11/16/18

On Wednesday, we discussed how to build a family of graphs $G_n^{r,s,N}$ out of Aztec Diamonds such that

$$x_n = \sum_{\substack{M \text{ perfect} \\ \text{matching of } G_n^{r,s,N}}} x(M)\, y(M)$$ where x_n satisfies the Gale-Robinson (from principal coeffs)

$$\boxed{x_n x_{n-N} = x_{n-r}\, x_{n-N+r} + \prod_{i=1}^{N} y_i^{\,d(n-N-i,\, s,\, N-s)}\, x_{n-s}\, x_{n-N+s}}.$$

An equivalent construction for $G_n^{r,s,N}$ can be given as:

1) Define $H_n^{(r,N)}$ to be the induced subgraph on tiling $\tau^{(r,s,N)}$, dual to GR quiver assoc. to (r,s,N), with unit height, width $2\left\lfloor \frac{n-N-1}{r} \right\rfloor + 1$, placed so the leftmost square is

$$\bar{n} := (n-N) \bmod r$$

[Called a Horizontal Strip]

E.g.'s for Somos-5 $(r=1, s=2, N=5)$

$H_6^{(1,5)} = \boxed{1}$, $H_7^{(1,5)} = \boxed{1}\boxed{2}$, $H_8^{(1,5)} = \boxed{1}\boxed{2}\boxed{3}$,

$H_9^{(1,5)} = \boxed{1}\boxed{2}\boxed{3}\boxed{4}$, $H_{10}^{(1,5)} = \boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{5}\boxed{1}$,

$H_{11}^{(1,5)} = \boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{5}\boxed{1}\boxed{2}$

For $r=2$, $N=7$, $H_{10}^{(2,7)} = \boxed{1}\boxed{3}$, $H_{11}^{(2,7)} = \boxed{2}\boxed{4}$,

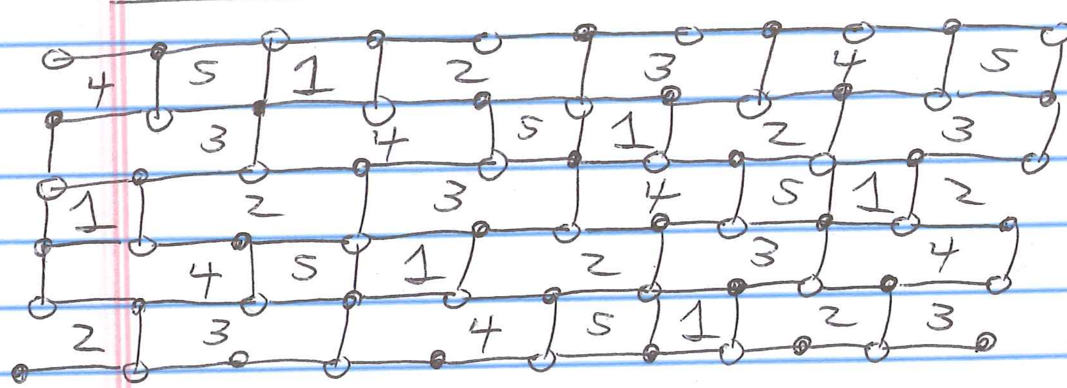
$H_{12}^{(2,7)} = \boxed{1}\boxed{3}\boxed{5}$, $H_{13}^{(2,7)} = \boxed{2}\boxed{4}\boxed{6}\boxed{1}$, $H_{14}^{(2,7)} = \boxed{1}\boxed{3}\boxed{5}\boxed{7}\boxed{2}$

2) Glue the strips together:

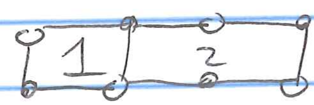
$H^{(r,N)}_{n-(N-s)}$ to the top of $H^{(r,N)}_n$, $H^{(r,N)}_{n-s}$ to the bottom of $H^{(r,N)}_n$

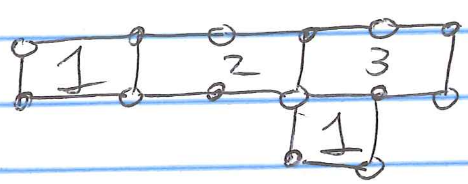
in the unique way so that is contained in $\tau^{(r,s,N)}$.
Then continue, w/ gluing $H^{(r,N)}_{n-2(N-s)}$ above, $H^{(r,N)}_{n-2s}$ below
etc. until $\underline{n-k(N-s)}$ and $\underline{n-ks}$ are below ~~zero~~ one
hence correspond to empty strips.

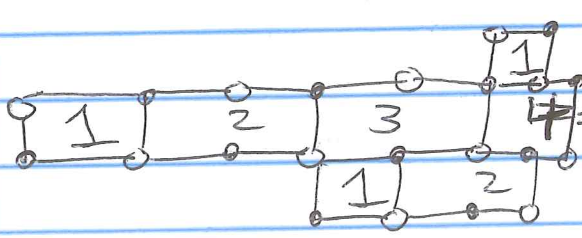
<u>Example (Somos 5)</u> Recall $\tau^{(1,2,5)}$ looks like



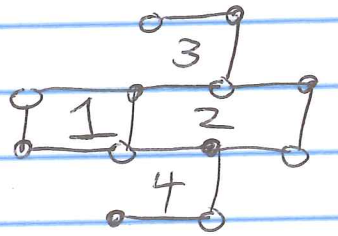
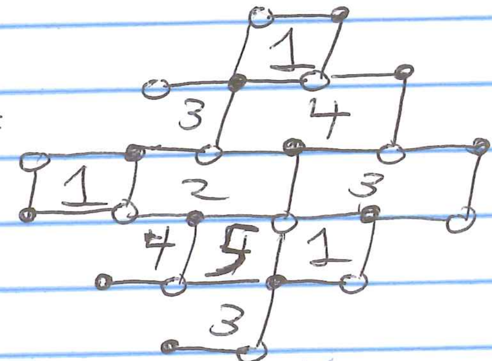
$G^{(1,2,5)}_6 = G_6 =$  $= H^{(1,5)}_6$

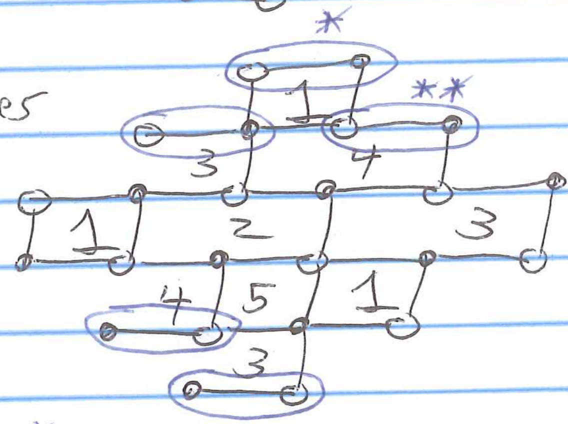
$G_7 =$  $= H^{(1,5)}_7$ $s=2,\ N-s=3,\ H^{(1,5)}_5 = H^{(1,5)}_4 = \emptyset$

$G_8 =$  $= H^{(1,5)}_8 \cup H^{(1,5)}_6$ (below)

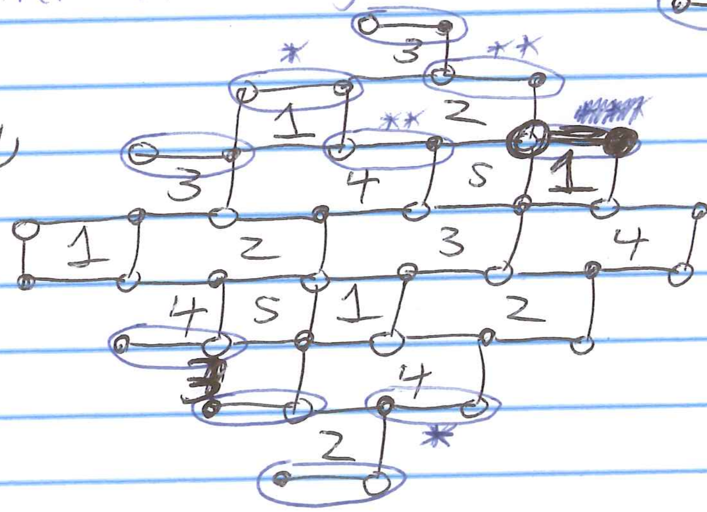
$G_9 =$  $= H^{(1,5)}_9 \cup \overset{\underline{s}}{H^{(1,5)}_7} \cup \overset{\underline{N-s}}{H^{(1,5)}_6}$
(below) (above)

(3) Equivalent to

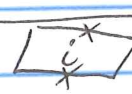
$G_7 =$  , $G_8 =$ 

Becomes 

edges * & ** (marked with) only removed after removal of the others lead to one of their vertices becoming leaves

Similarly, $G_9 =$ 

$Y(M)$ defined by recording what faces need to be twisted to get from $\boxed{M_0 = \text{minimal matching} = \text{unique matching with all horizontal edges}}$

multiply by y_i for each face  or  twisted

 OR  .

(4) $x(M)$ defined as

$$\frac{\displaystyle\prod_{\substack{\text{faces } F_i \\ \text{in } G_n^{r,s,N} \cup \partial G_n^{r,s,N} \\ \text{in } \tau^{(r,s,N)}}} x_i^{\,d-|M\cap \partial F_i|}}{\displaystyle\prod_{\substack{\text{closed faces } F_i \\ \text{in } G_n^{r,s,N}}} x_i}$$

(F_i) labeled i

where d defined from

closed face F_i in $G_n^{r,s,N}$ is a $2d$-gon

OR open face F_i in $\partial G_n^{r,s,N}$ borders $G_n^{r,s,N}$ in $2d$ or $2d-1$ edges

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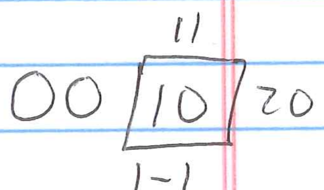
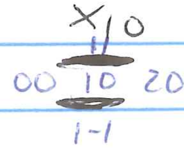
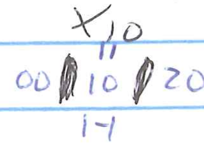
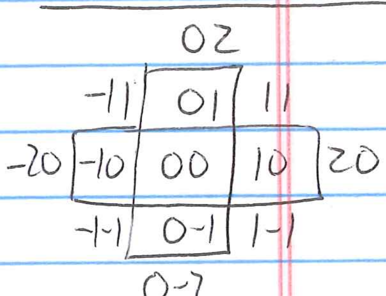
Aztec Diamond Examples of $x(M)y(M)$ w/ $\mathbb{Z}^2$-lattice



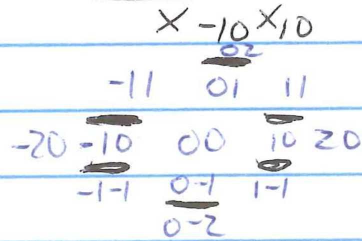
$$\frac{x_{00}\,x_{20}}{x_{10}} + \frac{x_{11}\,x_{1-1}}{x_{10}}\,y_{10}$$

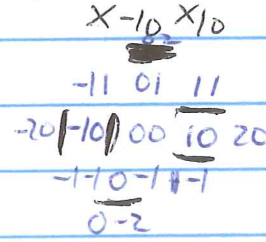


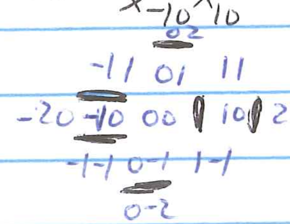




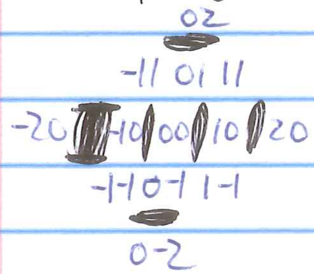
$$\frac{x_{-20}\,x_{00}\,x_{20}}{x_{-10}\,x_{10}} + \frac{x_{-11}\,x_{-1-1}\,x_{20}}{x_{-10}\,x_{10}}\,y_{-10} + \frac{x_{-20}\,x_{11}\,x_{1-1}}{x_{-10}\,x_{10}}\,y_{10}$$

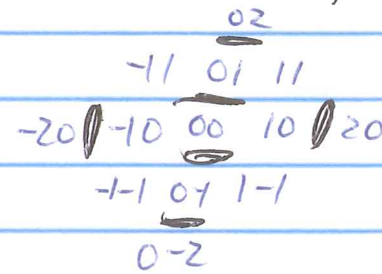




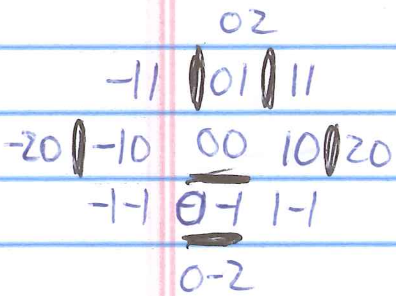


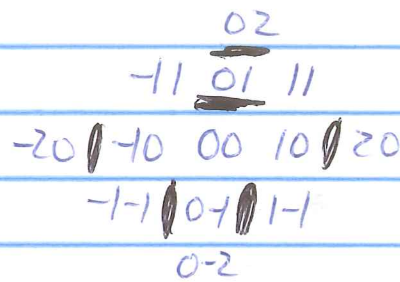
$$⑤ \; + \frac{X_{-11}X_{11}X_{1-1}X_{-1-1}}{x_{-10}x_{00}x_{10}} Y_{-10}Y_{10} + \frac{X_{-11}X_{11}X_{1-1}X_{-1-1}}{x_{0-1}x_{00}x_{01}} Y_{-10}Y_{00}Y_{10}$$



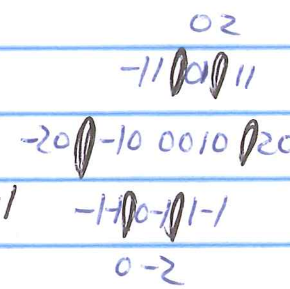


$$+ \frac{X_{02}X_{-1-1}X_{1-1}}{x_{0-1}x_{01}} Y_{-10}Y_{00}Y_{10}Y_{01} + \frac{X_{-11}X_{11}X_{0-2}}{x_{0-1}x_{01}} Y_{-10}Y_{00}Y_{10}Y_{0-1}$$





$$+ \frac{X_{02}X_{00}X_{0-2}}{x_{0-1}x_{01}} Y_{-10}Y_{00}Y_{10}Y_{01}Y_{0-1}$$



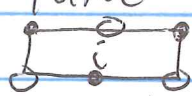
There are a number of approaches to proving this combinatorial interpretation in the literature. See

- Speyer '07 "Perfect Matchings and the Octahedron Recurrence"
- Bosquet-Melou-Propp-West '09 "Perf. matchings for the three-term Gale-Robinson seq's"
- Jeong-M-Zhang - FPSAC '12 Abstract "Gale-Robinson sequences and brane tilings"
- Di Francesco '13 "T-systems, networks and dimers"
- Vichitkunakorn '15 "Solutions to the T-systems with principal coefficients"
- R. Eager '10 "Brane Tilings and Non-commutative Geometry"

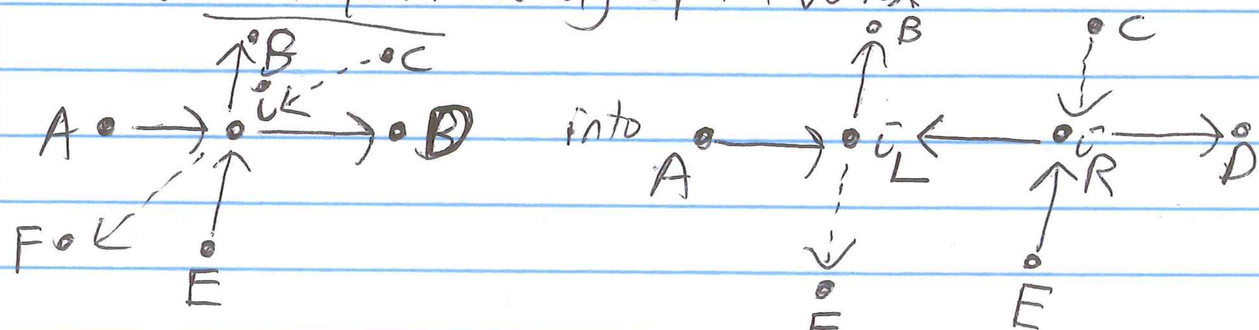
and - Glick-Weyman '17 "Gale-Robinson Quivers: From Representation to Comb. Formulas" using Kuo's Graphical Condensation

⑥ Since we already discussed the proof of this interpretation for $\mathbb{Z}^2$-lattice and Aztec Diamonds, like the Example we just discussed, we provide a method for reducing Gale-Robinson case to $\mathbb{Z}^2$-case like our approach reducing Pentagram Glick Quivers Q_n to $\mathbb{Z}^2$-case

By the way, see Kedem-VichitKunaKorn "T-systems and the Pentagram Map" '14 for a formal treatment of this reduction.

Take tiling $\mathcal{T}^{(r,s,N)}$ and split rectangular faces  into two squares  by imagining a vertical line bisecting them.

On the dual quiver level, split a vertex such as

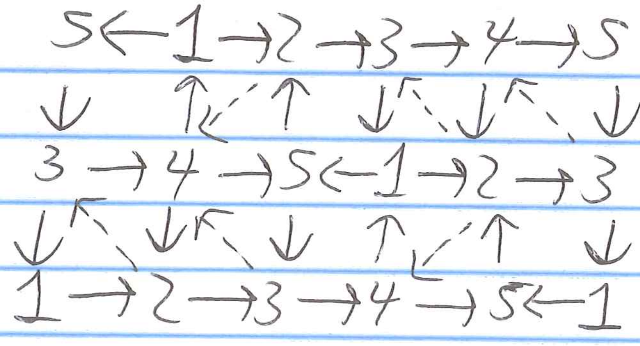


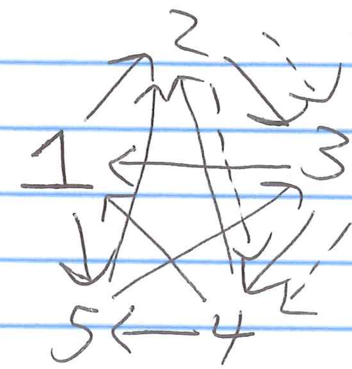
Note: as we go cyclically around vertex i, we see a pattern alternating in-out in both cases.

Vertical edge $i_L | i_R$ corresponds to new arrow $i_L \leftarrow i_R$.

Turns the Gale-Robinson Quiver into a projection of an Octahedron Recurrence / $\mathbb{Z}^2$-Quiver. In fact, will be isomorphic to a Glick/pentagram Q_m quiver.

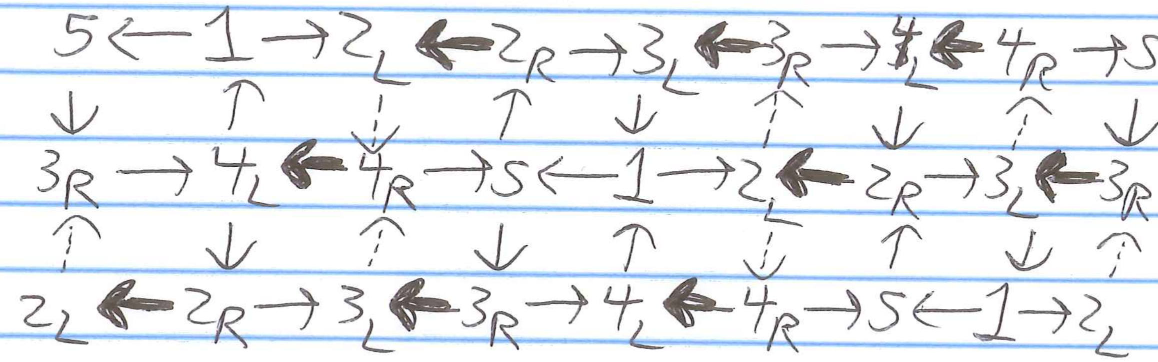
7 E.g. Somos-5





Becomes

$Q_4 \cong$ pentagram/Glick Quiver



We can thus break up our $\mu_1 \mu_2 \cdots \mu_N$ mutation sequence into

$\mu_{odd} \circ \mu_{even}$ | e.g. $\mu_{odd} = 1, 2_R, 3_R, 4_R$

$\mu_{even} = 2_L, 3_L, 4_L, 5$

In Speyer's Octahedron Recurrence Paper, he not only gives face weightings and $X(M)$ associated to matchings, as above, but also gives edge weightings to incorporate into the formula.

Setting the corresponding parameters assoc to $i_L \leftarrow i_R$ to zero degenerates to X_n's & G_n^{BSN} for Gale-Robinson sequence.

Then identify $x_{i_L} = x_{i_R} = x_i$ throughout, Y_i's analogous but more complicated