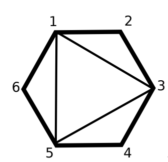
Math 8680: Cluster Algebras and their Variations

Homework 1 (Due Monday October 1st, 2018)

I encourage collaboration on the homework, as long as each person understands the solutions, writes them up in their own words, and indicates on the homework page their collaborators. You may use computer algebra packages for calculations but should also briefly describe your calculations in words in this case.

The following three problems, and their subproblems, showcase the various themes presented in the class thus far. Please do **at least ten** of the following subproblems. You may choose to complete all subproblems associated with a given problem or mix and match as you see fit.

1) Consider the following triangulation of a hexagon:  .

(a) What is the corresponding 9×3 exchange matrix B corresponding to this choice of triangulation?

(b) What is the corresponding quiver Q, including frozen vertices?

(c) Mutating in all possible directions, express all generators (i.e. cluster variables) of this cluster algebra $\mathcal{A}$ as Laurent polynomials in $x_1 = \Delta_{13}$, $x_2 = \Delta_{35}$, $x_3 = \Delta_{15}$, $y_1 = \Delta_{12}$, $y_2 = \Delta_{23}$, $y_3 = \Delta_{34}$, $y_4 = \Delta_{45}$, $y_5 = \Delta_{56}$, and $y_6 = \Delta_{16}$.

(d) What are combinatorial interpretations of these cluster variables using snake graphs?

(e) What is a compatible Poisson bracket $\{\cdot,\ \cdot\}$ (i.e. the skew-symmetric matrix Ω^x encoding this 2-form) for the associated initial cluster $x = \{x_1, x_2, x_3, y_1, y_2, y_3, y_4, y_5, y_6\}$?

HInt: You may use the fact that the rank of matrix B, from part (a), is 3.

(f) Illustrate that your Poisson bracket $\{\cdot,\ \cdot\}$, from part (d), is log-canonical.

(g) Illustrate how your Poisson bracket $\{\cdot,\ \cdot\}$, from part (d), changes after mutation in the 1st, 2nd, or 3rd direction.

(h) Illustrate that your mutated Poisson brackets $\{\cdot,\ \cdot\}$ from part (f) are still log-canonical.

2) (a) Consider a triangulation of a once-punctured torus. Construct a seed for a cluster algebra, either (X, B) or (X, Q), from your triangulation.

Hint: The fundamental domain of a torus can be given as a square where the vertical edges are identified (in an orientation-preserving manner), and the horizontal are also identified (in an orientation-preserving manner). The universal cover of the torus is the Euclidean plane. Lifting the unique marked point to this cover yields every point in $\mathbb{Z}^2$.

(b) Illustrate how your seed changes under mutation, and verify that this agrees with the corresponding changes on the triangulated surface.

(c) Argue why every Laurent polynomial appearing in this cluster algebra is homogenous? What grading are you using and what is the constant degree?

(d) If you label your initial cluster as $\{x, y, z\}$, argue why the Laurent polynomial $\frac{x^2+y^2+z^2}{xyz}$ can be expressed as a **positive** Laurent polynomial in every cluster of $\mathcal{A}$.

(e) Using (c) and (d) together, prove that for $\mathcal{A}$, its upper cluster algebra is strictly bigger (i.e.e not equal) to its cluster algebra.

Hint: What is the degree of a cluster monomial?

(f) Use snake graphs to give a combinatorial interpretation to the Laurent expansions of a few non-initial cluster variables in $\mathcal{A}$. In particular, pick 2 to 4 different arcs γ on this surface, chosen so that you get non-isomorphic snake graphs.

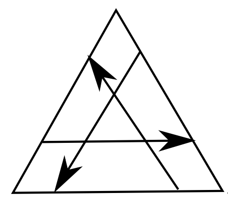
3) Let $\pi_{k,n}$ denote the Grassmann permutation $\begin{pmatrix} 1 & 2 & \dots & n-k & n-k+1 & \dots & n \\ k+1 & k+2 & \dots & n & 1 & \dots & k \end{pmatrix}$.

We may build the Quadrilateral Arrangement $A_{2,n}$ associated to $\pi_{2,n}$ by the following steps:

(i) Triangulate an n-gon using the chords

$$\Big\{(n, n-2), (n-2, 1), (1, n-3), (n-3, 2), \dots\Big\}$$

which continue in a zig-zag or snake-like pattern. Call the resulting triangulation $T_{2,n}$.

(ii) Inside of each triangle of $T_{2,n}$, draw paths that locally look like .

Gluing these paths together, we obtain the Quadrilateral Arrangement $A_{2,n}$, which is a collection of paths on the $2n$-gon, with vertices labeled $1, 1', 2, 2', 3, 3', \dots, n, n'$ in clockwise order, and such that vertex i is connected to vertex $\pi_{2,n}(i)'$. This $2n$-gon is overlayed onto $T_{2,n}$ so that vertex i (resp. i') in $A_{2,n}$ lies just to the right (resp. left) of vertex i in $T_{2,n}$.

Note: Even if it appears to have a bend in our drawing, we consider segments of paths between crossings to be single edges.

Given such an arrangement, its internal regions will have boundary oriented in a clockwise or counter-clockwise manner. Others are oriented in an alternating fashion. We call the latter of these even regions. Shade the cyclically oriented regions and label even regions with 2-subsets I of $\{1, 2, \dots, n\}$ where $i \in I$ if and only if the path emanating from vertex i strictly stays to the right of this region.

(a) Using this procedure, construct the Quadrilateral Arrangement $A_{2,6}$ and label the even regions accordingly.

(b) How does applying a quadrilateral flip to the a diagonal in the triangulation $T_{k,n}$ affect the arrangement $A_{k,n}$? What is the label of the new even region appearing in the new arrangement after mutation, and how does that label compare to the even region that had been mutated and its neighbors?

(c) What is the quiver associated to the Quadrilateral Arrangement $A_{2,6}$?

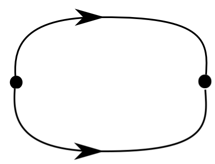
More generally, a Postnikov Arrangement, associated to $\pi_{k,n}$, is a collection of n oriented paths in the interior of the $2n$-gon (with labels $1', 1, 2', 2, \ldots, n', n$ clockwise) such that

0) Vertex i connects to $\pi_{k,n}(i)'$ and is oriented towards $\pi_{k,n}(i)'$.

1) No path intersects itself.

2) There are no triple or higher degree crossings in the diagram, i.e. all intersections are transversal.

3) As a path is traversed from its beginning to its end, the paths intersecting it alternate in orientation, first rightward, second leftward, etc. and ending in a rightward arc crossing it.

4) For any two paths, an unoriented lens  is forbidden.

(d) Verify that the Quadrilateral Arrangement $A_{2,6}$ you built in part (a) satisfies conditions (0)-(4).

(e) Build a Postnikov Arrangement associated to $\pi_{3,6}$ and label the even regions accordingly.

(f) What is the quiver associated to your choice of Postnikov arrangement for $\pi_{3,6}$?