11/19/18

Over the last few classes, we have seen a variety of cluster algebras and quivers on tori:

e.g. F_0 quiver

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r, Rightarrow] & \overset{3}{\bullet} \arrow[d, Rightarrow] \\ \underset{4}{\bullet} \arrow[u, Rightarrow] & \underset{2}{\bullet} \arrow[l, Rightarrow] \end{tikzcd}$$

Pentagram/Glick Quivers Q_n,

Somos-4, Somos-5, and Gale-Robinson Quivers.

Today we study integrable systems defined by such cluster algebras and quivers on tori. To towards this end, we utilize Poisson compatible structures we studied in the case of cluster algebras from surfaces/Teichmüller space.

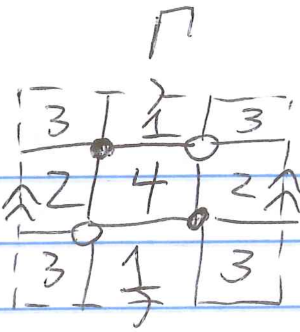
We Follow Goncharov-Kenyon "Dimers and cluster integrable systems".

Let Γ denote a bipartite & bicolored graph on a torus. Γ is embedded on the torus on a way so that it is a 2-dimensional cell complex (as opposed to just a 1-dim cell complex), using bounded regions (2d-gons) as (2-dim) Faces.

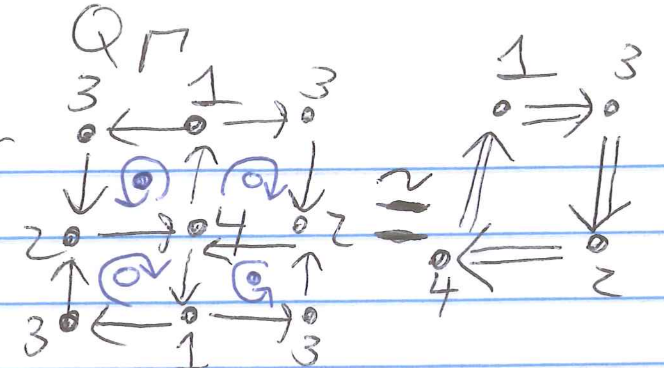
Q_Γ = Dual Quiver defined by

Γ $\longleftrightarrow$	Q_Γ
2d-Faces	vertices of valence 2d w/ arrows alternating in/out
edges $\bullet$—$\circ$	arrow $\bullet$- -$\uparrow$- -$\circ$
$\bullet$ vertices	cycle of arrows - counterclockwise in toroidal embedding
$\circ$ vertices	cycle of arrows - clockwise

② Example: Γ



has dual quiver Q_Γ



From a bipartite graph on a torus Γ, we will build $\mathcal{L}_\Gamma$, a "space of line bundles with connections" and give $\mathcal{L}_\Gamma$ a Poisson structure.

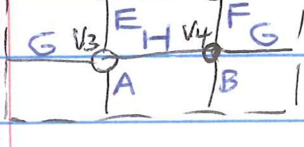
We will then relate this Poisson structure back to quiver Q_Γ, its associated cluster algebra and Y-system.

Def: Given a graph $\Gamma = (V, E)$, a line bundle is an assignment of a 1-dim Complex Vector Space V_v to each vertex $v \in V$.
An associated connection is a choice of isomorphisms $\phi_e = \phi_{vv'}$ for each edge $e \in E$ (connecting v and v').

$$\phi_e = \phi_{vv'} : \underset{\mathbb{C}^1}{\overset{\cong}{V_v}} \to \underset{\mathbb{C}^1}{\overset{\cong}{V_{v'}}} \quad \text{such that } \phi_{v'v} : V_{v'} \to V_v \text{ satisfies } \phi_{v'v} = \phi_{vv'}^{-1}$$

If Γ is a bipartite graph, given the condition $\phi_{v'v} = \phi_{vv'}^{-1}$, it is sufficient to define each connection assoc. to an edge $e = (v, v')$ orienting them all e.g. $\circ \longrightarrow \bullet$

Example: cont. $\Gamma =$



$$\mathcal{L}_\Gamma \cong \left\{ \begin{array}{l} \phi_A : V_{v_3} \to V_{v_1}, \ \phi_B : V_{v_2} \to V_{v_4}, \\ \phi_C : V_{v_2} \to V_{v_1}, \ \phi_D : V_{v_2} \to V_{v_1}, \\ \phi_E : V_{v_3} \to V_{v_1}, \ \phi_F : V_{v_2} \to V_{v_4}, \\ \phi_G : V_{v_3} \to V_{v_4}, \ \phi_H : V_{v_3} \to V_{v_4} \end{array} \right\} \Big/ \sim \ \cong (\mathbb{C}^*)^{|E|}$$

a choice of eight 1x1 invertible matrices.

$\circ \to \bullet$

(3) Assign a value $\lambda_e \in \mathbb{C}^*$ to each $e \in E$, i.e. signifying $\phi_e = [\lambda_e]$

We say two line bundles $\{V_v, \phi_e : v \in V, e \in E\}$ and $\{V'_v, \phi'_e\}$ with connections are <u>isomorphic</u>, <u>gauge equivalent</u> if there are isomorphisms $\psi_v : V_v \to V'_v$ for every vertex $v \in V$ s.t.

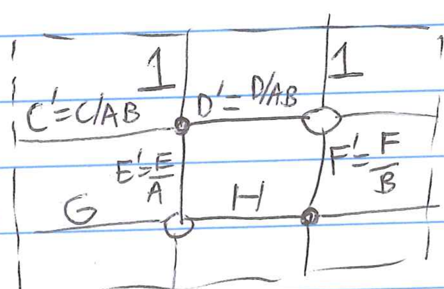
$$\phi'_e = \psi_{v_2} \circ \phi_e \circ \psi_{v_1}^{-1} \text{ for every edge } e = (v_1, v_2) \in E.$$

i.e. if we can pick $\alpha_v \in \mathbb{C}^*$ for every vertex $v \in V$ and $\lambda'_e = \alpha_{v_1} \alpha_{v_2} \lambda_e$ for every $e = (v_1, v_2) \in E$, then the $\{\lambda'_e\}$'s and $\{\lambda_e\}$'s are gauge equivalent.

<u>Example: conti</u> By abuse of notation, let $A, B, \ldots, G$ denote the scalars $\lambda \in \mathbb{C}^*$ associated to the edges w/ those labels.

Up to Equivalence, we can pick $\alpha_{v_1} = A^{-1}, \alpha_{v_2} = B^{-1}, \alpha_{v_3} = 1, \alpha_{v_4} = 1$

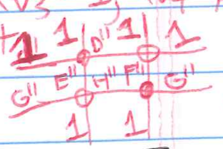
to get



. Hence $\mathcal{L}_\Gamma$ determined (up to equivalence) by the six parameters, at most, $C', D', E', F', G, H \in \mathbb{C}^*$.

<u>In Fact 5-dim'l space of such parameters up to equiv.</u>

Let $\alpha_{v_1} = \frac{1}{A}, \alpha_{v_2} = \frac{A}{C}$
$\alpha_{v_3} = 1, \alpha_{v_4} = \frac{C}{AB}$
to get



<u>$A'' = B'' = C'' = 1$</u>

<u>Claim:</u> For general bipartite Γ on a torus, up to gauge equivalence, $\mathcal{L}_\Gamma$ determined by $|F(\Gamma)| + 1$ $\mathbb{C}^*$ parameters,

one for each monodromy around all but one face + two for homology of torus

<u>Example:</u> $F_4 \leftrightarrow D E^{-1} H F^{-1} = D' E'^{-1} H F'^{-1}$, $z_{(1,0)} \leftrightarrow C D^{-1} = C' D'^{-1}$,
$z_{(0,1)} \leftrightarrow A E^{-1} = 1 \cdot E'^{-1}$

4

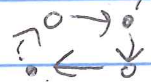
Let W_i be the parameter determined by monodromy around face F_i.

Z_1 & Z_2 " " meridian & equator of the torus

Hence $\mathcal{X}_\Gamma \cong \mathrm{Hom}(H_1(\Gamma, \mathbb{Z}), \mathbb{C}^*) = H^1(\Gamma, \mathbb{C}^*)$

where H_1 = 1st Homology of punctured torus, w/ a puncture for each face F_i of Γ.

Hence, thinking of X_i as the function that sends oriented cycle around Face F_i to parameter $W_i \in \mathbb{C}^*$

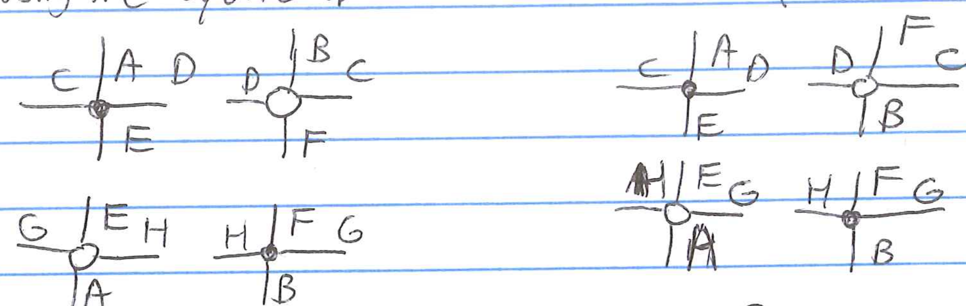


we get a <u>Poisson structure on these functions</u>

$$\{X_i, X_j\} = \varepsilon_{ij} X_i X_j \quad \text{for } \varepsilon_{ij} \in \mathbb{Z}.$$

We define ε_{ij} by considering the <u>conjugate</u> surface graph $\hat{\Gamma}$ defined by embedding a new graph onto a new surface based on Γ by keeping ordering around vertices the same, but reversing the cyclic orientation around vertices.

<u>E.g.</u>

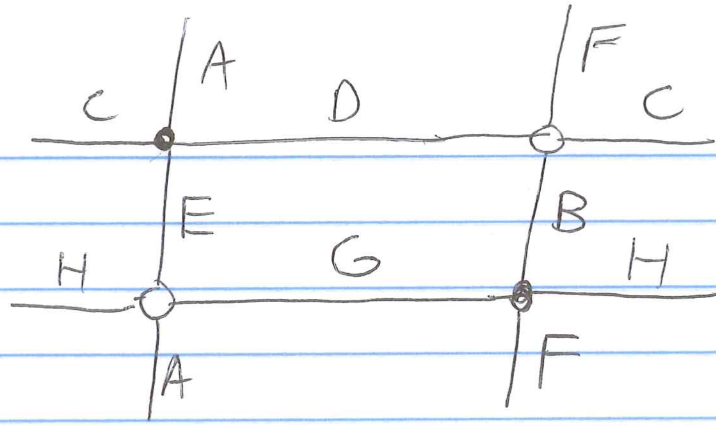


$$\Gamma \longrightarrow \hat{\Gamma}$$

In general, $\hat{\Gamma}$ might not be again on ~~the~~ torus. But in this case, we can glue $\hat{\Gamma}$ together as follows

⑤ Eg. $\hat{\Gamma}$ =

on $\hat{S}$

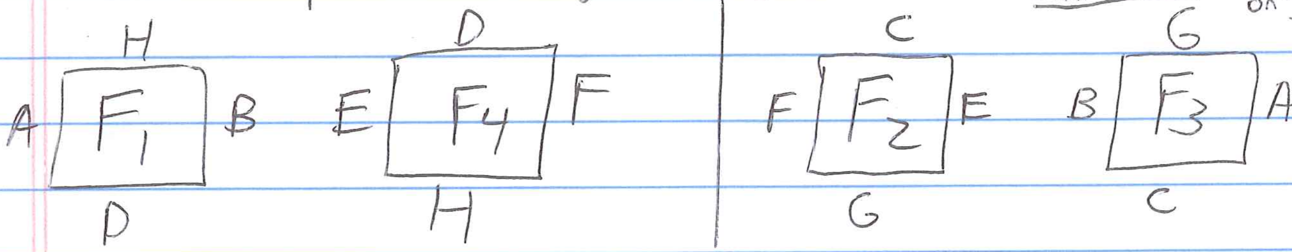


[Note the 180° rotation around bottom right • vertex to glue this together consistently]

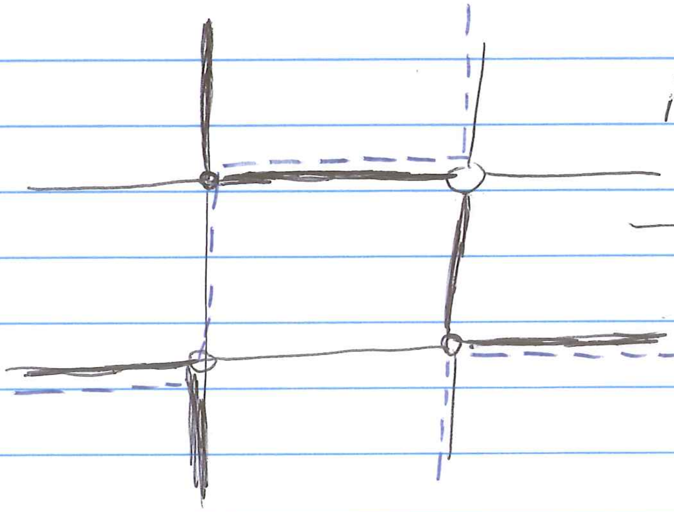
ε_{ij} is defined using the intersection product $\varepsilon: H_1(\hat{S},\mathbb{Z}) \wedge H(\hat{S},\mathbb{Z}) \to \mathbb{Z}$

Eg. The loops defined by Faces 1 and 4 on Γ on S become paths* through $\hat{S}$. (*really still loops but non-contractible loops on $\hat{S}$)

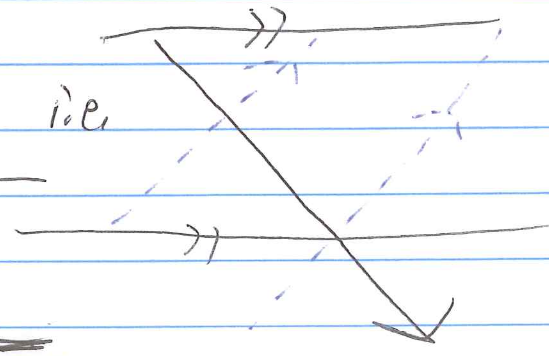
On $\Gamma \subset S$)



On $\hat{\Gamma} \subset \hat{S}$)



i.e.



2 intersections

Eg. continued) $\{X_1, X_4\} = -2^{X_1 X_4}_j \{X_1, X_3\} = +2^{X_1 X_3}_j \{X_1, X_2\} = 0_{X_1 X_2}$

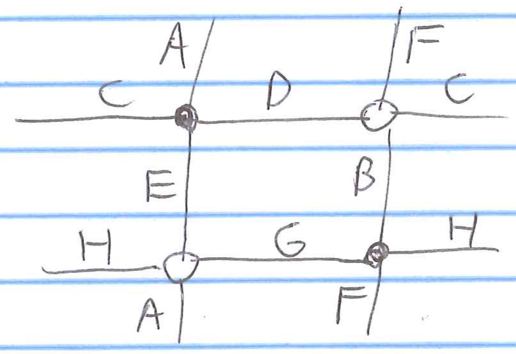
$\{X_2, X_3\} = -2^{X_2 X_3}_j \{X_2, X_4\} = +2^{X_2 X_4}_j \{X_3, X_4\} = 0_{X_3 X_4}$

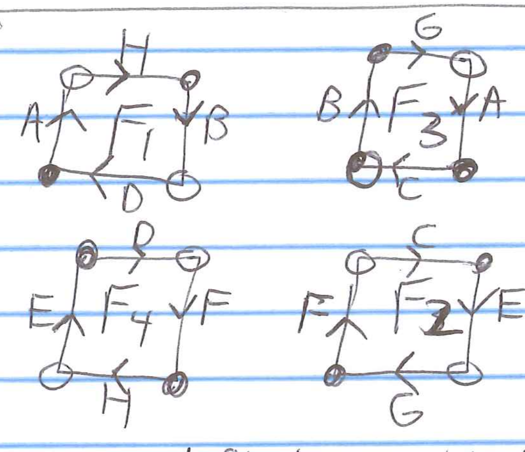
We also get $\{X_i, Z_j\} = \pm 1$ for $i = 1,2,3,4$, $j = 1,2$ in this case.

(using Z_1 has edges CD or GH, Z_2 has edges AE or BF)

(6) Computations of $\{x_i, x_j\}$ for our running example

$\hat{\Gamma}$ on $\hat{S}$

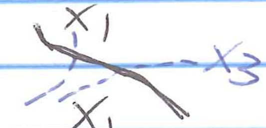




Γ on S

left at ○, right at •

Rem: We redraw $\overset{\text{clockwise}}{\text{loops}}$ F_i's on $\hat{\Gamma}$ (as "zig-zag paths") so become non-contractible loops with intersections invariant under homotopies (e.g. on Γ: F_1 F_3 $\simeq$ F_1 F_3 | on $\hat{\Gamma}$



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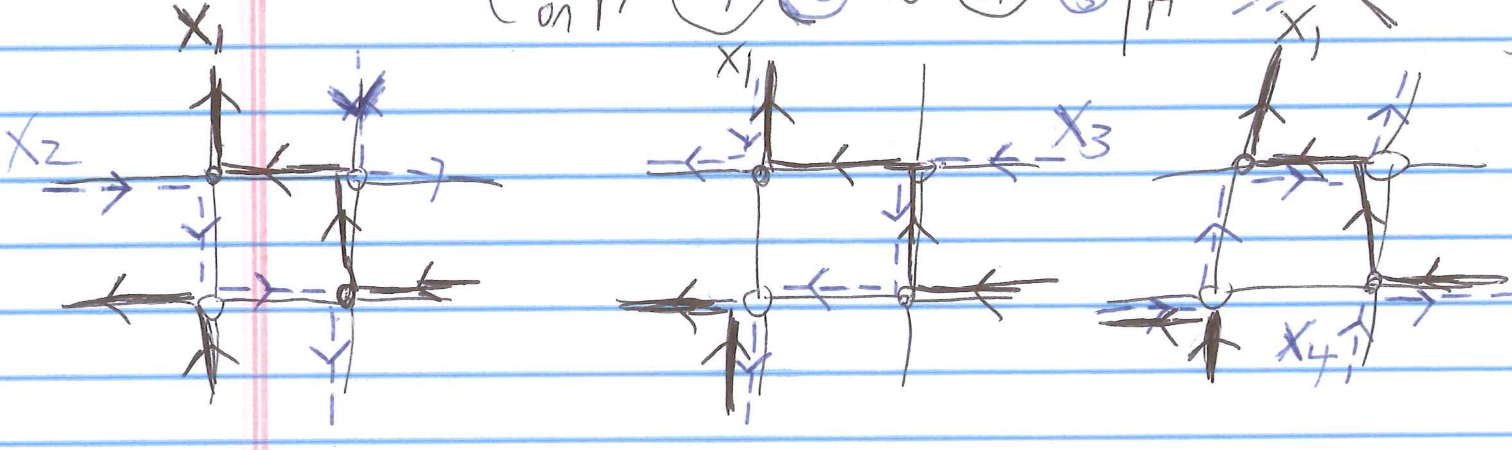

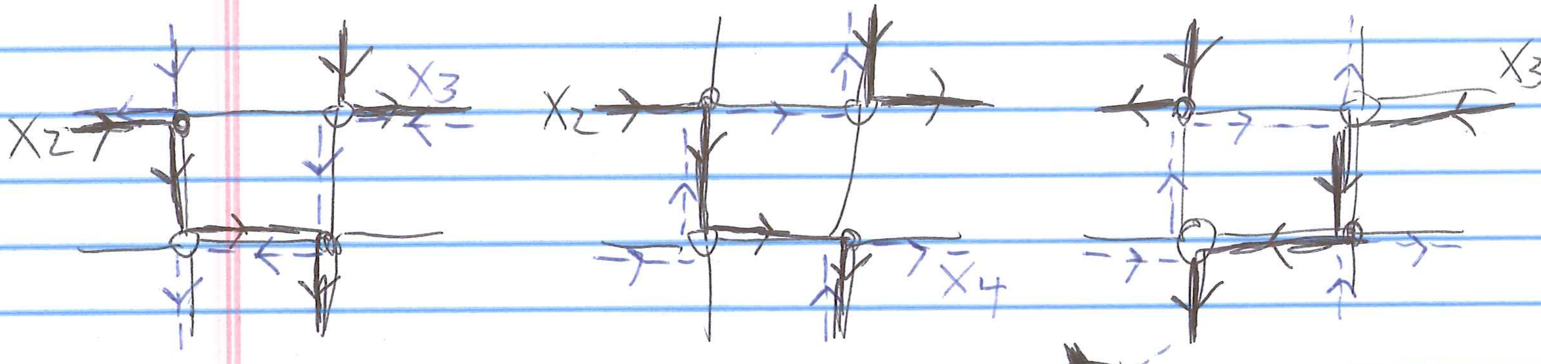

$\{x_1, x_2\} = 0x_1x_2$ $\quad$ $\{x_1, x_3\} = +2x_1x_3$ $\quad$ $\{x_1, x_4\} = -2x_1x_4$

$\{x_2, x_3\} = -2x_2x_3$ $\quad$ $\{x_2, x_4\} = +2x_2x_4$ $\quad$ $\{x_3, x_4\} = 0x_3x_4$