_j \prod_{i=1}^{n} x_i^{b_{ij}}$

$\begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} = \vec{g}_{j,n}$, $[b_{ij}] = B_{t_0}$

and denominator uses $\oplus$ in place of addition $+$.

Prop 3.13
Y-system Dynamics

$$Y_{j,t} = Y_{j,t}|_{\mathrm{Trop}} \cdot \prod_{i=1}^{n} F_{i,t}^{(b'_{ij})_t}$$ where $[b'_{ij}] = B_t$

$Y_{j,t}|_{\mathrm{Trop}}$ = replace $\oplus$ w/ tropical semifield addition

⑥ Example ($Q = 1 \leftarrow 2$)

[Example 3.6.11 of [FW16] and Table 3 of [Cl. Alg. IV]]

t	$y_{i,t}$		$y_{i,t}\vert_{\text{Trop}}$		$F_{i,t}$		B_t	$Y_{i,t}$	
0	y_1	y_2	y_1	y_2	1	1	$\begin{bmatrix}0&1\\-1&0\end{bmatrix}$ $\shortparallel B_0$	Y_1	Y_2
1	$\frac{1}{y_1}$	$\frac{y_1y_2}{y_1\oplus 1}$	$\frac{1}{y_1}$	y_1y_2	y_1+1	1	$\begin{bmatrix}0&-1\\1&0\end{bmatrix}$	$\frac{1}{Y_1}$	$\frac{Y_1Y_2}{Y_1+1}$
2	$\frac{y_2}{y_1y_2\oplus y_1\oplus 1}$	$\frac{y_1\oplus 1}{y_1y_2}$	y_2	$\frac{1}{y_1y_2}$	y_1+1	$y_1y_2+y_1+1$	$\begin{bmatrix}0&1\\-1&0\end{bmatrix}$	$\frac{Y_2}{Y_1Y_2+Y_1+1}$	$\frac{Y_1+1}{Y_1Y_2}$
3	$\frac{y_1y_2\oplus y_1\oplus 1}{y_2}$	$\frac{1}{y_1(y_2\oplus 1)}$	$\frac{1}{y_2}$	$\frac{1}{y_1}$	y_2+1	$y_1y_2+y_1+1$	$\begin{bmatrix}0&-1\\1&0\end{bmatrix}$	$\frac{Y_1Y_2+Y_1+1}{Y_2}$	$\frac{1}{Y_1(Y_2+1)}$
4	$\frac{1}{y_2}$	$y_1(y_2\oplus 1)$	$\frac{1}{y_2}$	y_1	y_2+1	1	$\begin{bmatrix}0&1\\-1&0\end{bmatrix}$	$\frac{1}{Y_2}$	$Y_1(Y_2+1)$
5	y_2	y_1	y_2	y_1	1	1	$\begin{bmatrix}0&-1\\1&0\end{bmatrix}$	Y_2	Y_1

using $Y_{j,t} = y_{j,t}\big|_{\text{Trop}} \cdot \prod_{i=1}^{n} F_{i,t}^{(b'_{ij})_t}$, e.g.

$Y_{1,1} = y_{1,1}\big\|_{\text{Trop}} \cdot F_{2,1}^{-1}$	$Y_{1,2} = y_{1,2}\big\|_{\text{Trop}} \cdot F_{2,2}^{+1}$	$Y_{1,3} = y_{1,3}\big\|_{\text{Trop}} \cdot F_{2,3}^{-1}$ $\cdots$
$Y_{2,1} = y_{2,1}\big\|_{\text{Trop}} \cdot F_{1,1}^{+1}$	$Y_{2,2} = y_{2,2}\big\|_{\text{Trop}} \cdot F_{1,2}^{-1}$	$Y_{2,3} = y_{2,3}\big\|_{\text{Trop}} \cdot F_{1,3}^{+1}$

$B_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\hat{y}_1 = y_1 x_1^0 x_2^{-1}$, $\hat{y}_2 = y_2 x_1^{+1} x_2^0$

$= y_1/x_2$ $\quad = y_2 x_1$

⑦ Example $(Q = 1 \leftarrow 2)$ continued

t	$F_{i,t}$		$F_{i,t}(\hat{y}_1,\hat{y}_2)$		$\vec{g}_{i,t}$		$X_{i,t}$	
0	1	1	1	1	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}0\\1\end{bmatrix}$	x_1	x_2
1	y_1+1	1	$\frac{y_1+x_2}{x_2}$	1	$\begin{bmatrix}-1\\1\end{bmatrix}$	$\begin{bmatrix}0\\1\end{bmatrix}$	$\frac{y_1+x_2}{x_1(y_1\oplus 1)}$	x_2
2	y_1+1	$y_1y_2+y_1+1$	$\frac{y_1+x_2}{x_2}$	$\frac{x_1y_1y_2+y_1+x_2}{x_2}$	$\begin{bmatrix}-1\\1\end{bmatrix}$	$\begin{bmatrix}-1\\0\end{bmatrix}$	$\frac{y_1+x_2}{x_1(y_1\oplus 1)}$	$\frac{x_1y_1y_2+y_1+x_2}{x_1x_2(y_1y_2\oplus y_1\oplus 1)}$
3	y_2+1	$y_1y_2+y_1+1$	x_1y_2+1	$\frac{x_1y_1y_2+y_1+x_2}{x_2}$	$\begin{bmatrix}0\\-1\end{bmatrix}$	$\begin{bmatrix}-1\\0\end{bmatrix}$	$\frac{x_1y_2+1}{x_2(y_2\oplus 1)}$	$\frac{x_1y_1y_2+y_1+x_2}{x_1x_2(y_1y_2\oplus y_1\oplus 1)}$
4	y_2+1	1	x_1y_2+1	1	$\begin{bmatrix}0\\-1\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\frac{x_1y_2+1}{x_2(y_2\oplus 1)}$	x_1
5	1	1	1	1	$\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$	x_2	x_1

Using $X_{i,t} = \dfrac{F_{i,t}(\hat{y}_1,\dots,\hat{y}_n)\, x_1^{g_1}\cdots x_n^{g_n}}{F_{i,t}|_{\mathbb{P}}(y_1,\dots,y_n)}$ where $\vec{g}_{i,t} = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}$

For Principal coeffs, $F_{i,t}|_{Trop} = 1 \ \forall\ 1 \le i \le n$.

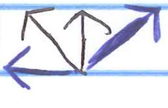
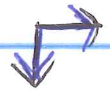
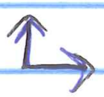
$y_i = u_1^0 u_2^0 \cdots u_i^1 \cdots u_n^0$
for $1 \le i \le n$

⑧ Given the case of principal coeffs, i.e. $\tilde{B} = \left[\begin{array}{c} \overset{B_0}{\overset{\parallel}{B}} \\ \hline I \end{array}\right]$

we define $\tilde{B}_t = \left[\begin{array}{c} B_t \\ \hline C_t \end{array}\right]$ and let $C_t = \left[[\vec{c}_1]_t, [\vec{c}_2]_t, \ldots, [\vec{c}_n]_t \right]$

the <u>c-vectors</u> associated to <u>seed t</u>.

<u>E.g. ($Q = 1 \leftarrow 2$)</u>

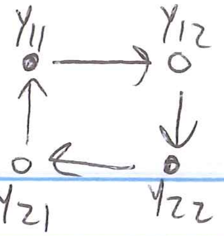
t	$y_{j;t}\vert_{Trop}$	$\tilde{B}_t$	c-vectors	g-vectors
0	$y_1 \quad y_2$	$\left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ 
1	$\frac{1}{y_1} \quad y_1 y_2$	$\left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \\ \hline -1 & 1 \\ 0 & 1 \end{array}\right]$	$\begin{bmatrix} -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ 
2	$y_2 \quad \frac{1}{y_1 y_2}$	$\left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \\ \hline 0 & -1 \\ 1 & -1 \end{array}\right]$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ 
3	$\frac{1}{y_2} \quad \frac{1}{y_1}$	$\left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \\ \hline 0 & -1 \\ -1 & 0 \end{array}\right]$	$\begin{bmatrix} 0 \\ -1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ 
4	$\frac{1}{y_2} \quad y_1$	$\left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \\ \hline 0 & 1 \\ -1 & 0 \end{array}\right]$	$\begin{bmatrix} 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 
5	$y_2 \quad y_1$	$\left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \\ \hline 0 & 1 \\ 1 & 0 \end{array}\right]$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 

$\langle \vec{c}_1, \ldots, \vec{c}_n \rangle \perp \langle \vec{g}_1, \ldots, \vec{g}_n \rangle$

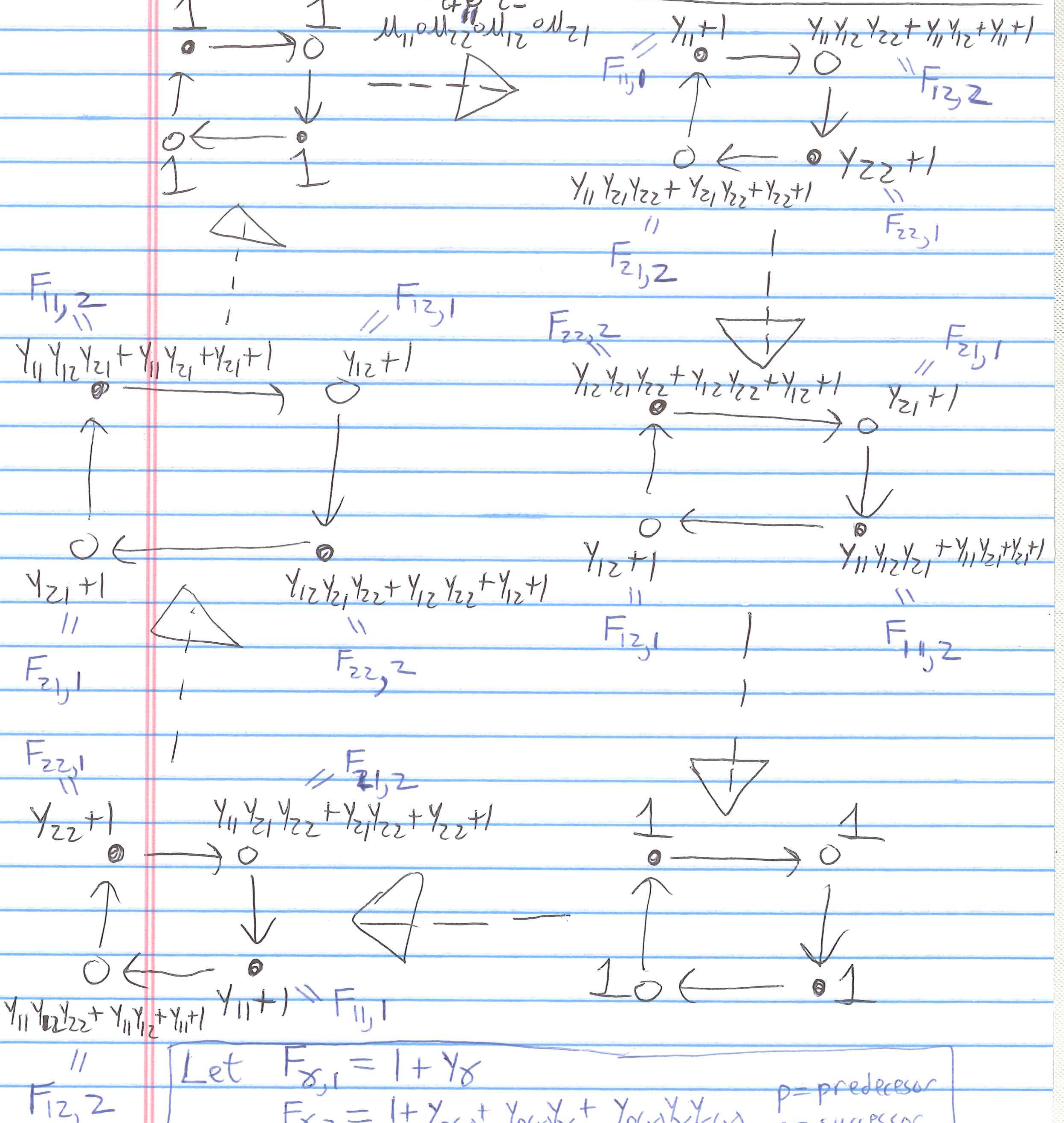
i.e. $\left[C^T\right]^{-1} = G$

$\underset{\begin{bmatrix} c_{i1} \\ c_{i2} \\ \vdots \\ c_{in} \end{bmatrix}}{\underset{\parallel}{(\vec{c}_i)_t}} \Rightarrow y_1^{c_{i1}} y_2^{c_{i2}} \cdots y_n^{c_{in}} = y_{j;t}\vert_{Trop}$

⑨ F-polynomials for $A_2 \times A_2$



mutating periodically $\mu_{11}\circ\mu_{22}\circ\mu_{12}\circ\mu_{21}$



$$\boxed{\begin{aligned} \text{Let } F_{\gamma,1} &= 1 + y_\gamma \\ F_{\gamma,2} &= 1 + y_{p(\gamma)} + y_{p(\gamma)}y_\gamma + y_{p(\gamma)}y_\gamma y_{s(\gamma)} \end{aligned} \quad \begin{aligned} p &= \text{predecessor} \\ s &= \text{successor} \end{aligned}}$$