Before reviewing Speyer's definition of weight(M) as Laurent monomials in the face weights of $AD_{k+1}(i,j)$ we set all initial $x_{ij0} = x_{ij1} = 1$ to

Focus on the F-polynomials.

$$\boxed{\text{Claim: } F_{ijk} = \sum_{\substack{M \text{ is a perfect} \\ \text{matching of } AD_k(i,j)}} \text{height}(M).}$$

Cor: Max Glick's formula for $F_{j,k} = \sum_{\substack{\text{order ideals} \\ I \text{ in poset } P_k}} \prod_{(r,s,t)\in I} Y_{3r+s+t+j}$ (subscripts taken mod $2n$)

PF: Project down doubly-infinite quiver modding by lattice $\left\langle \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2n \end{bmatrix} \right\rangle$ yields pentagram quiver Q_n on $2n$ vertices, associated to a convex n-gon. ∎

Warm-up: Computing c-vectors along this specific choice of mutation sequence on the doubly-infinite quiver.

Let us first consider the projection down to the 4-vertex quiver

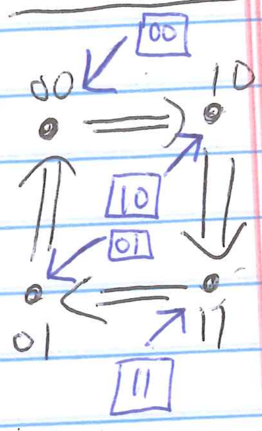
$$\begin{tikzcd} 00 \arrow[r, Rightarrow] & 10 \arrow[d, Rightarrow] \\ 01 \arrow[u, Rightarrow] & 11 \arrow[l, Rightarrow] \end{tikzcd}$$

to view this as a finite problem.

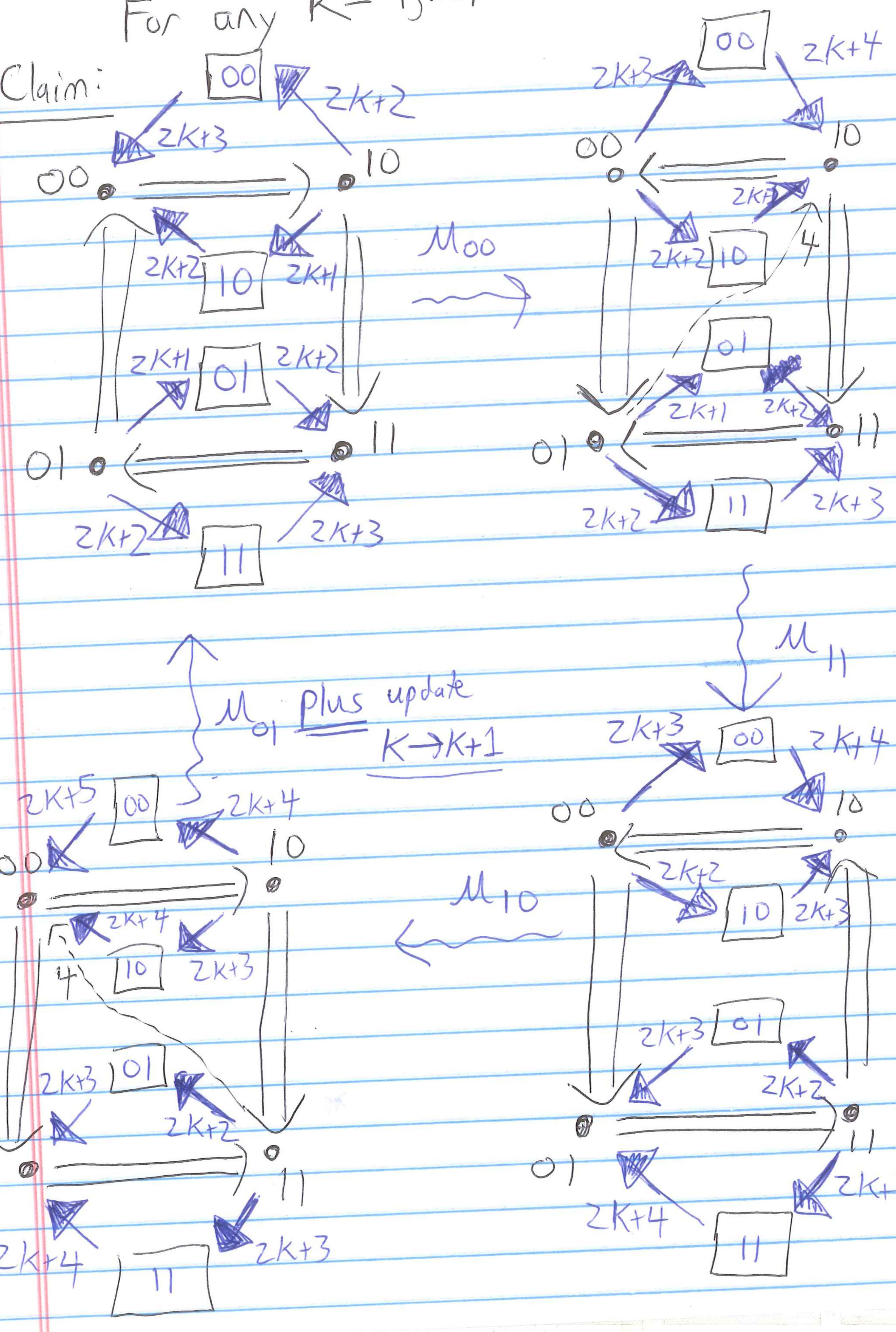
For any $K \geq -1$, we prove by induction:

Claim:

$K = -1$ case



corresponds to initial quiver with principal coeffs



E.g. in this first case, we get c-vectors

$$\begin{array}{c} 00 \\ 10 \\ 11 \\ 01 \end{array} \begin{bmatrix} 2k+3 \\ 2k+2 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -2k-2 \\ -2k-1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 2k+3 \\ 2k+2 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ -2k-2 \\ -2k-1 \end{bmatrix}$$

and we get the other three cases as we iterate mutation.

Label the initial cluster as $\{x_{000}, x_{101}, x_{110}, x_{011}\}$

and mutate in order $\mu_{00}, \mu_{11}, \mu_{10}, \mu_{01}$ w/ $\underline{x_{ij\ell} \to x_{ij(\ell+2)}}$
Then with principal coeffs, we have the recurrences

Observe y-monomials involve y_{ij} and $y_{\bar{i}j}$ $(\bar{0}=1, \bar{1}=0)$ but j-coordinate unchanged. exponents match.

$$x_{0,0,\ell}\, x_{0,0,\ell+2} = x_{1,0,\ell+1}\, x_{1,0,\ell+1} + y_{00}^{2k+3}\, y_{10}^{2k+2}\, x_{0,1,\ell+1}\, x_{0,1,\ell+1}$$

$$x_{1,1,\ell}\, x_{1,1,\ell+2} = x_{0,1,\ell+1}\, x_{0,1,\ell+1} + y_{11}^{2k+3}\, y_{01}^{2k+2}\, x_{1,0,\ell+1}\, x_{1,0,\ell+1}$$

$$x_{1,0,\ell+1}\, x_{1,0,\ell+3} = x_{0,0,\ell+2}\, x_{0,0,\ell+2} + y_{10}^{2k+3}\, y_{00}^{2k+4}\, x_{1,1,\ell+2}\, x_{1,1,\ell+2}$$

$$x_{0,1,\ell+1}\, x_{0,1,\ell+3} = x_{1,1,\ell+2}\, x_{1,1,\ell+2} + y_{01}^{2k+3}\, y_{11}^{2k+4}\, x_{0,0,\ell+2}\, x_{0,0,\ell+2}$$

where $\underline{\ell \geq 0 \text{ is even}}$, $\underline{k = (\ell-2)/2} \geq -1$

Unfolding to the infinite quiver, we get (for any $\ell \geq 0$, even or odd)

$$\boxed{x_{ij\ell}\, x_{ij(\ell+2)} = x_{i-1,\bar{j},\ell+1}\, x_{i+1,\bar{j},\ell+1} + \underline{y_{i-\ell,j} \cdots y_{i+\ell,j}}\, x_{\bar{i},j-1,\ell+1}\, x_{\bar{i},j+1,\ell+1}}$$

We also get c-vectors along this mutation sequence have the form
$$\vec{c}_{(a,b)} = \begin{cases} 1 & \text{if } |a-i| \le l,\ b=j \\ 0 & \text{o.w.} \end{cases}$$

(compare with $\prod_{i=-k}^{k} y_{j+3i}$ of Glick under the pentagram quiver specialization.)

To finish the proof $F_{ijk} = \sum_{M \text{ is a perf. matching of } AD_k(i,j)} \text{height}(M)$)

we let all $x_{ij} = 1$ to get simple recurrence

$$\boxed{F_{ijl}\,F_{ij(l+2)} = F_{i-1,j,l+1}\,F_{i+1,j,l+1} + y_{i-l,j}\,y_{i-l+1,j} \cdots y_{i+l,j}\,F_{i,j-1,l+1}\,F_{i,j+1,l+1}}$$

We use Kuo's method of Graphical Condensation (which we will prove next week)

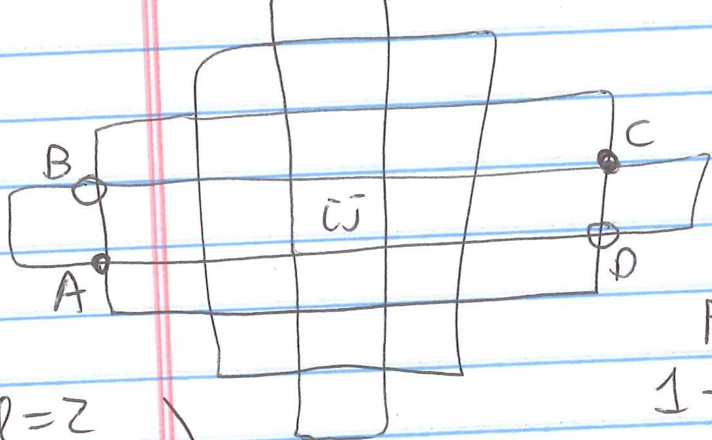
Thm (Kuo) Let $G=(V,E)$ be a bicolored planar bipartite graph where $V = V_1 \sqcup V_2$ & $|V_1| = |V_2|$.
V_1: ●, V_2: □

Also assume $A, C \in V_1$, $B, D \in V_2$ & A, B, C, D all lie on the same face (possibly the infinite unbounded face) in cyclic order. Then we have a weight-preserving bijection

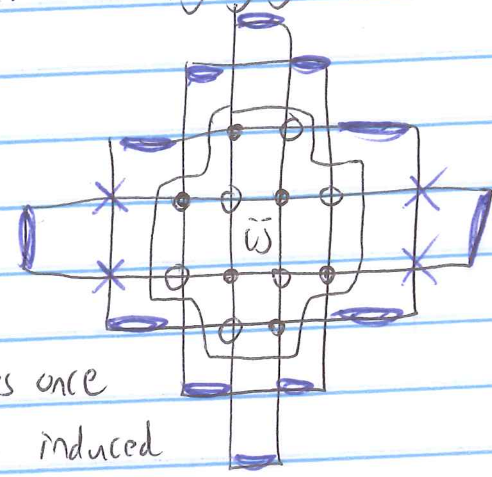
$$M(G)\cdot M(G-\{A,B,C,D\}) = M(G-\{A,B\})\cdot M(G-\{C,D\}) + M(G-\{A,D\})\cdot M(G-\{B,C\})$$

$M(G)$ = # perf. matchings in induced subgraph G

For the case of Aztec Diamonds, choose vertizes A, B, C, D as follows



$\Rightarrow$

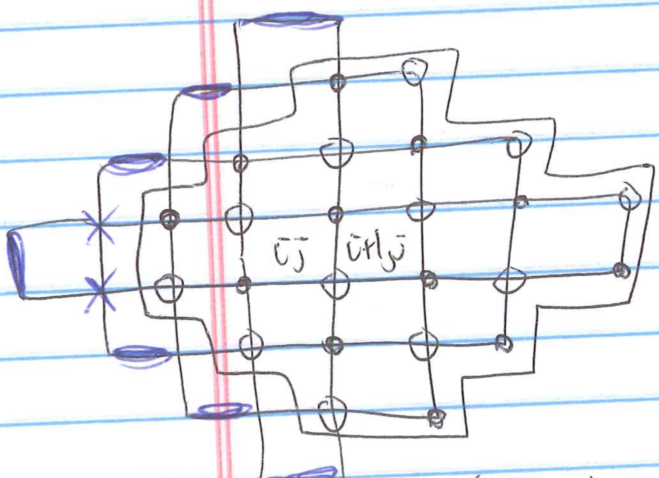


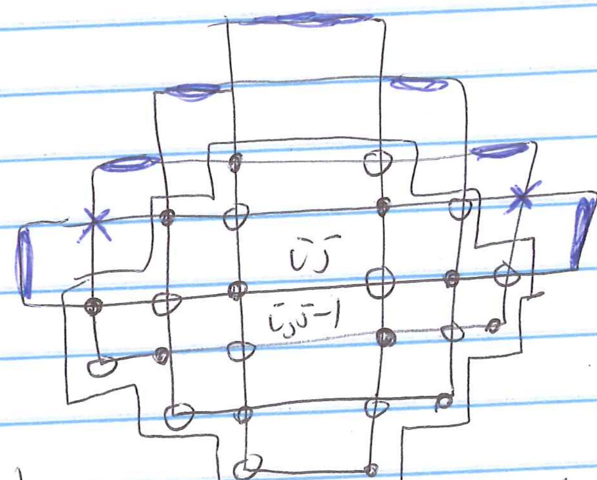
($\ell = 2$ illustrated)

$G = AD_{\ell+2}(i,j)$

Forced edges once 1-valent in induced subgraph

$G - \{A, B, C, D\} = AD_{\ell}(i,j)$





$G - \{A, B\} = AD_{\ell+1}(i+1, j)$

$G - \{C, D\} = AD_{\ell+1}(i-1, j)$ analogous

$G - \{B, C\} = AD_{\ell+1}(i, j-1)$

$G - \{A, D\} = AD_{\ell+1}(i, j+1)$ analog

Step 1: verify $F_{ij2} = 1 + y_{ij}$ is gen func for $AD_1(i,j)$ (for $i+j$ even)

F_{ij3} = gen func for $AD_2(i,j)$ (for $i+j$ odd)

Base case ✓

Step 2: Kuo Graphical condensation essentially proves the inductive step since

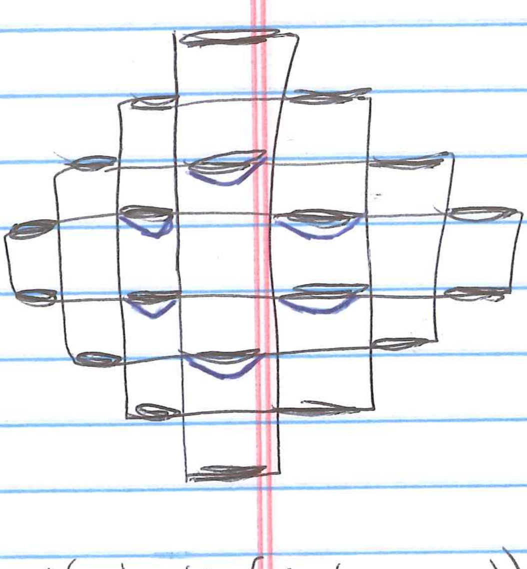
$$M(G)\cdot M(G-\{A,B,C,D\}) = M(G-\{A,B\})\,M(G-\{C,D\}) + M(G-\{A,D\})\,M(G-\{B,C\})$$

becomes

$$M(AD_{\ell+2}(i,j))\cdot M(AD_{\ell}(i,j)) = M(AD_{\ell+1}(i+1,j))\,M(AD_{\ell+1}(i-1,j)) + M(AD_{\ell+1}(i,j+1))\cdot M(AD_{\ell+1}(i,j-1))$$

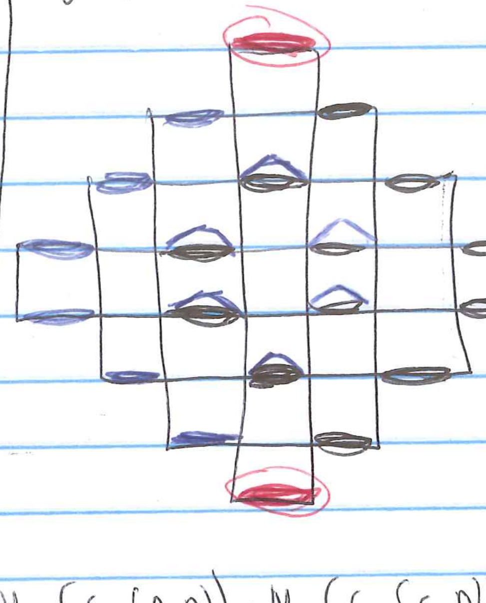
almost agreeing w/ the F-poly recurrence (if $F_{i,j,\ell} = M(AD_\ell(i,j))$). except for the $Y_{i-\ell,j}\cdot\ldots\cdot Y_{i+\ell,j}$ factor

Step 3: We look at superpositions of minimal matchings M_0 in Kuo's graphical condensation.

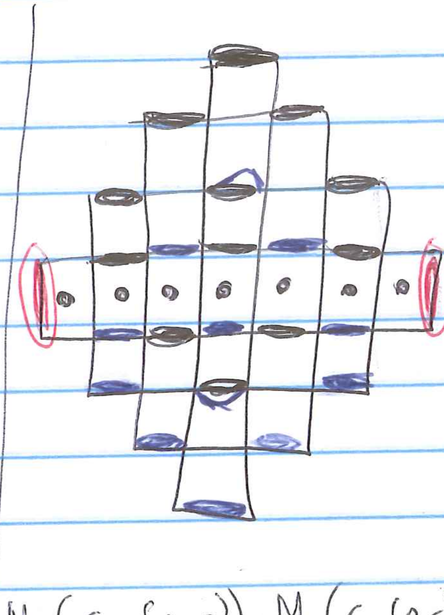


$M_0(G)\cdot M_0(G-\{A,B,C,D\})$

Inductive step $\boxed{\checkmark}$



$M_0(G-\{A,B\})\cdot M_0(G-\{C,D\})$



$M_0(G-\{A,D\})\cdot M_0(G-\{B,C\})$

twisted all squares in central row $\leftrightarrow$ $Y_{i-\ell,j}\cdots Y_{i+\ell,j}$