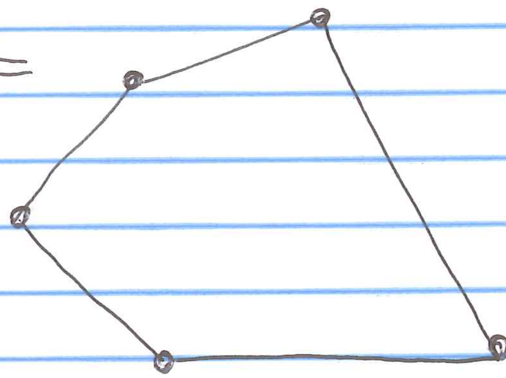
10/31/18 **The pentagram map** (We follow Sec 6.2 of [GR18])

First defined by Richard Schwartz in 1992

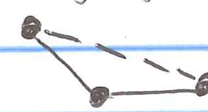
Given a pentagon $P =$



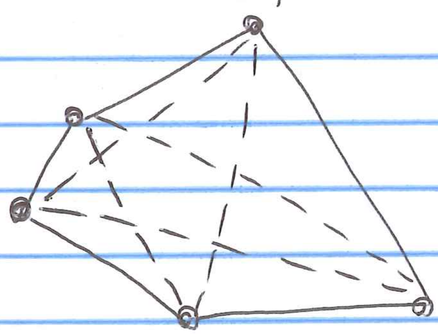
we build a new pentagon

$P' = T(P)$ by

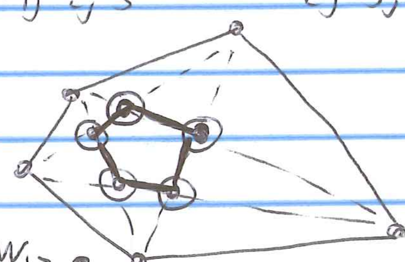
Step 1: For each consecutive triple of vertices v_1, v_2, v_3 connect v_1 and v_3



Happy Halloween!



Step 2: Let W_{23} be the intersection of the line segments for v_1, v_2, v_3 and v_2, v_3, v_4.



Step 3: P' defined as the convex hull of $W_{23}, W_{34}, W_{45}, W_{51}, W_{12}$.

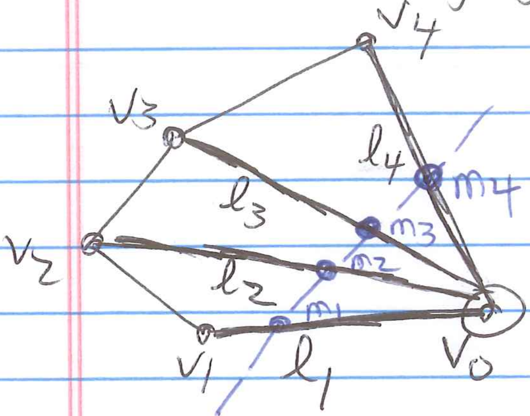
Rem: Generically, this map can be iterated but in affine coordinates, pentagons shrinking as map applied.

Rem: Natural extension to any polygons, not just pentagons.

We consider vertices of polygons in projective coordinates (rather than affine) to allow dilations & P, $T(P)$ of same relative "size".

(2) Let us first review some facts from affine and projective geometry:

Consider a pentagon, denote one of its 5 points as v_0 and define l_1, l_2, l_3, l_4 as the lines $\overline{V_0 V_i}$.



Let s_i = slope of l_i $\in \mathbb{R} \cup \{\infty\}$

This pentagon and choice of v_0 determines a cross-ratio

$$\chi(s_1, s_2, s_3, s_4) := \frac{(s_1 - s_2)(s_3 - s_4)}{(s_1 - s_3)(s_2 - s_4)}.$$

Recall from when we were discussing hyperbolic geometry:

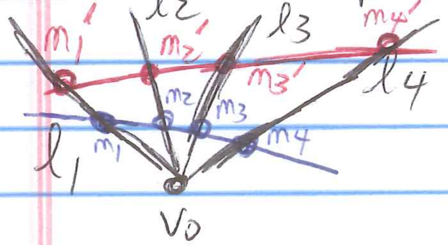
If F is a Möbius transformation, i.e. $F(x) = \frac{ax+b}{cx+d}$ where $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL_2(\mathbb{R})$, then

$$\chi(F(s_1), F(s_2), F(s_3), F(s_4)) = \chi(s_1, s_2, s_3, s_4).$$

Claim: If we intersect l_1, l_2, l_3, l_4 with any line l, then $\chi(s_1, s_2, s_3, s_4) = \frac{\|m_1 - m_2\| \cdot \|m_3 - m_4\|}{\|m_1 - m_3\| \cdot \|m_2 - m_4\|}$ where $\|m_i - m_j\|$ = length of line segment.

m_i = intersection point of l and l_i.

③ Varying line ℓ applies a Möbius transformation to the four points $\{m_1, m_2, m_3, m_4\}$, preserving cross-ratio χ_0

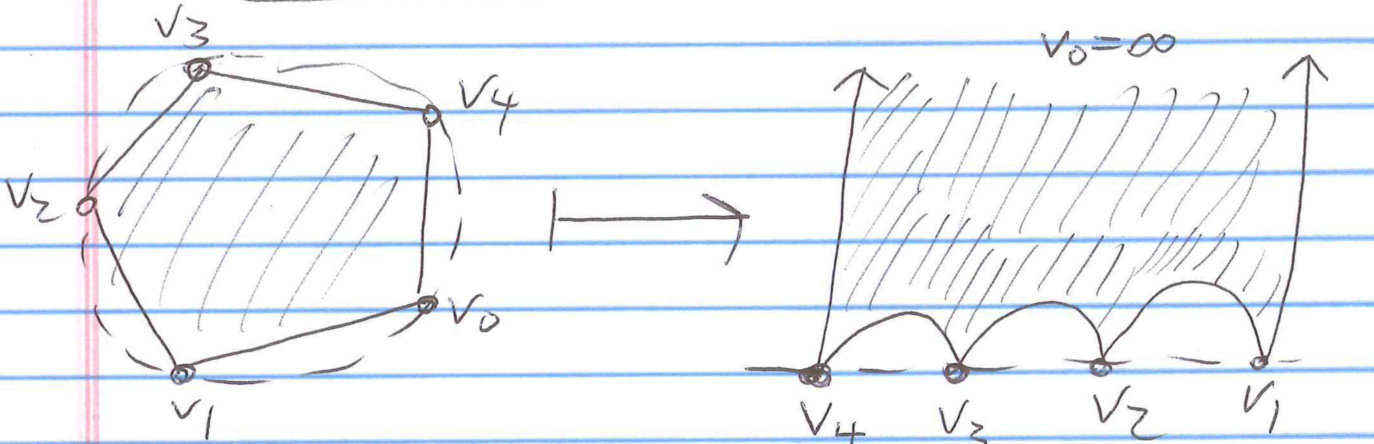


Rem: Projective space has a duality

lines meeting at a point $\approx$ collinear points

(just as) two lines determine a point $\approx$ two points determine a line

Rem: This definition of cross-ratio for 4 vertices of a pentagon also can be interpreted as map from Poincaré Disk to Upper Half Plane w/ pentagon inscribed in a circle:



thinking of $v_1, v_2, v_3, v_4 \in \mathbb{R} \cup \{\infty\}$, i.e. collinear

$\chi(v_1, v_2, v_3, v_4) = \chi(m_1, m_2, m_3, m_4)$ from above collinear pts.

We call $\chi(v_1, v_2, v_3, v_4)$ the corner invariant of pentagon P at vertex v_0. Denote as X_0.

Up to projective equiv., P is determined by $X_0, X_1, \ldots, X_4$.

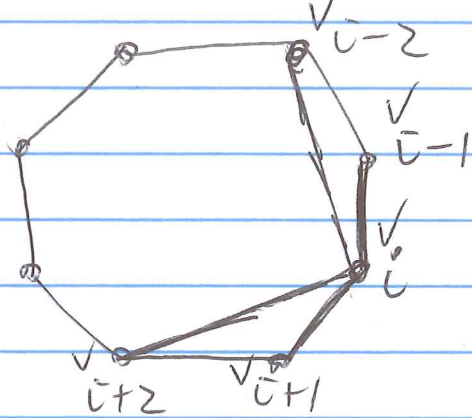
Claim: For a convex pentagon, $X_i(P') = X_i(P)$ for $i = 0, 1, 2, 3, 4$ and $P' = T(P)$, the image after one iteration of the pentagram map.

(4) For more general <u>convex n-gons</u>, we define corner invariants $X_0, X_1, \dots, X_{n-1}$ by

counter-clockwise cyclic order

$$X_i := \chi(s_{i-1}, s_{i-2}, s_{i+2}, s_{i+1})$$

where



s_j = slope of the line $\overline{V_i V_j}$

In this case, $X_i(P') \neq X_i(P)$ for $P' = T(P)$ and hence we have a more interesting dynamical system.

Next time, we see how M. Glick used Aztec Diamonds, Y-systems, and F-polynomials to use cluster algebra theory to coordinatize the dynamics.