10/22/18

Discrete Integrable Systems and Cluster Algebras

At the end of last week, we mentioned Zamolodchikov periodicity related to Dynkin Diagrams.

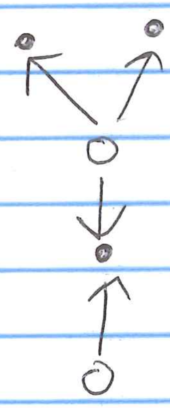
We talk about a more general case today, corresponding to the "product" of two Dynkin Diagrams.

For this case, periodicity proven by B. Keller in 2013 using categorification. (Published in Annals of Mathematics)

Let Δ and Δ' be two Dynkin diagrams with vertex sets I and I', both oriented in a bipartite i.e. alternating fashion.

e.g. A_4 $\bullet \longrightarrow \circ \longleftarrow \bullet \longrightarrow \circ$

and D_5



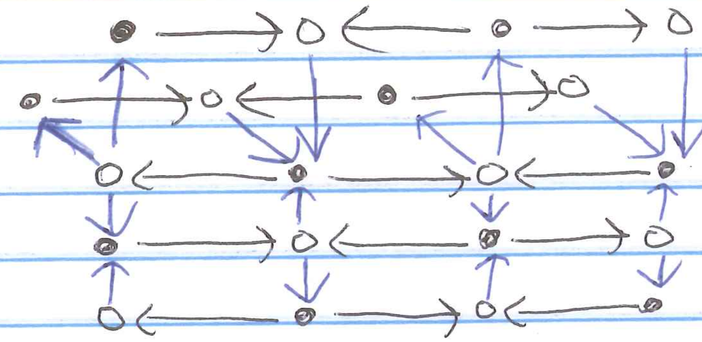
We define the quiver associated to $\Delta \times \Delta'$ by arranging
Δ horizontally
Δ' vertically
and making a new graph on vertices $(i, i') \in I \times I'$

We color vertex (i,i') as $\bullet$ if $i \in \Delta$ & $i' \in \Delta'$ have the same color

and as $\circ$ if " " opposite colors.

(2) we orient arrows of quiver associated to $\Delta \times \Delta'$ by $\bullet \longrightarrow \circ$ in Δ and $\circ \longrightarrow \bullet$ in Δ'.

E.g. $A_4 \times D_5$



Def: Cartan matrices associated to Δ, Δ' come from root systems/reflection group theory, essentially symmetrized version of exch. matrices assoc. to quiver/cl. alg. Can define Cartan matrices for types B_n, C_n, F_4, G_2 also where skew-symmetrizable matrix but not a quiver.

E.g. For Δ type A_5, Cartan matrix $= \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & 0 \\ & -1 & 2 & -1 & \\ 0 & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$

always have +2's on diagonal
non-positive off-diagonal

E.g. Δ type B_5

$$\begin{bmatrix} 2 & -1 & & & \\ -2 & 2 & -1 & & 0 \\ & -1 & 2 & -1 & \\ 0 & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$$

Δ type C_5

$$\begin{bmatrix} 2 & -2 & & & \\ -1 & 2 & -1 & & 0 \\ & -1 & 2 & -1 & \\ 0 & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$$

Δ type G_2

$$\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

(3) Y-system associated to $(\Delta \times \Delta')$ defined as

$\{ Y_{i,i',t} : (i,i') \in \Delta \times \Delta', t \in \mathbb{Z} \}$ satisfying

$$Y_{i,i',t+1} \cdot Y_{i,i',t-1} = \frac{\prod_{j \in I} (1 + Y_{j,i',t})^{-a_{ij}}}{\prod_{j' \in I'} (1 + Y^{-1}_{i,j',t})^{-a'_{i'j'}}}$$

for t odd (resp. even) if $(i,i') = \circ$ (resp. $\bullet$)

where a_{ij} = ijth entry of Cartan matrix for Δ
$a'_{i'j'}$ = $i'j'$th entry of " " Δ'

(exponents are nonnegative since off-diagonal entries of Cartan matrices are nonpositive.)

Let h (resp. h') be the Coxeter number associated to Δ (resp. Δ') defined as the period applying a certain cyclic transformation (Coxeter transf.) to the root system.

E.g. type A_n (corresponds to symm. group S_{n+1})
let s_i = transposition $(i, i+1)$ for $1 \le i \le n$.
let $c = s_1 s_3 s_5 \cdots s_{2m+1} \; s_2 s_4 \cdots s_{2m}$ if $m = \lfloor \frac{n}{2} \rfloor$

c acts on a two-dim plane, has order $n+1 = h$.

(4) **Thm (Keller) [Periodicity Conjecture]**

$$Y_{i,i',t+2(h+h')} = Y_{i,i',t} \quad \forall\, i \in \Delta,\ i' \in \Delta',\ t \in \mathbb{Z}.$$

History: Zamolodchikov conjectured ≈ 1991 [Thermodynamic Bethe Ansatz] for $\Delta \times A_1$ case w/ Δ simply-laced (ADE).

This was the case discussed the end of last week.

periodicity $= 2(h+2)$ in this case (since $h'=1$ for $\Delta' = A_1$)

For $\Delta = A_n$, $\Delta' = A_1$, first proven w/ explicit solutions in Frenkel–Szenes ≈ 1995.

independently by Gliozzi–Tateo (using volumes of 3-folds and triangulations) ≈ 1996

$\Delta \times A_1$ for any Δ (not necc. simply-laced) using cluster algebras $\approx 2001/2002$ by Fomin–Zelevinsky.

$A_n \times A_m$ case by Volkov ≈ 2006 (explicit formulas using cross-ratios)

E.g. Looks like (in $A_n \times A_m$ case)

$$Y_{i,i',t+1}\, Y_{i,i',t-1} = \frac{(1+Y_{i,i'-1,t})(1+Y_{i,i'+1,t})}{(1+Y^{-1}_{i-1,i',t})(1+Y^{-1}_{i+1,i',t})}$$

for $1 \le i \le n$, $1 \le i' \le m$

setting boundary conds for $i = 0, n+1$ & $i' = 0, m+1$.

(5) <u>Relation with cluster algebras</u>: Define coloring of vertices of quiver assoc. to $\Delta \times \Delta'$ by $\bullet \not\in \circ$ as above.

<u>Define</u>: τ_+ = Mutate at all $\bullet$

τ_- = Mutate at all $\circ$.

We think of τ_+ (resp. τ_-) as multiple mutations simultaneously (equivalently sequentially by any order).

[Since no vertices of same colors border each other, such mutations commute.]

Result after τ_+) all arrows reversed.

Following by τ_-) reverses all arrow back to the original.

We define $Y_{i,i',t-1}$ as the Y-system value at (i,i') $\bullet/\circ$

After applying τ_+/τ_-, let $Y_{i,i',t}$ be the updated value.

Here $Y_{i,i',t+1}$ is value after $\tau_+\tau_-/\tau_-\tau_+$.
Same quiver and thus can iterate or run backwards to get values for all $t \in \mathbb{Z}$.

(6) By definition of Y-seed mutation

After τ_+ all $\bullet$ vertices are $Y^{-1}_{i,i',t-1}$

their neighbors (j,j') $\circ$ vertices updated to

$$Y_{j,j',t-1} \xrightarrow{\mu_{\bullet(i,i')}} \begin{cases} Y_{j,j',t-1} * (1+Y_{i,i',t-1}) & \text{if } (j,j') \circ \leftarrow \bullet (i,i') \\ Y_{j,j',t-1} * (1+Y^{-1}_{i,i',t-1})^{-1} & \text{if } (j,j') \circ \rightarrow \bullet (i,i') \end{cases}$$

mutation at a single vertex

Taking the full composite mutation sequence

$$Y_{j,j',t-1} \xrightarrow{\tau_+} Y_{j,j',t-1} * \frac{\prod_{(j,j')\circ \leftarrow \bullet(i,i')} (1+Y_{i,i',t-1})}{\prod_{(j,j')\circ \rightarrow \bullet(i,i')} (1+Y^{-1}_{i,i',t-1})}$$

$$Y_{j,j',t} = Y_{j,j',t-1} * \frac{\prod_{(j,j')\circ \leftarrow \bullet(i,i')} (1+Y_{i,i',t-1})}{\prod_{(j,j')\circ \rightarrow \bullet(i,i')} (1+Y^{-1}_{i,i',t-1})}$$

We then apply τ_-

all $\circ$ vertices are inverted

$$\Rightarrow Y_{j,j',t+1} = Y^{-1}_{j,j',t-1} \frac{\prod_{(j,j')\circ \rightarrow \bullet(i,i')} (1+Y^{-1}_{i,i',t-1})}{\prod_{(j,j')\circ \leftarrow \bullet(i,i')} (1+Y_{i,i',t-1})}$$

⑦ Lastly, we note that $Y_{i,i',t} = Y^{-1}_{i,i',t-1}$ for (i,i') •

and so rewrite <u>for $(j,j') = \circ$</u>

$$Y_{j,j',t+1} \times Y_{j,j',t-1} = \frac{\prod_{(j,j') \circ \to (i,i') \bullet} (1 + Y_{i,i',t})}{\prod_{(j,j') \circ \leftarrow (i,i') \bullet} (1 + Y^{-1}_{i,i',t})}$$

as desired

The proof for $Y_{i,i',t+1}$'s where <u>$(i,i') = \bullet$</u> is analogous except we need t even rather than odd this time (so that we mutate τ_- then τ_+ instead).

Note that even in the <u>$A_2 \times A_2$ case</u> the Y-system already non-trivial.

$h = h' = 3$ for this case. By the Thm

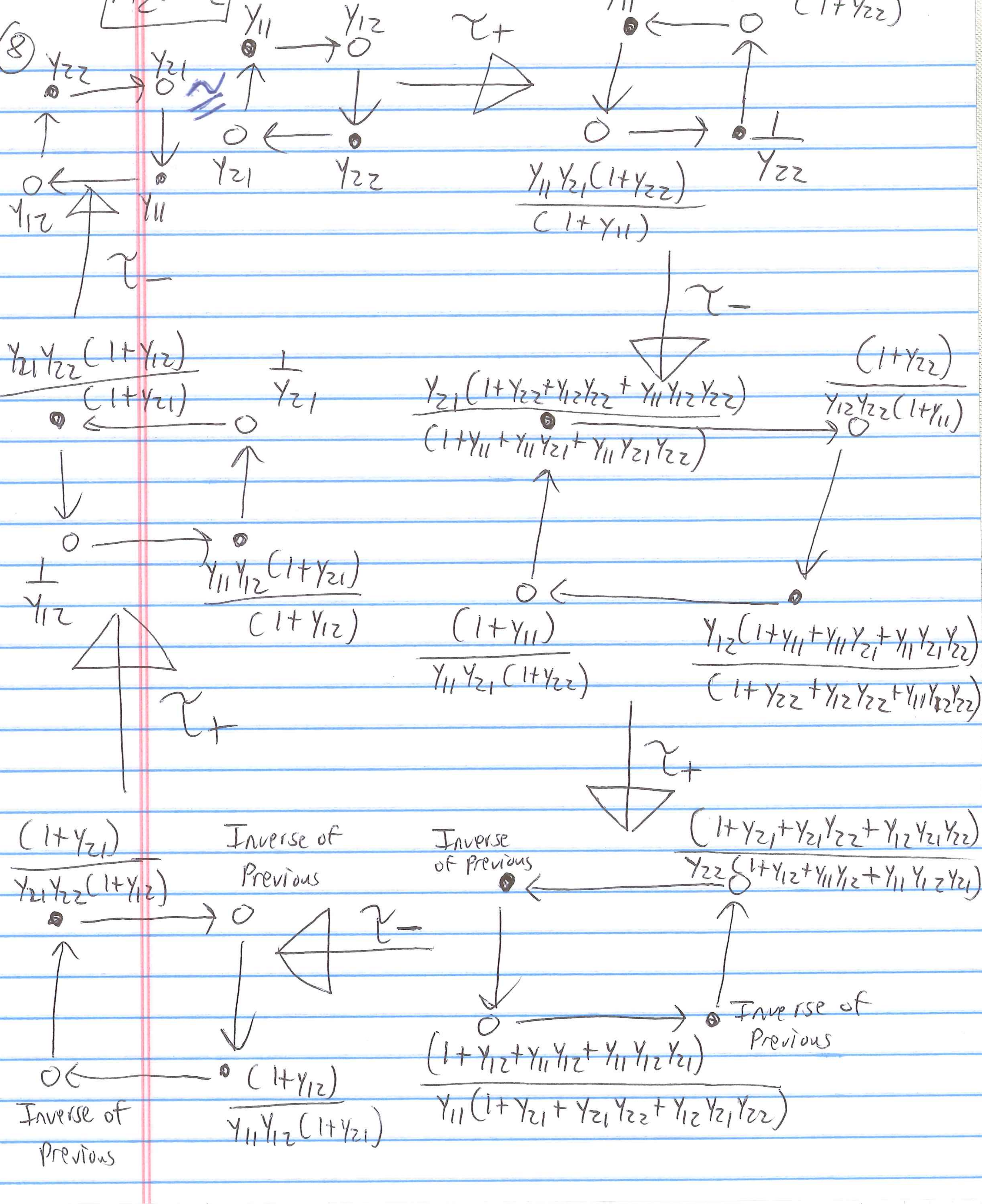
$\Rightarrow$ period for this Y-system is $2(h+h') = 12$.

we illustrate the first six steps

which corresponds to a 180°-rotation.

$A_2 \times A_2$

(8)



Y22
Y21
Y11
Y12
Y11
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Y21
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≅
τ+
1/Y11
Y12Y22(1+Y11)/(1+Y22)
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Y11Y21(1+Y22)/(1+Y11)
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Y21Y22(1+Y12)/(1+Y21)
1/Y21
1/Y12
Y11Y12(1+Y21)/(1+Y12)
Y21(1+Y22+Y12Y22+Y11Y12Y22)/(1+Y11+Y11Y21+Y11Y21Y22)
(1+Y22)/Y12Y22(1+Y11)
(1+Y11)/Y11Y21(1+Y22)
Y12(1+Y11+Y11Y21+Y11Y21Y22)/(1+Y22+Y12Y22+Y11Y12Y22)
τ+
τ+
(1+Y21)/Y21Y22(1+Y12)
Inverse of Previous
Inverse of previous
(1+Y21+Y21Y22+Y12Y21Y22)/Y22(1+Y12+Y11Y12+Y11Y12Y21)
τ−
Inverse of Previous
(1+Y12)/Y11Y12(1+Y21)
(1+Y12+Y11Y12+Y11Y12Y21)/Y11(1+Y21+Y21Y22+Y12Y21Y22)
Inverse of Previous

(9) Sketch of Proofs

For original $\Delta \times A_1$ (Zamolodchikov) case,

There exists a root system associated to Δ

(e.g. $\{e_i - e_j \mid 1 \le i, j \le n+1\}$ for A_n case)

Letting $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3, \ldots, \alpha_n = e_n - e_{n+1}$
we can rewrite $e_i - e_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$.

Each mutation μ_i in sequence $\tau_+ \circ \tau_-$ looks
locally like $\longleftrightarrow i \longleftrightarrow$

after mutation

$$X_{i,t-1} \cdot X_{i,t+1} = X_{i-1,t} \cdot X_{i+1,t} + 1$$

so parity of (vertex label)+(time) conserved.

We let inital cluster be

$$\{x_1, x_2, \ldots, x_n\} = \{X_{1,0}, X_{2,1}, X_{3,0}, \ldots, X_{2m,1}, X_{2m+1,0}\}$$

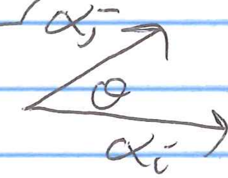
After $(\tau_+ \tau_-)^k$ $\{X_{1,2k}, X_{2,2k+1}, X_{3,2k}, \ldots, X_{2m,2k+1}, X_{2m+1,2k}\}$

Initial cluster $\longleftrightarrow$ roots $\{-\alpha_1, -\alpha_2, -\alpha_3, \ldots, -\alpha_{n-1}, -\alpha_n\}$

For non-initial $x_{i,}''''$ let $X_{i,t} = \dfrac{P_{i,t}(x_1, \ldots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}}$.

(10) Let s_i = root system reflection (of $\alpha_i \leftrightarrow -\alpha_i$)

$$s_i(\alpha_j) := \alpha_j - \frac{2\langle \alpha_i, \alpha_j\rangle}{\langle \alpha_i, \alpha_i\rangle}\alpha_i$$



Rem: $a_{ij} = \frac{2\langle \alpha_i, \alpha_j\rangle}{\langle \alpha_i, \alpha_i\rangle} = \frac{\|\alpha_j\|}{\|\alpha_i\|} 2\cos\theta$

are the values of the associated Cartan matrix.

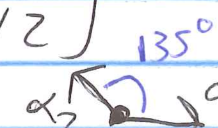
Can be shown: If $t = 2K$, & $X_{i,t}$ as above, then

$$d_1\alpha_1 + d_2\alpha_2 + \cdots + d_n\alpha_n = (s_1 \circ s_3 \circ \cdots \circ s_{2m+1} \circ s_2 \circ s_4 \circ \cdots \circ s_{2m})^K[-\alpha_i]$$

Hence periodicity of sequence of Y-system, follows by periodicity " " cluster variables, which follows by periodicity / finiteness of the root system assoc. to Δ.

Example (A_2) $\overset{1}{\bullet} \to \overset{2}{\bullet}$ w/ Cartan matrix $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

We can embed the assoc. root system in $\mathbb{R}^2$ as



Thus $-\alpha_1 \xrightarrow{s_1} \alpha_1 \xrightarrow{s_2} \alpha_1 + \alpha_2 \xrightarrow{s_1} \alpha_2 \xrightarrow{s_2} -\alpha_2$

For more complicated examples we use piecewise-linear analogue

$$\sigma_i(\alpha) := \begin{cases} s_i(\alpha) & \text{unless } \alpha = -\alpha_j \text{ for } j \neq i \\ \alpha & \text{if } \alpha = \alpha_j \text{ for } j \neq i \end{cases}$$

$\langle \tau_+ \tau_- \rangle$-orbits (using $\sigma_i(\alpha)$'s) yield all pos roots & periodicity of cl. vars & Y-systems (from negative simples).

(11) For full $\Delta \times \Delta'$ case, Keller uses category theory to mimic root system combinatorics

Auslander-Reiten translation τ on bounded derived category

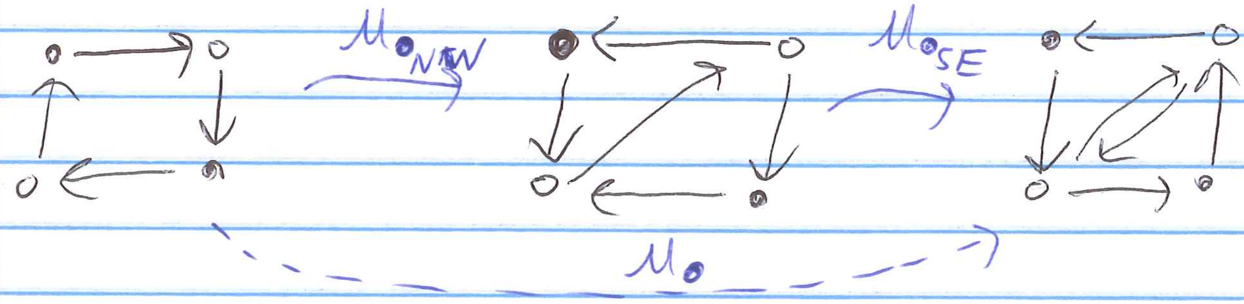
$\mathcal{D}^b(kQ)$ of modules over the path algebra KQ

plays the role of $\tau_+ \tau_- = c$ (Coxeter elt) in refl. gp.

Thms of Gabriel & Happel $\Rightarrow$ periodicity of AR trans. acting of Grothendieck group $K_0(\mathcal{D}^b(kQ))$

Another nuance: since $\Delta \times \Delta'$ contains cycles

e.g.



use quivers w/ potentials to deal w/ 2-cycles categorically (not just simply reversing sources to sinks throughout)