Math 8680: Cluster Algebras and their Variations

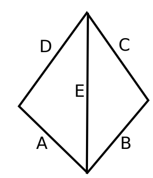
Homework 2 (Due Monday November 19th, 2018)

I encourage collaboration on the homework, as long as each person understands the solutions, writes them up in their own words, and indicates on the homework page their collaborators. You may use computer algebra packages for calculations but should also briefly describe your calculations in words in this case.

The following four problems, and their subproblems, showcase the various themes presented in the class thus far. Please do **at least ten** of the following subproblems. You may choose to complete all subproblems associated with a given problem or mix and match as you see fit.

1) Given a marked surface (S, M) with a triangulation T, with no self-folded triangles, and a function $f : T \to \mathbb{R}$, taking values on each edge e of T, the associated Weil-Petersson form can be defined as

$$\omega = \frac{1}{2} \sum_{E \in T} (dx_A \wedge dx_E - dx_B \wedge dx_E + dx_C \wedge dx_E - dx_D \wedge dx_E)$$



where A, B, C, and D are the four sides of the quadrilateral inscribing E as illustrated and x_e is shorthand for $\log f(e)$.

a) Use logarithmic differentiation, skew-symmetry of $\wedge$, and rearrangement/regrouping of the sum to show that we can rewrite the Weil-Peterson form as

$$\omega = \sum_{B \text{ follows } A \text{ immediately in the counter-clockwise order around a vertex}} \frac{df(A)}{f(A)} \wedge \frac{df(B)}{f(B)},$$

where we sum over appropriate pairs of edges in T.

Recall that $\lambda(e) = e^{\ell(e)/2}$, i.e. using Penner's λ-lengths for a choice of horocycles centered at each marked point in M. Here, $\ell(e)$ is defined as the hyperbolic length between horocycles.

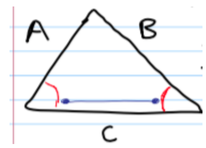
b) Using either of these expressions for the Weil-Petersson form, and letting $f(e) = \lambda(e)$ for all $e \in T$ (given a fixed choice of horocycles), demonstrate that ω is independent under the quadrilateral flip that replaces vertical arc E with horizontal arc E'.

Hint: Describe how $\lambda(E')$ can be expressed in terms of $\lambda(A)$, $\lambda(B)$, $\lambda(C)$, $\lambda(D)$, and $\lambda(E)$.

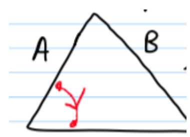
c) Consider a triangulated pentagon inscribed inside a 10-gon, i.e. seven internal arcs and ten boundary segments. Defining the λ-length as the constant 1 for every boundary arc, construct the Weil-Petersson form for this marked surface. Express the coefficients of this form as a 7-by-7 skew-symmetric matrix Ω such that

$$\omega = \sum_{\leq i<j\leq} \Omega_{ij} dx_i \wedge dx_j.$$

2) Given an arc or a closed loop γ, the hyperbolic length, and hence the lambda length, can be computed by breaking up into small segments of the following forms:

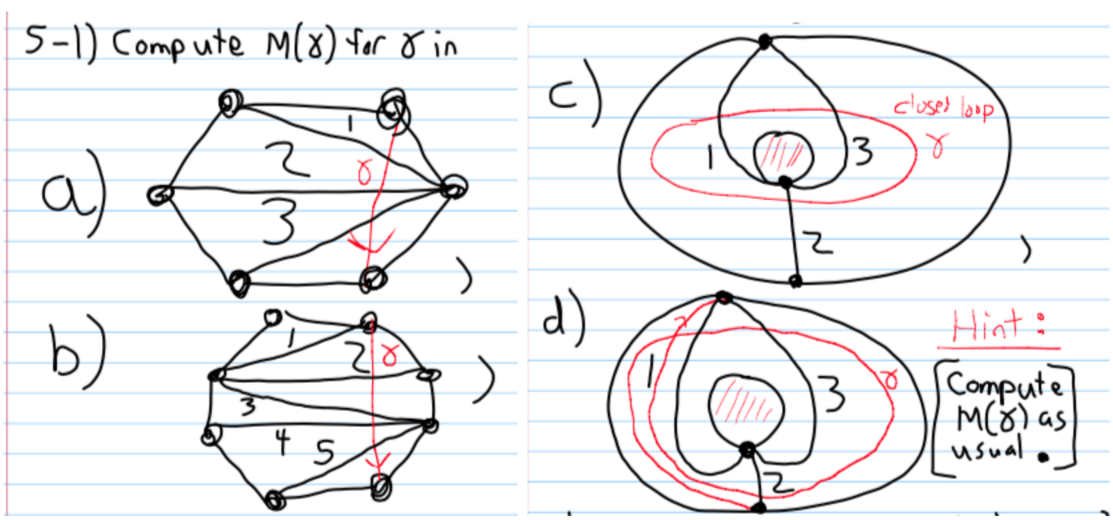
For a step enclosed inside a triangle traversing a segment between two horocycles 

associate the matrix $\begin{bmatrix} 0 & C \\ -1/C & 0 \end{bmatrix}$.

For a step following a portion of a horocycle as in 

associate the matrix $\begin{bmatrix} 1 & 0 \\ B/AC & 1 \end{bmatrix}$.

For any arc or closed loop γ, it is possible to break it up into segments that retract onto step of one of the following two types. Define 2-by-2 matrices for each such step and multiply them together in order from left-to-right. Call the result $M(\gamma)$.



a-d) Compute the 2-by-2 matrices $M(\gamma)$ for the above four examples. In cases (a), (b) and (d), where γ is an arc, what is the significance of the top right entry? In the case (c) where γ is a closed loop, what is the significance of the trace of the resulting matrix?

e) Let γ be an arc connecting punctures p and q in the marked surface (S, M). Let $\gamma^{(p)}$ denote the tagged version of this arc with a notch at puncture p. Let $\gamma^{(pq)}$ denote the tagged version notched at both punctures. Let $L(h_p)$ and $L(h_q)$ denote the hyberbolic lengths of the horocycles around punctures p and q respectively. Use λ-lengths to prove the Laurent expansions of $x_{\gamma^{(p)}}$, $x_{\gamma^{(pq)}}$, x_γ, $L(h_p)$, and $L(h_q)$ satisfy

$$x_{\gamma^{(p)}} = x_\gamma \cdot L(h_p)$$

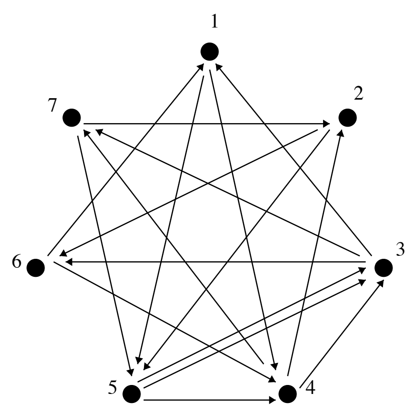
$$x_{\gamma^{(pq)}} = x_\gamma \cdot L(h_p) \cdot L(h_q)$$

Consider the once-punctured bi-gon with boundary segments b_1 and b_2 triangulated with radii r and s. The corresponding cluster algebra from this surface is of type $D_2 = A_1 \times A_1$ and has four cluster variables.

f) Use cluster mutation to write the two non-initial cluster variables in terms of b_1, b_2, x_r and x_s. (Here x_r and x_s are the cluster variables corresponding to arcs r and s, and by abuse of notation we leave the frozen variables labeled as b_1 and b_2.)

g) Let ℓ_1 and ℓ_2 be the loops in this triangulated bigons that you get by flipping the radii r and s. Use the skein relation to expand $x_{\ell_1} x_{\ell_2}$ as a sum of three non-zero terms. (**Note:** a contractible monogon corresponds to zero in the cluster algebra.)

h) Comparing your computation in part (b) to the product of the four cluster variables, from part (f), conclude that the value of a closed loop around a puncture is simply the number 2 in the cluster algebra.

3) Consider the seven vertex quiver $Q =$ 

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a) Prove that this quiver is 1-periodic in the Fordy-Marsh sense, meaning that if we mutate at vertex 1, the resulting quiver is equivalent to the original up to a cyclic relabeling of the vertices.

b) Show that if we mutate iteratively in order by $1, 2, 3, 4, 5, 6, 7, 1, 2, 3, \ldots$, and let $x_8, x_9, x_{10}, \ldots$, denote the resulting cluster variables, then we get a one-parameter of cluster variables $\{x_n : n \in \mathbb{Z}\}$ where the product $x_n x_{n-7}$ can be expressed in terms of $x_{n-1}, x_{n-2}, \ldots, x_{n-6}$.

Recall that a Poisson bracket $\{\cdot, \cdot\}$ is compatible with the cluster structure if $\{x_j, x_k\} = \omega_{jk} x_j x_k$ for any two cluster variables x_j and x_k in the same cluster.

c*) Find a compatible Poisson bracket and express the associated matrix Ω for the initial cluster for Q.

d*) How does Ω look after a single (resp. two or three) mutations by 1 (resp. $1, 2$ or $1, 2, 3$)?

e*) Defining x_n for $n \in \mathbb{Z}$ as in part (b), compute c_{jk}'s (as a function of j and k) so that $\{x_j, x_k\} = c_{jk} x_j x_k$ for any $j, k \in \mathbb{Z}$ (even for cluster variables from different clusters).

f*) A function f is known as a Casimir for a Poisson bracket if it satisfies $\{f, g\} = 0$ for all other functions g. Construct Casimirs (Laurent polynomials in the x_i's) for the Poisson bracket associated to Q.

g) Decompose quiver Q into primitive 1-periodic quivers.

h) Let B be the 7-by-7 exchange matrix associated to Q. Let $\widetilde{B} = [B\ I]^T$ denote the associated extended exchange matrix for the cluster algebra with principal coefficients. By mutating iteratively in order by $1, 2, 3, 4, 5, 6, 7, 1, 2, 3, \ldots$, find a description of the c-vectors obtained along this mutation sequence.

i) Give a compact formula for the associated Y-parameters, occurring along this mutation sequence, in terms of $\{y_1, y_2, \ldots, y_7\}$ and F-polynomials that is analogous to Glick's formula in the case of the pentagram map.

j) Can you give a combinatorial interpretation for the F-polynomials for the first few examples in this case?

4) (a-c) Do Problems 4-1 (a-c) from Lecture 4 of MSRI-2011.

(d-f) Do Problems 4-2 (a-c) from Lecture 4 of MSRI-2011.