10/3/18

Math 8680 A pre-symplectic structure ω

is a closed differential 2-form (could be degenerate) on an $(n+m)$-dim'l rational manifold.

Since ω possibly degenerate, we don't necessarily have a Poisson bracket w/ which to define log-canonicality

So we instead say functions $g_1, \dots, g_{n+m}$ are log-canonical w.r.t ω

if $$\omega = \sum_{i,j=1}^{n+m} \omega_{ij} \frac{dg_i}{g_i} \wedge \frac{dg_j}{g_j}$$ where ω_{ij} are constants.

we still define $\Omega^g = (\omega_{ij})$ as the coeff matrix. Ω^g is skew-symmetric by construction.

Def: pre-sympl. structure ω on a rational manifold is compatible w/ the cl. alg. $\mathcal{A}$ if all clusters in $\mathcal{A}$ are log-canonical w.r.t. ω.

Example 6.1 Let $\tilde{x} = \{x_1, x_2, x_3\}$, $\tilde{B} = B = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

No non-trivial Poisson bracket compatible w/ this cluster algebra, but if we let

$$\omega = \lambda \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} + \mu \frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3} + \nu \frac{dx_1}{x_1} \wedge \frac{dx_3}{x_3},$$

we can determine when this pre-symp. str. compatible w/ $\mathcal{A}$.

(2) Observe $x_1' = \frac{x_2+x_3}{x_1}$.

Claim: $\frac{dx_1}{x_1} = \frac{-dx_1'}{x_1'} + \frac{dx_2}{x_2+x_3} + \frac{dx_3}{x_2+x_3}$.

PF: $d\left(\frac{f}{g}\right) = \frac{g\,df - f\,dg}{g^2}$ by quotient rule

$\left[\text{or using } d\left(\frac{1}{g}\right) = \frac{-dg}{g^2} \text{ \& product rule } d(f\tilde{g}) = f\,d\tilde{g} + \tilde{g}\,df,\ \tilde{g} = 1/g\right]$

So $dx_1' = \frac{x_1 \overbrace{(dx_2+dx_3)}^{d(x_2+x_3)} - dx_1(x_2+x_3)}{x_1^2}$

$\Rightarrow \frac{dx_1}{x_1} = \frac{-dx_1' \cdot x_1}{x_2+x_3} + \frac{dx_2+dx_3}{x_2+x_3}$

and $\frac{x_1}{x_2+x_3} = \frac{1}{x_1'}$ so we get the desired equality. ▨

Hence we can rewrite w in terms of x_1', x_2, x_3 as

$$w = -\lambda \frac{dx_1'}{x_1'} \wedge \frac{dx_2}{x_2} + \left(u + \frac{\lambda x_3 + v x_2}{x_2+x_3}\right)\frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3} - v \frac{dx_1'}{x_1'} \wedge \frac{dx_3}{x_3}.$$

(using $dx_2 \wedge dx_2 = 0$, $dx_3 \wedge dx_3 = 0$, $dx_3 \wedge dx_2 = -dx_2 \wedge dx_3$)

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More details:

$$\omega = \lambda\left(\frac{-dx_1'}{x_1'} + \frac{dx_2}{x_2+x_3} + \frac{dx_3}{x_2+x_3}\right) \wedge \frac{dx_2}{x_2} + \mu \frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3}$$

$$+ \nu\left(\frac{-dx_1'}{x_1'} + \frac{dx_2}{x_2+x_3} + \frac{dx_3}{x_2+x_3}\right) \wedge \frac{dx_3}{x_3}$$

$$= -\lambda \frac{dx_1'}{x_1'} \wedge \frac{dx_2}{x_2} + \cancel{\lambda \frac{dx_2}{x_2+x_3} \wedge \frac{dx_2}{x_2}}^{\,0} + \lambda \frac{dx_3}{x_2+x_3} \wedge \frac{dx_2}{x_2}$$

$$+ \mu \frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3} - \nu \frac{dx_1'}{x_1'} \wedge \frac{dx_3}{x_3} + \nu \frac{dx_2}{x_2+x_3} \wedge \frac{dx_3}{x_3}$$

$$+ \cancel{\nu \frac{dx_3}{x_2+x_3} \wedge \frac{dx_3}{x_3}}^{\,0}$$

minus sign from $dx_3 \wedge dx_2 = -dx_2 \wedge dx_3$

$$= -\lambda \frac{dx_1'}{x_1'} \wedge \frac{dx_2}{x_2} + \left(\mu - \frac{\lambda x_3 + \nu x_2}{x_2+x_3}\right) \frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3}$$

$$- \nu \frac{dx_1'}{x_1'} \wedge \frac{dx_3}{x_3}.$$

Hence, this form is compatible only if $\left(\mu + \frac{-\lambda x_3 + \nu x_2}{x_2+x_3}\right)$ is a constant, i.e. only if $\boxed{\lambda = -\nu}$.

Using mutations in the other directions can yield $\boxed{\lambda = \mu}$ and we'll get $\Omega^x = \lambda B$ if compatible.

(4) Analogous to Thm 4.5 regarding compatible Poisson brackets, we get Thm 6.2 which classifies the vector space of possible <u>pre-symplectic structures</u>.

(Part of) <u>Thm 4.5</u> If rank $\tilde{B} = n$ ($\tilde{B}$ is $(n+m)\times n$), all compatible Poisson brackets form a vector space of dimension $r(B) + \binom{m}{2}$ where $r(B) = 1$ if B irreducible
= # blocks in decomp if reducible

<u>Thm 6.2</u> As long as $\tilde{B}$ has no zero rows, all compatible pre-sympl. structures form a vec sp. of dimension $r(B) + \binom{m}{2}$. [i.e. the same dimension if rank $\tilde{B} = n$]

<u>Cor.</u> When B irreducible & $\tilde{B} = B$ (i.e. $m=0$) then $r(B) = 1$, $\binom{m}{2} = 0 \Rightarrow$ 1-dimensional space of pre-sympl. structures, i.e. <u>scalar multiple of $\tilde{B}$ as E.g. 6.1</u>.

Note also that if

$$w = \sum_{j,k=1}^{n+m} w_{jk} \frac{dx_j}{x_j} \wedge \frac{dx_k}{x_k} = \sum_{j,k=1}^{n+m} w_{jk}' \frac{dx_j'}{x_j'} \wedge \frac{dx_k'}{x_k'}$$

after mutation in the ith direction

$$\{x_1, \ldots, x_{m+n}\} \xmapsto{\mu_i} \{x_1', \ldots, x_{m+n}'\}$$

then $w_{ij}' = -w_{ij}$ & for $j \neq i, k \neq i$,

$$w_{jk}' = w_{jk} + w_{ik} b_{ij} \text{ if } \begin{matrix} b_{ij} \cdot b_{ik} < 0 \\ (\text{e.g. } b_{ij} > 0 \text{ \& } b_{ik} < 0) \end{matrix}$$

↓ continued

(5) and $w_{jk}' = w_{jk}$ if $b_{ij} \cdot b_{ik} \geq 0$.

Furthermore $w_{ij}\, b_{ki} = w_{ik}\, b_{ji}$ in this case, i.e. if b_{ji} & b_{ki} are both positive

$$\frac{w_{ij}}{b_{ji}} = \frac{w_{ik}}{b_{ki}} = \mu_i .$$

Hence $\Omega^{\tilde{x}}[n+m, n] = \mathrm{diag}(\mu_1, \ldots, \mu_n)\, \tilde{B}$.

__Corollary 6.4__ Suppose $m=0$, so $\tilde{B}=B$, and assume B is skew-symmetric. Further assume B is irreducible. Then up to a scalar, there exists a unique __closed 2-form__ w on what's called the __secondary cluster manifold__ compatible w/ $A(B)$. __This form is symplectic.__

$\boxed{\text{Called the Weil-Peterssen form associated w/ } A(B).}$

i.e. $$w = \sum_{j,k} b_{kj} \frac{dx_j}{x_j} \wedge \frac{dx_k}{x_k} .$$

Also get τ-coordinates on secondary cluster manifold as

$$\tau_j = \prod_{k=1}^{n} x_k^{b_{kj}} .$$ [Although no longer necessarily __functionally independent.__]

__Next time:__ Teichmüller theory & Weil-Peterssen form in that case.