Crash Course on Symplectic Manifolds & Poisson algebras

A symplectic manifold is a

manifold M plus "geometric structure" ω

(Analogous to a Riemannian manifold where extra geom. structure is a metric)

Warm-up e.g. A symplectic vector space is a

real vector space V (say of dim d) i.e. $V \cong \mathbb{R}^d$ and a

non-degenerate 2-form ω s.t.

$\omega(\vec{v}, \vec{w}) = \vec{v}^T A \vec{w}$ where A is ~~a~~ $d \times d$ matrix satisfying $A^T = -A$ and $\det A \neq 0$.

e.g. ~~A~~ $J = \left[\begin{array}{c|c} 0 & -I \\ \hline I & 0 \end{array}\right]$ is an example of such a possible A.

Fact: Every symplectic vec. space is even dim ($d = 2n$) and has a basis $\{p_1, p_2, \dots, p_n, q_1, q_2, \dots, q_n\}$ s.t.

$\omega(\vec{v}, \vec{w}) = \vec{v}^T J \vec{w}$.

Thus $\omega(p_i, q_j) = -\omega(q_i, p_j) = \delta_{ij}$ ← Kronecker delta

and $\omega = dp_1 \wedge dq_1 + \dots + dp_n \wedge dq_n$.

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Warning: If W is a subspace of symplectic vec. space V, it is possible for $W^{\perp} := \{\vec{v} \in V : w(\vec{v}, \vec{w}) = 0 \ \forall \vec{w} \in W\}$ to satisfy $W \cap W^{\perp} \neq \{0\}$.

In fact, for any $\vec{v} \in V$, $w(\vec{v}, \vec{v}) = 0$

$\Rightarrow$ If we let $W = \langle \vec{v} \rangle$, one-dim subspace,

$\langle \vec{v} \rangle^{\perp} \ni \vec{v}$ & $V \neq \langle \vec{v} \rangle \oplus \langle \vec{v} \rangle^{\perp}$, for e.g.

Subspace W of symplectic vec space V is

- symplectic if $W \cap W^{\perp} = \{0\}$
  (i.e. form restricted to W and $W^{\perp}$ each non-degenerate)
- Lagrangian if $W = W^{\perp}$
  (i.e. form w restricted to W is zero)
- isotropic if $W \subset W^{\perp}$, or
- co-isotropic if $W^{\perp} \subset W$.

e.g. For any $\overset{\text{non zero}}{\vec{v}} \in V$, $\langle \vec{v} \rangle$ is isotropic.

e.g. any hyperplane (codim 1) is co-isotropic.

③ e.g. In $\mathbb{R}^4$ w/ coordinates p_1, p_2, q_1, q_2 and

$w = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$, i.e.

$w(\vec{v}, \vec{w}) = \vec{v}^T \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \vec{w}$, then the

<u>2-planes</u> $\langle p_1, p_2 \rangle$ & $\langle q_1, q_2 \rangle$ are <u>Lagrangian</u>

while the <u>2-planes</u> $\langle p_1, q_1 \rangle$ & $\langle p_2, q_2 \rangle$ are <u>symplectic</u>.

$p_1 \leftrightarrow [1000]$
$p_2 \leftrightarrow [0100]$
$q_1 \leftrightarrow [0010]$
$q_2 \leftrightarrow [0001]$

so $w(p_1, \vec{w}) = [1000] \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_{p_1} \\ w_{p_2} \\ w_{q_1} \\ w_{q_2} \end{bmatrix} = -w_{q_1}$

So $\langle p_1, p_2 \rangle^{\perp} = \langle p_1, p_2 \rangle$ since $w(p_1, q_1) = -1 \neq 0$
<u>Lagrangian</u> ☑ $\quad w(p_2, q_2) = -1 \neq 0$

Similarly $\langle p_1, q_1 \rangle^{\perp} = \langle p_2, q_2 \rangle$ since

$w(p_1, \vec{w}) = -w_{q_1}$, $w(q_1, \vec{w}) = +w_{p_1}$

So <u>symplectic</u> ☑.

<u>Rem</u>: Can get all symplectic 2-planes of $\mathbb{R}^4$ using Plücker coords.

<u>"Def"</u>: A symplectic manifold "locally looks like" symplectic (real) vec space.

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Def:

Given a symplectic manifold (M, ω) and two smooth functions $F, G : M \to \mathbb{R}$, the Poisson bracket $\{F, G\}$ is defined as

another function whose value at point x is

$$\{F, G\}(x) := \frac{d}{dt} F(\varphi_t(x))\Big|_{t=0}$$

where φ_t is the Hamiltonian (local) flow of X_G

Hamiltonian vec field.

In particular,

$$\{F, G\}(x) = dF(x)\left(X_G(x)\right),$$

$$\{F, G\} = \omega(X_G, X_F) = -\omega(X_F, X_G)$$

Cor: $\{G, F\} = -\{F, G\}$.

In the (p, q)-words for $\mathbb{R}^{2n}$ case,

$$\{F, G\} := \left[G_q^T \,\middle|\, -G_p^T \right] \left[\begin{array}{c|c} 0 & -I \\ \hline I & 0 \end{array} \right] \left[\begin{array}{c} F_q \\ \hline -F_q \end{array} \right]$$

$$= \sum_{i=1}^{n} \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i}.$$

(5) As motivated, an (abstract) Poisson bracket is a skew-symmetric bilinear map $\{\cdot,\cdot\}$ satisfying the Leibniz identity $\{f_1 f_2, f_3\} = f_1\{f_2, f_3\} + \{f_1, f_3\} f_2$ and the Jacobi identity

$$\{f_1,\{f_2, f_3\}\} + \{f_2,\{f_3,f_1\}\} + \{f_3,\{f_1, f_2\}\} = 0.$$

Def: A Poisson algebra is a commutative associative algebra g equipped w/ a Poisson bracket

$$\{\cdot,\cdot\} : g \times g \to g.$$

Remark: If we let $g = \mathcal{C}^\infty(M)$, the alg. of smooth functions on a symplectic manifold, the usual product & chain rules of differentiation verify these two identities.

Rem: Given an assoc. alg. A, letting $\{x,y\}$ be defined as $[x,y] = xy - yx$, the commutator, yields a Lie algebra w/ Lie bracket $[\cdot,\cdot]$ and together w/ the mult. structure of A, this is a Poisson algebra.

Rem: The tensor algebra of a Lie algebra is a Poisson alg.

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Def:

For $H \in \mathfrak{g}$, if $\{F, H\} = 0$, $F \in \mathfrak{g}$ is called an integral of the motion H.

(thinking of X_F & X_H as vec. fields)

i.e. $[X_F, X_H] = X_{-\{F,H\}}$

Also said that they commute.

Def: An element $c \in \mathfrak{g}$ s.t. $\{c, f\} = 0 \ \forall f \in \mathfrak{g}$ is called a Casimir element or conserved quantity.

We will see later on the course that Poisson alg's and Casimir elts/conserved quantities have roles in cluster algebra theory.