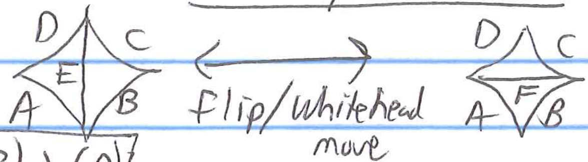
10/12/18 ① Last time we ended with a Proof of Ptolemy's Relation using Hyperbolic Geometry:



$$\boxed{\lambda(E)\lambda(F) = \lambda(A)\lambda(C) + \lambda(B)\lambda(D)}$$

regardless of the choices of horocycles.

Since we have homeomorphism $\widetilde{\mathcal{T}}(S,M) \to \mathbb{R}_{>0}^{n+c}$ via λ-lengths
(where $n = \#$ arcs in a triangulation, $c = \#$ boundary segments)

we can recover Laurent expansions of Cl. Vars as Hyperbolic λ-lengths as follows:

1) Given an initial choice of triangulation (w/o self-folded triangles) T, assign indeterminates $x_1, \ldots, x_{n+c} \in \mathbb{R}_{>0}$ to the extended cluster corresponding to $T \cup \{\text{Boundary Segments}\}$

Pick the unique pt in $\widetilde{\mathcal{T}}(S,M)$ [i.e. hyperbolic metric and horocycles at M]

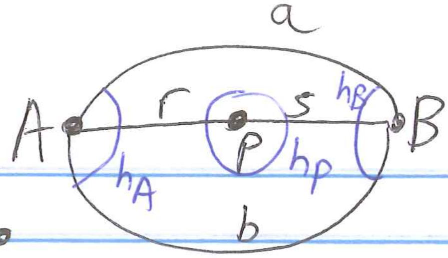
s.t. $\lambda(E_i) = x_i$ for $E_i \in T$

$\lambda(b_i) = x_{n+i}$ for boundary segment b_i.

2) Then w/ the same metric & horocycles, if arc E reachable from T by a sequence of flips (w/o self-folded triangles) then

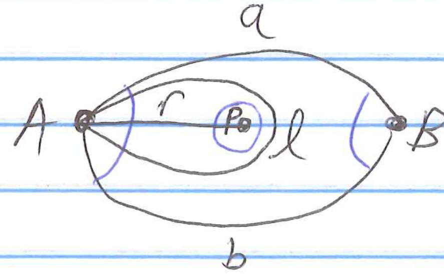
Ptolemy Relation matches Cluster Mutation $\Rightarrow \lambda(E) =$ associated cl. var (regardless of mut. seq.)

(2) Consider punctured bigon

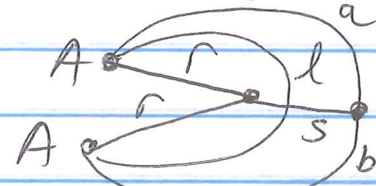


w/ $A, B, P \in M$ and pick horocycles.

Flipping s yields ideal triangulation

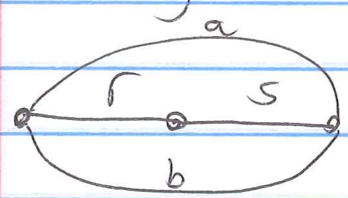


By Ptolemy Relation

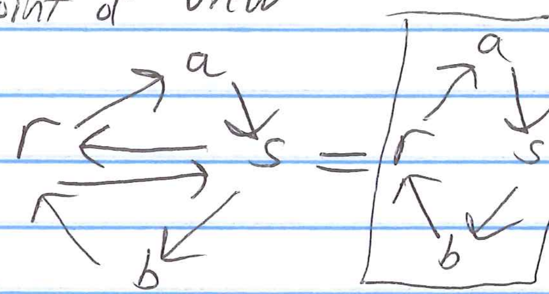


(*) $\lambda(l)\lambda(s) = \lambda(a)\lambda(r) + \lambda(b)\lambda(r)$.

However, from cluster algebra point of view

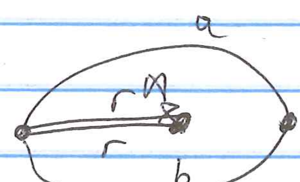


has local quiver



$X_{s'} X_s = X_a + X_b$

Claim: $X_{s'} = X_{r^{\bowtie}}$



in terms of tagged triangulations and we can define

$$\lambda(r^{\bowtie}) = \frac{\lambda(l)}{\lambda(r)}.$$

Hence Eqn (*) above can be rewritten as

$$\lambda(r)\lambda(r^{\bowtie})\lambda(s) = \lambda(r)\lambda(a) + \lambda(r)\lambda(b)$$

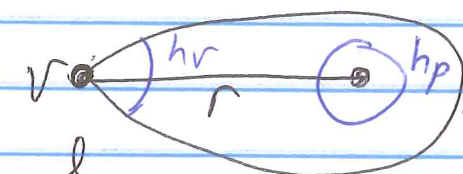
$\Rightarrow$ $\lambda(r^{\bowtie})\lambda(s) = \lambda(a) + \lambda(b)$ as desired.

Furthermore, we will give a geometric meaning to $\lambda(r^{\bowtie})$ to prove our claim.

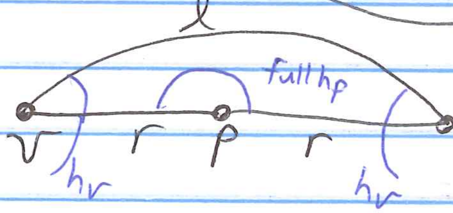
③ <u>Def</u>: Given an point in $\widetilde{\mathcal{T}}(S,M)$ and an <u>arc r</u> touching the <u>puncture p</u> with horocycle h_p, the <u>conjugate horocycle</u> $h_p^{\bowtie}$ defined as the choice of horocycle based at p s.t. $L(h_p^{\bowtie}) = \dfrac{1}{L(h_p)}$.

Here $L(h_p)$ is the full hyperbolic length of the h_p as a circle.

Then we define <u>$\lambda(r^{\bowtie})$ as λ-length using $h_p^{\bowtie}$ at p</u>.

<u>Observe</u>: In configuration 

,

which we can open up to 

)

$$L(h_p) = \frac{\lambda(\ell)}{\lambda(r)\cdot\lambda(r)}$$

Similarly, $\underbrace{L(h_p^{\bowtie})}_{\frac{1}{L(h_p)}} = \dfrac{\lambda(\ell)}{\lambda(r^{\bowtie})\lambda(r^{\bowtie})}$ ← by definition

$$\Rightarrow \frac{\lambda(r)\lambda(r)}{\lambda(\ell)} = \frac{\lambda(\ell)}{\lambda(r^{\bowtie})\lambda(r^{\bowtie})} \Rightarrow$$

$$\left[\lambda(r)\lambda(r^{\bowtie})\right]^2 = \left[\lambda(\ell)\right]^2$$ [Taking square-roots, since all λ-lengths > 0]

We conclude $\lambda(\ell) = \lambda(r)\lambda(r^{\bowtie})$ as desired. ▨

(4) Hence we can extend hyperbolic geometric interpretation to <u>tagged triangulations</u> of <u>punctured surfaces</u>

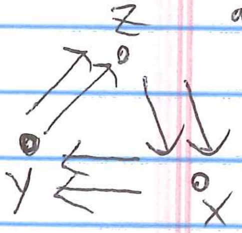
and using $p \overset{r}{\circ\!\!-\!\!-\!\!\circ} q \longmapsto \lambda(r)$

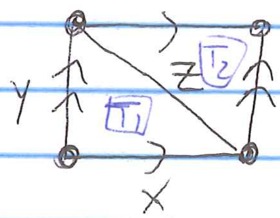
$p \overset{r^{\bowtie}}{\circ\!\bowtie\!\!-\!\!-\!\!\circ} q \longmapsto \lambda(r^{\bowtie})$ use conjugate horocycle $h_p^{\bowtie}$

$p \overset{r^{\bowtie\bowtie}}{\circ\!\bowtie\!\!-\!\!-\!\!\bowtie\!\circ} q \longmapsto \lambda(r^{\bowtie\bowtie})$ use both conjugate horocycles $h_p^{\bowtie}$ & $h_q^{\bowtie}$

<u>hyperbolic geometry</u> gives Laurent expansions for all cl. vars even for initial triangulations w/ self-folded triangles/tagged arcs or ending in " ".

<u>Application:</u> Consider the <u>Markoff Cluster Algebra</u> associated to <u>once-punctured torus</u> as on HW 1.

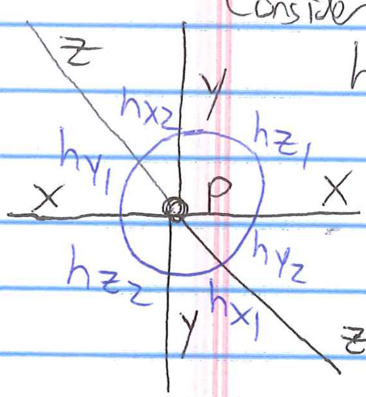




Letting $\lambda(e) = E$

i.e. $\lambda(x)=X$, $\lambda(y)=Y$, $\lambda(z)=Z$

Consider the Laurent Polynomial defined as $L(h_p)$ where h_p is the full horocycle around unique puncture p.



$$L(h_p) = L(h_{x_1}) + L(h_{y_2}) + L(h_{z_1}) + L(h_{x_2}) + L(h_{y_1}) + L(h_{z_2})$$

$$= \frac{X}{YZ} + \frac{Y}{XZ} + \frac{Z}{XY} + \text{" "} \quad \text{" "}$$

$$= 2\left(\frac{X^2+Y^2+Z^2}{XYZ}\right) \longleftarrow$$ (invariant under change of triangulation!)

In fact, for more geneal (S,M), can obtain Laurent polys associated to L (closed curve). (Sometimes in cl. alg. Sometimes in upper cl. alg. only)

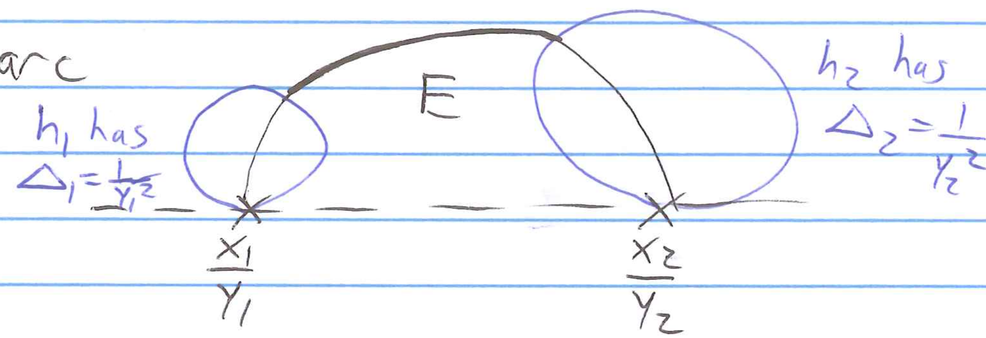
(5) Note: We can also encode a choice of ideal pt in $\mathbb{RP}^1$ plus a horocycle

as $(x,y) \in \mathbb{R}^2 \longmapsto$ ideal pt x/y w/ horocycle of diameter $\Delta = 1/y^2$.

Claim: Under this correspondence,

$$\lambda_E = \left| \det \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \right|$$

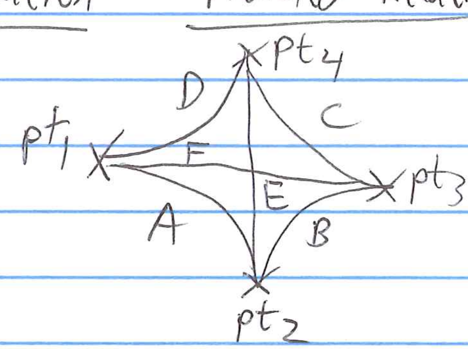
if $E =$ arc



Pf: Easy algebra of $\lambda_E = \dfrac{\left(\frac{x_2}{y_2} - \frac{x_1}{y_1}\right)}{\sqrt{\frac{1}{y_1^2}} \cdot \sqrt{\frac{1}{y_2^2}}} = x_2 y_1 - x_1 y_2$.

Via this translation, Ptolemy Relation = Plücker Relation

for $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix}$



λ-lengths) $\lambda_E \lambda_F = \lambda_A \lambda_C + \lambda_B \lambda_D$

$\Updownarrow$

Plückers) $P_{24} P_{13} = P_{12} P_{34} + P_{23} P_{14}$