A Combinatorial Formula for Birational Rowmotion on Rectangular Posets

Gregg Musiker (UMN)* and Tom Roby (UCONN)

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Michigan State

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http://math.umn.edu/∼musiker/Birational17.pdf

Outline

1. Standard Young Tableaux and Promotion
2. Classical Rowmotion
3. Birational Rowmotion
4. Formula in terms of Lattice Paths
5. Sketch of Proof

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the 2015 AIM workshop on Dynamical Algebraic Combinatorics.

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Standard Young Tableaux and Promotion

Recall that a filling of a Standard Young Tableaux is an assignment from $\{1, 2, \ldots, n\}$ that is row-increasing and column-increasing.

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One can also study Standard Young Tableaux of skew shapes.

$$\left\{ \begin{array}{cccc} & & \boxed{1}&\boxed{2}\\ \boxed{3}&\boxed{4}& & \end{array}, \begin{array}{cccc} & & \boxed{1}&\boxed{3}\\ \boxed{2}&\boxed{4}& & \end{array}, \begin{array}{cccc} & & \boxed{1}&\boxed{4}\\ \boxed{2}&\boxed{3}& & \end{array}, \begin{array}{cccc} & & \boxed{2}&\boxed{3}\\ \boxed{1}&\boxed{4}& & \end{array}, \begin{array}{cccc} & & \boxed{2}&\boxed{4}\\ \boxed{1}&\boxed{3}& & \end{array}, \begin{array}{cccc} & & \boxed{3}&\boxed{4}\\ \boxed{1}&\boxed{2}& & \end{array} \right\}$$

Standard Young Tableaux and Promotion

There is a certain dynamic one can apply, known as *Jeu de taquin* or *Promotion*.

1	2	3
4	5	6

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1	2	3
4	5	$\bullet$

Step 1: Replace the largest element with an empty square.

Standard Young Tableaux and Promotion

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1	2	3
4	$\bullet$	5

Step 2: Move smaller entries into empty square one at a time.

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$\bullet$	3	4
2	5	6

Step 3: Add one to all entries.

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1	3	4
2	5	6

Step 4: Replace empty square with 1.

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$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array}$$

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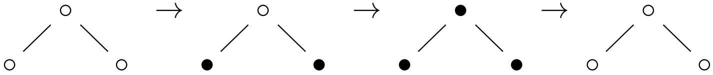
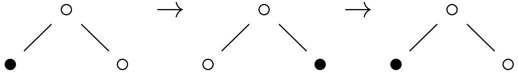
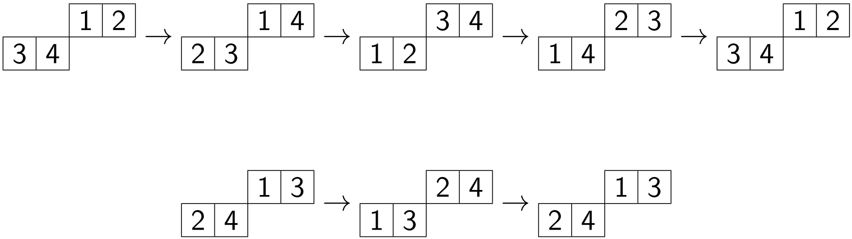
A Related Dynamic on Order Ideals

We can think of these orbits also as a dynamic on order ideals.





A Related Dynamic on Order Ideals (skew case)



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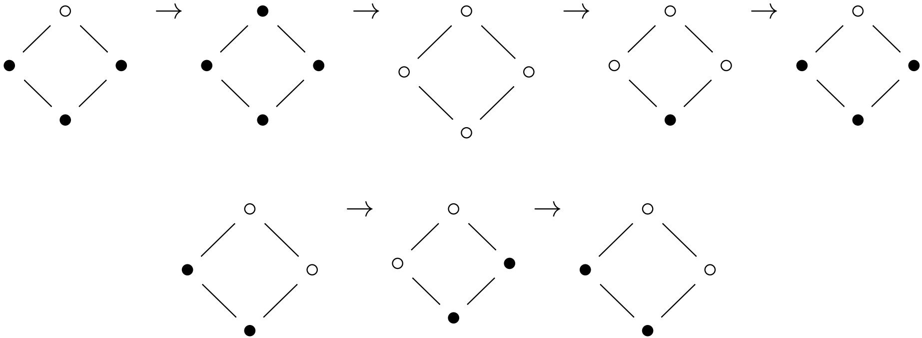

Classical rowmotion

Classical rowmotion is the rowmotion studied by Striker-Williams (`arXiv:1108.1172`). It has appeared many times before, under different guises:

- Brouwer-Schrijver (1974) (as a permutation of the antichains),
- Fon-der-Flaass (1993) (as a permutation of the antichains),
- Cameron-Fon-der-Flaass (1995) (as a permutation of the monotone Boolean functions),
- Panyushev (2008), Armstrong-Stump-Thomas (2011) (as a permutation of the antichains or "nonnesting partitions", with relations to Lie theory).

Let P be a finite poset. **Classical rowmotion** is the map $\mathbf{r} : J(P) \longrightarrow J(P)$ sending every order ideal S to a new order ideal $\mathbf{r}(S)$ generated by the minimal elements of $P \setminus S$.

Example: Let S be the following order ideal
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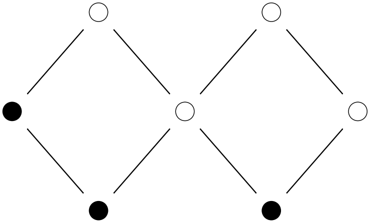

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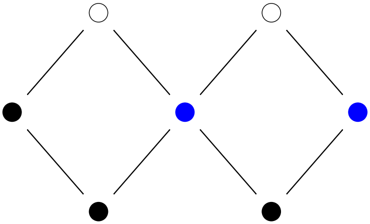
Example: Let S be the following order ideal
Mark M (the minimal elements of the complement) in blue.



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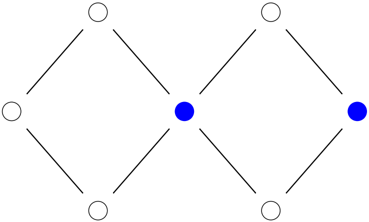
Example: Let S be the following order ideal
Remove the old order ideal:



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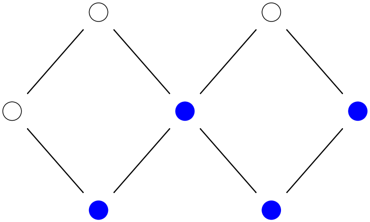
Example: Let S be the following order ideal
$\mathbf{r}(S)$ is the order ideal generated by M ("everything below M"):



Earlier Examples Revisited

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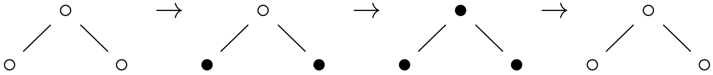

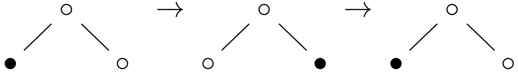

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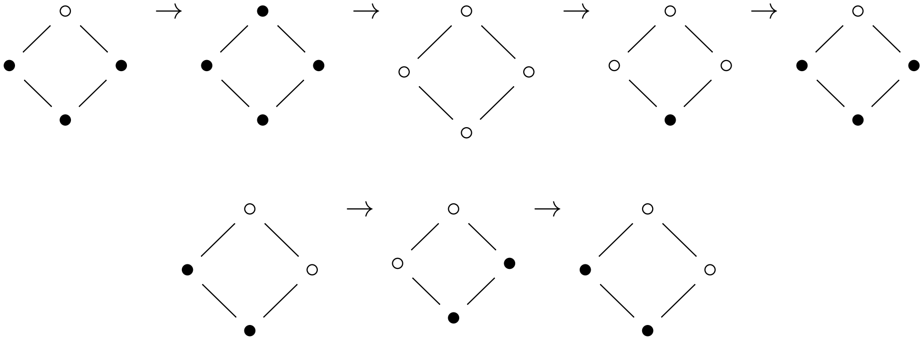

Classical rowmotion: properties

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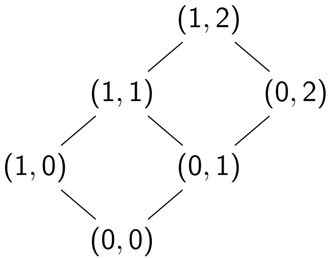
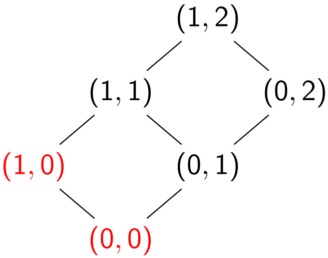
However, **for some types of** P, the order can be explicitly computed or bounded from above.
See Striker-Williams for an exposition of known results.

- If P is a $p \times q$-rectangle:



(shown here for $p = 2$ and $q = 3$), then $\text{ord}\,(\mathbf{r}) = p + q$.

Example:
Let S be the order ideal of the 2×3-rectangle $[0,1] \times [0,2]$ given by:



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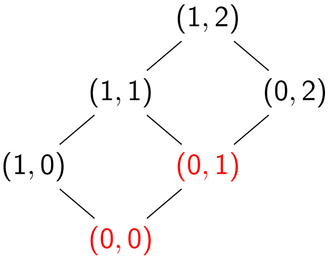
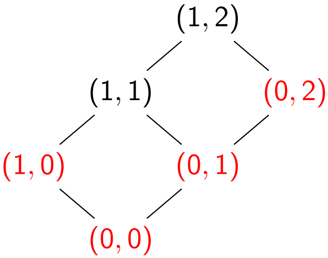


Example:
$\mathbf{r}^2(S)$ is



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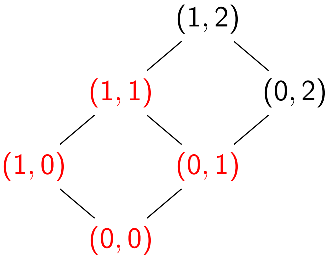

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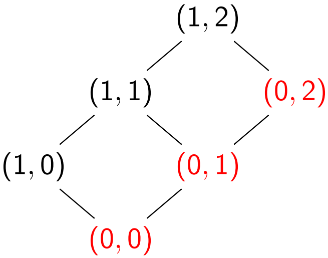

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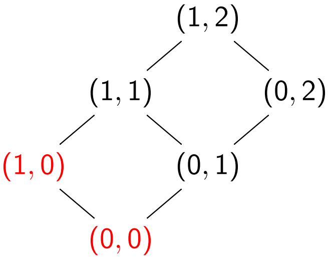
Example:
$\mathbf{r}^5(S)$ is



which is precisely the S we started with.

$\text{ord}(\mathbf{r}) = p + q = 2 + 3 = 5.$

There is an alternative definition of rowmotion, which splits it into many small operations, each an involution.

- Define $\mathbf{t}_v(S)$ as:
  - $S \triangle \{v\}$ (symmetric difference) if this is an order ideal;
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  ("Try to add or remove v from S, as long as the result remains an order ideal, i.e. within $J(P)$; otherwise, leave S fixed.")
- More formally, if P is a poset and $v \in P$, then the v-**toggle** is the map $\mathbf{t}_v : J(P) \to J(P)$ which takes every order ideal S to:
  - $S \cup \{v\}$, if v is not in S but all elements of P covered by v are in S already;
  - $S \setminus \{v\}$, if v is in S but none of the elements of P covering v is in S;
  - S otherwise.
- Note that $\mathbf{t}_v^2 = \text{id}$.

Classical rowmotion: the toggling definition

- Let $(v_1, v_2, ..., v_n)$ be a **linear extension** of P; this means a list of all elements of P (each only once) such that $i < j$ whenever $v_i < v_j$.
- Cameron and Fon-der-Flaass showed that

$$\mathbf{r} = \mathbf{t}_{v_1} \circ \mathbf{t}_{v_2} \circ ... \circ \mathbf{t}_{v_n}.$$

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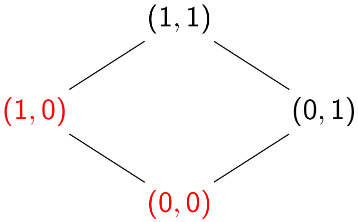

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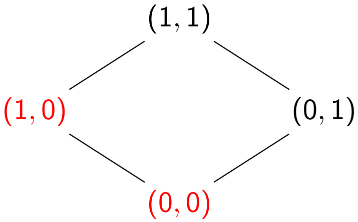
Example:
First apply $\mathbf{t}_{(1,1)}$, which changes nothing:



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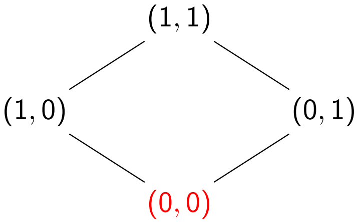
Then apply $\mathbf{t}_{(1,0)}$, which removes $(1,0)$ from the order ideal:



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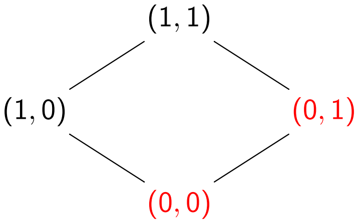
Example:

Then apply $\mathbf{t}_{(0,1)}$, which adds $(0,1)$ to the order ideal:



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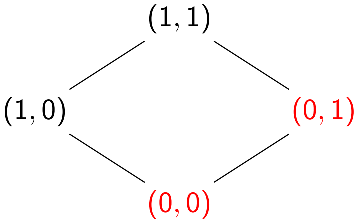

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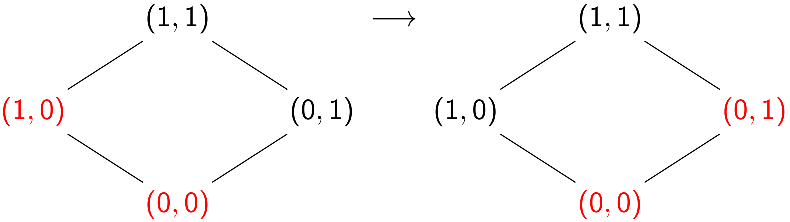
Example:

So this is $S \longrightarrow \mathbf{r}(S)$:



Generalizing to the piece-wise linear setting

The decomposition of classical rowmotion into toggles allows us to define a **piecewise-linear (PL)** version of rowmotion acting on functions on a poset. Let P be a poset, with an extra minimal element $\hat{0}$ and an extra maximal element $\hat{1}$ adjoined.

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The **order polytope** $\mathcal{O}(P)$ (introduced by R. Stanley) is the set of functions $f : P \to [0,1]$ with $f(\hat{0}) = 0$, $f(\hat{1}) = 1$, and $f(x) \leq f(y)$ whenever $x \leq_P y$.

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For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending f to the unique f' satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \cdot > x} f(z) + \max_{w < \cdot\, x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

where $z \cdot > x$ means z covers x and $w < \cdot\, x$ means x covers w.

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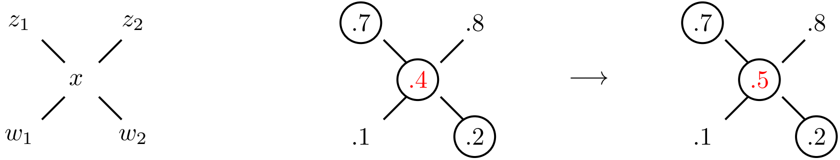
Note that the interval $[\min_{z \cdot > x} f(z), \max_{w < \cdot x} f(w)]$ is precisely the set of values that $f'(x)$ could have so as to satisfy the order-preserving condition.

if $f'(y) = f(y)$ for all $y \neq x$, the map that sends

$$f(x) \text{ to } \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x)$$

is just the affine involution that swaps the endpoints.

Example of flipping at a node

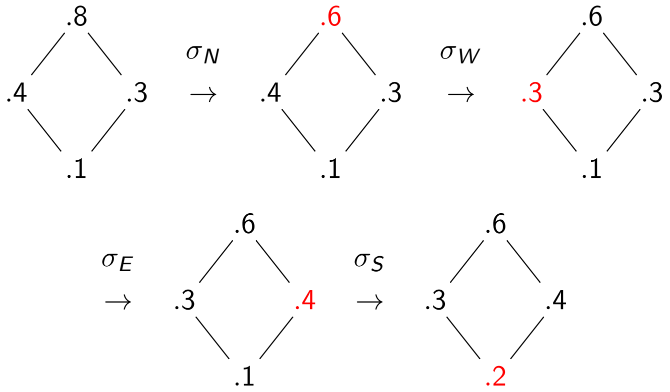


$$\min_{z \cdot> x} f(z) + \max_{w <\cdot x} f(w) = .7 + .2 = .9$$

$$f(x) + f'(x) = .4 + .5 = .9$$

Composing flips

Just as we can apply toggle-maps from top to bottom, we can apply flip-maps from top to bottom, to get *piecewise-linear rowmotion*:



(We successively flip at $N = (1, 1)$, $W = (1, 0)$, $E = (0, 1)$, and $S = (0, 0)$ in order.)

How PL rowmotion generalizes classical rowmotion

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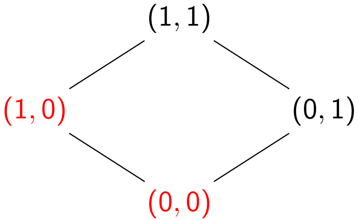

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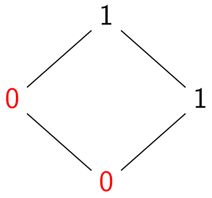
Example:

Translated to the PL setting:



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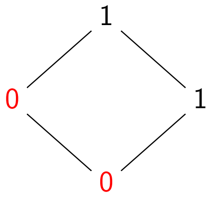
$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \cdot> x} f(z) + \max_{w <\cdot x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

where $z \cdot> x$ means z covers x and $w <\cdot x$ means x covers w.

Example:

First apply $\mathbf{t}_{(1,1)}$, which changes nothing:



How PL rowmotion generalizes classical rowmotion

For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending f to the unique f' satisfying

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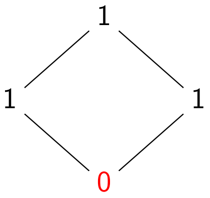
where $z \cdot > x$ means z covers x and $w < \cdot x$ means x covers w.

Example:

Then apply $\mathbf{t}_{(1,0)}$, which removes $(1,0)$ from the order ideal:



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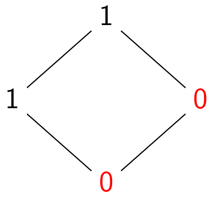
where $z \cdot > x$ means z covers x and $w < \cdot x$ means x covers w.

Example:

Then apply $\mathbf{t}_{(0,1)}$, which adds $(0, 1)$ to the order ideal:



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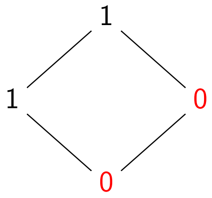
where $z \cdot > x$ means z covers x and $w < \cdot x$ means x covers w.

Example:

Finally apply $\mathbf{t}_{(0,0)}$, which changes nothing:



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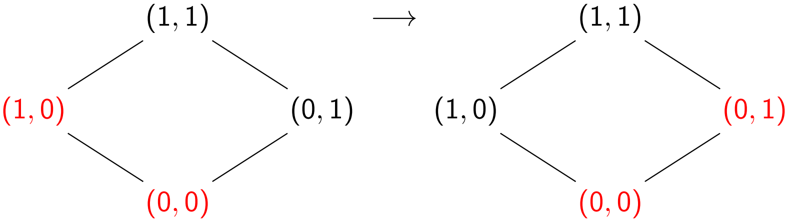
where $z \cdot > x$ means z covers x and $w < \cdot x$ means x covers w.

Example:

So this is $S \longrightarrow \mathbf{r}(S)$:



De-tropicalizing to birational maps

In the so-called *tropical semiring*, one replaces the standard binary ring operations $(+, \cdot)$ with the tropical operations $(\max, +)$. In the piecewise-linear (PL) category of the order polytope studied above, our flipping-map at x replaced the value of a function $f : P \to [0, 1]$ at a point $x \in P$ with f', where

$$f'(x) := \min_{z \cdot > x} f(z) + \max_{w < \cdot\, x} f(w) - f(x)$$

We can "detropicalize" this flip map and apply it to an assignment $f : P \to \mathbb{R}(x)$ of *rational functions* to the nodes of the poset, using that $\min(z_i) = -\max(-z_i)$, to get

$$f'(x) = \frac{\sum_{w < \cdot\, x} f(w)}{f(x) \sum_{z \cdot > x} \frac{1}{f(z)}}$$

Generalizing to the birational setting

- The rowmotion map **r** is a map of 0-1 labelings of P. It has a natural generalization to labelings of P by real numbers in $[0, 1]$, i.e., the *order polytope* of P. Toggles get replaced by piecewise-linear toggling operations that involve max, min, and $+$.
- *Detropicalizing* these toggles leads to the definition below of birational toggling. Results at the birational level imply those at the order polytope and combinatorial level.
- This is originally due to Einstein and Propp [EiPr13, EiPr14]. Another exposition of these ideas can be found in [Rob16], from the IMA volume *Recent Trends in Combinatorics*.

Birational rowmotion

- Let P be a finite poset. We define $\widehat{P}$ to be the poset obtained by adjoining two new elements 0 and 1 to P and forcing
  - 0 to be less than every other element, and
  - 1 to be greater than every other element.
- Let $\mathbb{K}$ be a field.
- A **$\mathbb{K}$-labelling of** P will mean a function $\widehat{P} \to \mathbb{K}$.
- The values of such a function will be called the **labels** of the labelling.
- We will represent labellings by drawing the labels on the vertices of the Hasse diagram of $\widehat{P}$.

Birational rowmotion

- For any $v \in P$, define the **birational v-toggle** as the rational map $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ defined by

$$(T_v f)(w) = \begin{cases} f(w), & \text{if } w \neq v; \\ \dfrac{1}{f(v)} \cdot \dfrac{\sum\limits_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u)}{\sum\limits_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \dfrac{1}{f(u)}}, & \text{if } w = v \end{cases}$$

for all $w \in \widehat{P}$.

- That is,
  - **invert** the label at v,
  - **multiply** by the **sum** of the labels at vertices **covered by** v,
  - **multiply** by the **parallel sum** of the labels at vertices **covering** v.

- For any $v \in P$, define the **birational v-toggle** as the rational map $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ defined by

$$(T_v f)(w) = \begin{cases} f(w), & \text{if } w \neq v; \\ \dfrac{1}{f(v)} \cdot \dfrac{\sum\limits_{\substack{u \in \widehat{P};\\ u \lessdot v}} f(u)}{\sum\limits_{\substack{u \in \widehat{P};\\ u \gtrdot v}} \dfrac{1}{f(u)}}, & \text{if } w = v \end{cases}$$

  for all $w \in \widehat{P}$.

- Notice that this is a **local change** to the label at v; all other labels stay the same.
- We have $T_v^2 = \mathrm{id}$ (on the range of T_v), and T_v is a birational map.

Birational rowmotion: definition

- We define **birational rowmotion** as the rational map

$$\rho_B := T_{v_1} \circ T_{v_2} \circ ... \circ T_{v_n} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}},$$

  where $(v_1, v_2, ..., v_n)$ is a linear extension of P.
- This is indeed independent of the linear extension, because:

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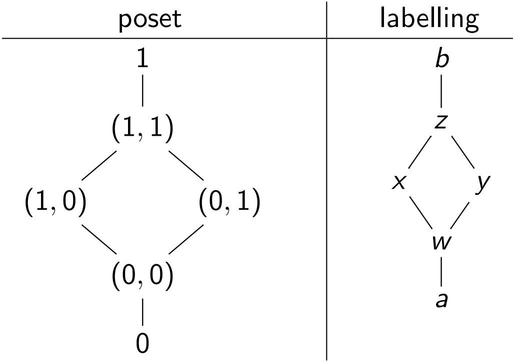
  where $(v_1, v_2, \ldots, v_n)$ is a linear extension of P.
- This is indeed independent of the linear extension, because:
  - T_v and T_w commute whenever v and w are incomparable (even whenever they are not adjacent in the Hasse diagram of P);
  - we can get from any linear extension to any other by switching incomparable adjacent elements.

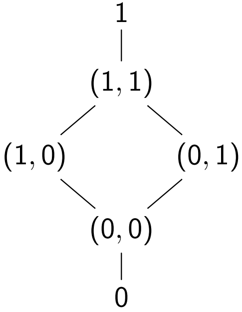
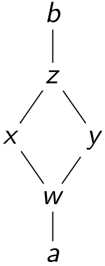
Example:
Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:



Birational rowmotion: example

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

poset	labelling
	

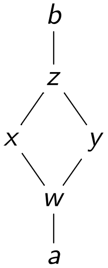
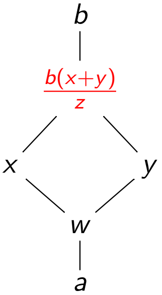
We have $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$ using the linear extension $((1,1),(1,0),(0,1),(0,0))$.

That is, toggle in the order "top, left, right, bottom".

Birational rowmotion: example

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(1,1)} f$
	

We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$.

Birational rowmotion: example

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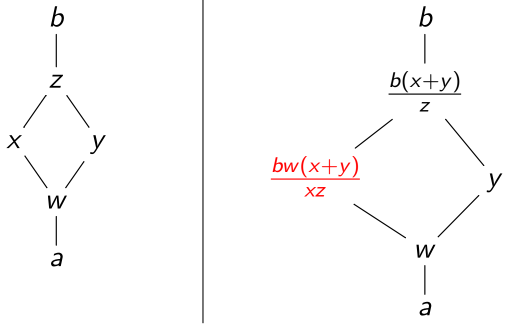

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Birational rowmotion: example

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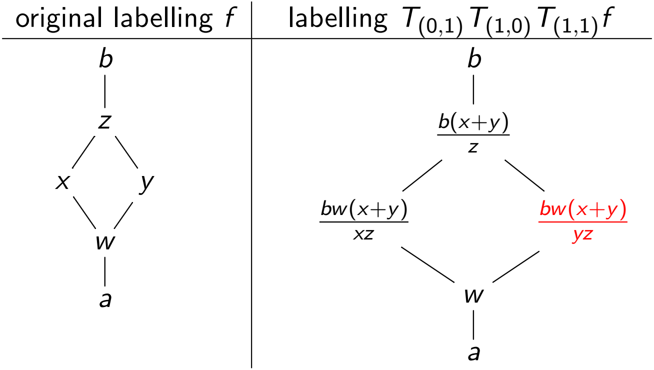
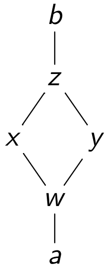
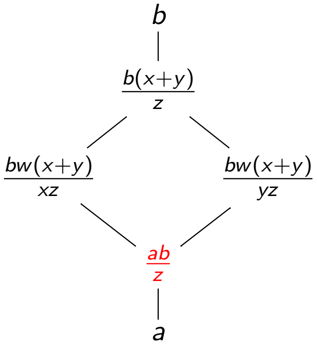


We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$.

Birational rowmotion: example

Example:
Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(0,0)} T_{(0,1)} T_{(1,0)} T_{(1,1)} f = \rho_B f$
	

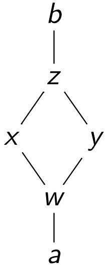
We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$.

Birational rowmotion orbit on a product of chains

Example:

Iteratively apply ρ_B to a labelling of the 2×2-rectangle.

$\rho_B^0 f =$



Birational rowmotion orbit on a product of chains

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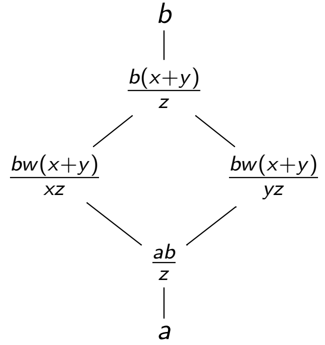
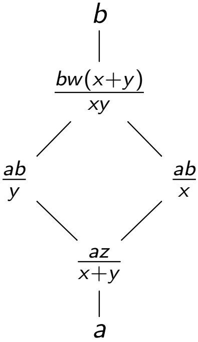


Birational rowmotion orbit on a product of chains

Example:
Iteratively apply ρ_B to a labelling of the 2×2-rectangle.
$\rho_B^2 f =$

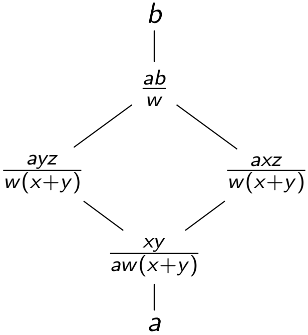


Birational rowmotion orbit on a product of chains

Example:

Iteratively apply ρ_B to a labelling of the 2×2-rectangle.

$\rho_B^3 f =$



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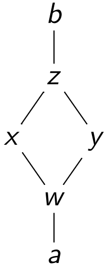
$\rho_B^4 f =$



Birational rowmotion orbit on a product of chains

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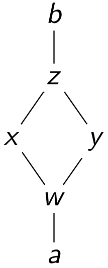

So we are back where we started.

$$\mathrm{ord}(\rho_B) = 4.$$

Generalizes $\rho_B^{r+s+2} f = f$ for $[0, r] \times [0, s]$, from [Grinberg-Roby 2015].

Birational Rowmotion on the Rectangular Poset

We now give a rational function formula for the values of iterated birational rowmotion $\rho_B^{k+1}(i,j)$ for $(i,j) \in [0,r] \times [0,s]$ and $k \in [0, r+s+1]$.

Birational Rowmotion on the Rectangular Poset

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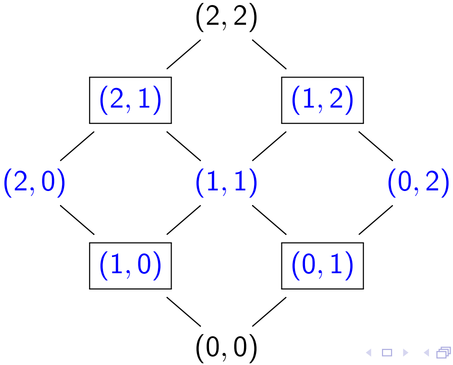
1) Let $\bigvee_{(m,n)} := \{(u,v) : (u,v) \geq (m,n)\}$ be the *principal order filter at* (m,n), $\hexagon^k_{(m,n)}$be the *rank-selected subposet*, of elements in $\bigvee_{(m,n)}$ whose rank (within $\bigvee_{(m,n)}$) is at least $k-1$ and whose corank is at most $k-1$.



2) Let $s_1, s_2, \ldots, s_k$ be the k minimal elements and let $t_1, t_2, \ldots, t_k$ be the k maximal elements of $\hexagon^k_{(m,n)}$.

Birational Rowmotion on the Rectangular Poset

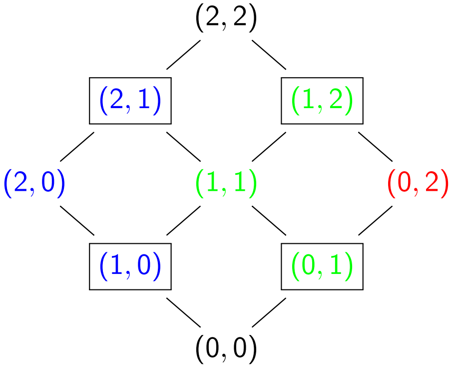
2) Let $s_1, s_2, \ldots, s_k$ be the k minimal elements and let $t_1, t_2, \ldots, t_k$ be the k maximal elements of $\hexagon^k_{(m,n)}$.

Let $A_{ij} := \frac{\sum_{z \lessdot (i,j)} x_z}{x_{(i,j)}} = \frac{x_{i,j-1}+x_{i-1,j}}{x_{ij}}$. We set $x_{i,j} = 0$ for $(i,j) \notin P$ and $A_{00} = \frac{1}{x_{00}}$ (working in $\widehat{P}$).

Given a triple $(k, m, n) \in \mathbb{N}^3$, we define a polynomial $\varphi_{\mathbf{k}}(\mathbf{m}, \mathbf{n})$ in terms of the A_{ij}'s as follows.

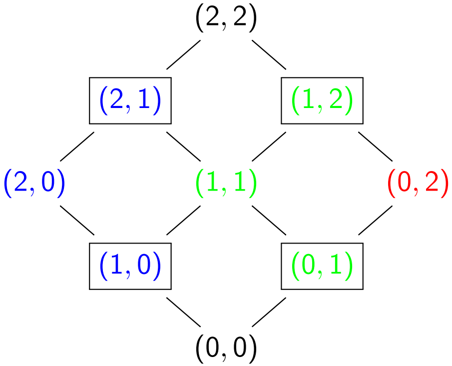
Birational Rowmotion on the Rectangular Poset

We define a **lattice path of length k** within $P = [0, r] \times [0, s]$ to be a sequence $v_1, v_2, \ldots, v_k$ of elements of P such that each difference of successive elements $v_i - v_{i-1}$ is either $(1, 0)$ or $(0, 1)$ for each $i \in [k]$. We call a collection of lattice paths **non-intersecting** if no two of them share a common vertex.



Birational Rowmotion on the Rectangular Poset

3) Let $S_k(m,n)$ be the set of non-intersecting lattice paths in $\hexagon^k_{(m,n)}$, from $\{s_1, s_2, \ldots, s_k\}$ to $\{t_1, t_2, \ldots, t_k\}$. Let $\mathcal{L} = (L_1, L_2, \ldots L_k) \in S^k_k(m,n)$ denote a k-tuple of such lattice paths.



Birational Rowmotion on the Rectangular Poset

4) Define

$$\varphi_k(m,n) := \sum_{\mathcal{L} \in S_k^k(m,n)} \prod_{\substack{(i,j) \in \hexagon^k_{(m,n)} \\ (i,j) \notin L_1 \cup L_2 \cup \cdots \cup L_k}} A_{ij}.$$

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5) Finally, set $[\alpha]_+ := \max\{\alpha, 0\}$ and let $\mu^{(a,b)}$ be the operator that takes a rational function in $\{A_{(u,v)}\}$ and simply shifts each index in each factor of each term: $A_{(u,v)} \mapsto A_{(u-a,v-b)}$

Main Theorem (M-Roby 2017+)

Fix $k \in [0, r+s+1]$, and let $\rho_B^{k+1}(i,j)$ denote the rational function associated to the poset element (i,j) after $(k+1)$ applications of the birational rowmotion map to the generic initial labeling of $P = [0,r] \times [0,s]$. Set $M = [k-i]_+ + [k-j]_+$.

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(a1) When $M = 0$, i.e. $(i-k, j-k)$ is still in the poset $[0,r] \times [0,s]$:

$$\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k, j-k)}{\varphi_{k+1}(i-k, j-k)}$$

where $\varphi_t(v,w)$ is as defined in 4) above.

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where $\varphi_t(v,w)$ is as defined in 4) above.

(a2) When $0 < M \leq k$:

$$\rho_B^{k+1}(i,j) = \mu^{([k-j]_+, [k-i]_+)} \left(\frac{\varphi_{k-M}(i-k+M, j-k+M)}{\varphi_{k-M+1}(i-k+M, j-k+M)} \right)$$

where $\mu^{(a,b)}$ is as defined in 5) above.

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Fix $k \in [0, r+s+1]$, and let $\rho_B^{k+1}(i,j)$ denote the rational function associated to the poset element (i,j) after $(k+1)$ applications of the birational rowmotion map to the generic initial labeling of $P = [0,r] \times [0,s]$. Set $M = [k-i]_+ + [k-j]_+$. We obtain the following formula for $\rho_B^{k+1}(i,j)$:

(a) When $0 \leq M \leq k$:

$$\rho_B^{k+1}(i,j) = \mu^{([k-j]_+,[k-i]_+)}\left(\frac{\varphi_{k-M}(i-k+M, j-k+M)}{\varphi_{k-M+1}(i-k+M, j-k+M)}\right)$$

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(a) When $0 \leq M \leq k$:

$$\rho_B^{k+1}(i,j) = \mu^{([k-j]_+,[k-i]_+)} \left(\frac{\varphi_{k-M}(i-k+M, j-k+M)}{\varphi_{k-M+1}(i-k+M, j-k+M)} \right)$$

where $\varphi_t(v,w)$ and $\mu^{(a,b)}$ are as defined in 4) and 5) above.

(b) When $M \geq k$: $\rho_B^{k+1}(i,j) = 1/\rho_B^{k-i-j}(r-i, s-j)$, which is well-defined by part (a).

Remark: Note that our formulae in (a) and (b) agree when $M = k$. Also, we have $\rho_B^{r+s+2+d} = \rho_B^d$ by periodicity on $[0,r] \times [0,s]$ so this gives a formula for **all** iterations of the birational rowmotion map.

Example 1: If $k = 0$, we recover the images after a single rowmotion are $\rho_B^1 f(i,j) = \frac{\varphi_0(i,j)}{\varphi_1(i,j)}$ where

$$\varphi_0(i,j) = \prod_{\substack{i \le p \le r \\ (p,q): j \le q \le s}} A_{pq} \text{ and } \varphi_1(i,j) = \sum_{\text{Lattice Path L}:(i,j)\mapsto(r,s)} \prod_{(p,q)\notin L} A_{pq}.$$

Examples

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Example 2: If $k = 1$, we recover the images after two rowmotion a

$$\rho_B^2 f(i,j) = \frac{\varphi_1(i-1,j-1)}{\varphi_2(i-1,j-1)},\ \varphi_1(i-1,j-1) = \sum_{L:(i-1,j-1) \mapsto (r,s)} \prod_{(p,q) \notin L} A_{pq};$$

$$\varphi_2(i-1,j-1) = \sum_{L_1 \& L_2 : \{(i,j-1),(i-1,j)\} \mapsto \{(r-1,s),(r,s-1)\}} \prod_{(p,q) \notin L_1 \cup L_2} A_{pq}$$

where $\{L_1, L_2\}$ is a family of non-intersecting lattice paths.

Example in Further Depth

In the "generic" case where shifting $(i,j) \mapsto (i-k, j-k)$ (straight down by $2k$ ranks) still gives a point in P, we get the following simpler formula

Corollary: For $k \leq \min\{i,j\}$, $\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k,j-k)}{\varphi_{k+1}(i-k,j-k)}$

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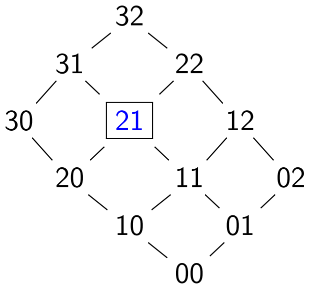
Example 3: We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for $k = 0,1,2,3,4,5,6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.



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When $\mathbf{k = 0}$, $M = 0$ and we get

$$\rho_B^1(2,1) = \frac{\varphi_0(2,1)}{\varphi_1(2,1)} = \frac{A_{21}A_{22}A_{31}A_{32}}{A_{22}+A_{31}}.$$

Example in Further Depth

In the "generic" case where shifting $(i,j) \mapsto (i-k, j-k)$ (straight down by $2k$ ranks) still gives a point in P, we get the following simpler formula

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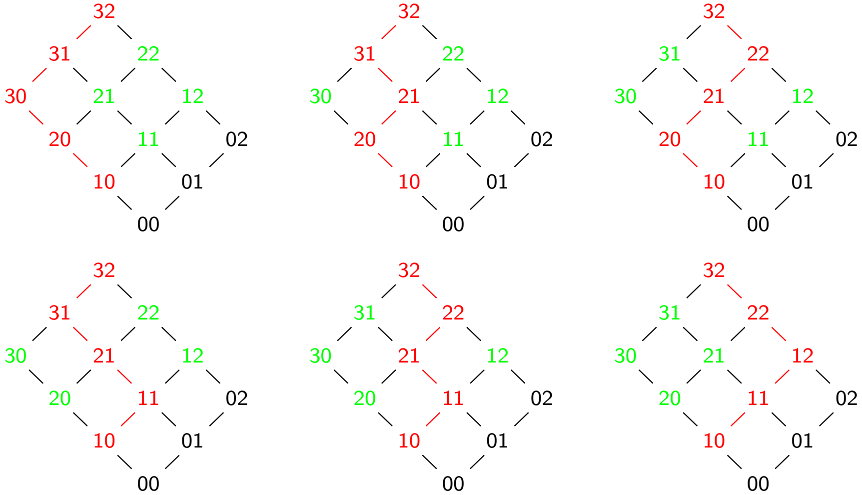
When $\mathbf{k} = \mathbf{1}$, we still have $M = 0$, and $\rho_B^2(2,1) = \frac{\varphi_1(1,0)}{\varphi_2(1,0)} =$

$$\frac{A_{11}A_{12}A_{21}A_{22} + A_{11}A_{12}A_{22}A_{30} + A_{11}A_{12}A_{30}A_{31} + A_{12}A_{20}A_{22}A_{30} + A_{12}A_{20}A_{30}A_{31} + A_{20}A_{21}A_{30}A_{31}}{A_{12} + A_{21} + A_{30}}.$$

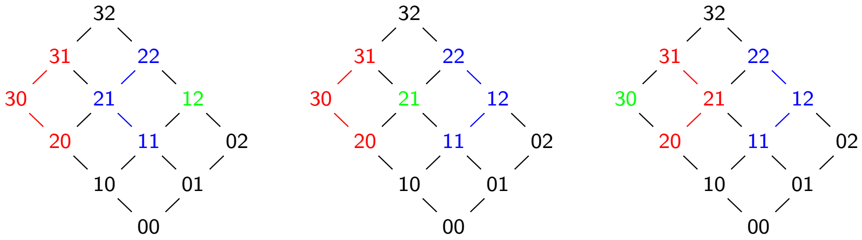
For the numerator, $s_1 = (1,0)$, $t_1 = (3,2)$, and there are six lattice paths from s_1 to t_1, each of which covers 5 elements and leaves 4 uncovered.

For the denominator, $s_1 = (2,0)$, $s_2 = (1,1)$, $t_1 = (3,1)$, and $t_2 = (2,2)$, and each pair of lattice paths leaves exactly one element uncovered.

Example in Further Depth



Example in Further Depth



Example in Further Depth

In the "generic" case where shifting $(i,j) \mapsto (i-k, j-k)$ (straight down by $2k$ ranks) still gives a point in P, we get the following simpler formula

Corollary: For $k \leq \min\{i,j\}$, $\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k,j-k)}{\varphi_{k+1}(i-k,j-k)}$

Example 3: We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for $k = 0,1,2,3,4,5,6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

When k = 2, we get $M = [2-2]_+ + [2-1]_+ = 1 \leq 2 = k$. So by part (a) of the main theorem we have

$$\rho_B^3(2,1) = \mu^{(1,0)}\left[\frac{\varphi_1(1,0)}{\varphi_2(1,0)}\right] = \text{(just shifting indices in the } k=1 \text{ formula)}$$

$$\frac{A_{01}A_{02}A_{11}A_{12} + A_{01}A_{02}A_{12}A_{20} + A_{01}A_{02}A_{20}A_{21} + A_{02}A_{10}A_{12}A_{20} + A_{02}A_{10}A_{20}A_{21} + A_{10}A_{11}A_{20}A_{21}}{A_{02} + A_{11} + A_{20}}.$$

Example in Further Depth

In the "generic" case where shifting $(i,j) \mapsto (i-k, j-k)$ (straight down by $2k$ ranks) still gives a point in P, we get the following simpler formula

Corollary: For $k \leq \min\{i,j\}$, $\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k,j-k)}{\varphi_{k+1}(i-k,j-k)}$

Example 3: We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for $k = 0,1,2,3,4,5,6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

When $\mathbf{k} = \mathbf{3}$, we get $M = [3-2]_+ + [3-1]_+ = 3 = k$. Therefore,

$$\rho_B^4(2,1) = \mu^{(2,1)}\left[\frac{\varphi_0(2,1)}{\varphi_1(2,1)}\right] = \mu^{(2,1)}\left[\frac{A_{21}A_{22}A_{31}A_{32}}{A_{22}+A_{31}}\right] = \frac{A_{00}A_{01}A_{10}A_{11}}{A_{01}+A_{10}}.$$

In this situation, we can also use part (b) of the main theorem to get

$$\rho_B^4(2,1) = 1/\rho_B^{3-2-1}(3-2, 2-1) = 1/\rho_B^0(1,1) = \frac{1}{x_{11}}.$$

The equality between these two expressions is easily checked.

Example in Further Depth

In the "generic" case where shifting $(i,j) \mapsto (i-k, j-k)$ (straight down by $2k$ ranks) still gives a point in P, we get the following simpler formula

Corollary: For $k \leq \min\{i,j\}$, $\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k,j-k)}{\varphi_{k+1}(i-k,j-k)}$

Example 3: We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for $k = 0,1,2,3,4,5,6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

When $\mathbf{k = 4}$, we get $M = [4-2]_+ + [4-1]_+ = 5 > k$. Therefore, by part (b) of the main theorem, then part (a),

$$\rho_B^5(2,1) = 1/\rho_B^{4-2-1}(3-2, 2-1) = 1/\rho_B^1(1,1) = \frac{\varphi_1(1,1)}{\varphi_0(1,1)} = \frac{A_{12}A_{22} + A_{12}A_{31} + A_{21}A_{31}}{A_{11}A_{12}A_{21}A_{22}A_{31}A_{32}}.$$

Each term in the numerator is associated with one of the three lattice paths from $(1,1)$ to $(3,2)$ in P, while the denominator is just the product of all A-variables in the principal order filter $\bigvee(1,1)$.

Example in Further Depth

In the "generic" case where shifting $(i,j) \mapsto (i-k, j-k)$ (straight down by $2k$ ranks) still gives a point in P, we get the following simpler formula

Corollary: For $k \leq \min\{i,j\}$, $\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k,j-k)}{\varphi_{k+1}(i-k,j-k)}$

Example 3: We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for $k = 0,1,2,3,4,5,6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

When $\mathbf{k = 5}$, we get $M = [5-2]_+ + [5-1]_+ = 7 > k$. Therefore, by part (b) of the main theorem, then part (a),

$$\rho_B^6(2,1) = 1/\rho_B^{5-2-1}(3-2,2-1) = 1/\rho_B^2(1,1) = \frac{\varphi_2(1,1)}{\varphi_1(1,1)} = \frac{1}{A_{12}A_{22} + A_{12}A_{31} + A_{21}A_{31}}.$$

The numerator here represents the empty product, since the unique (unordered) pair of lattice paths from $s_1 = (2,1)$ and $s_2 = (1,2)$ to $t_1 = (3,1)$ and $t_2 = (2,2)$ covers **all** elements of $\hexagon^2_{(1,1)}$. The denominator here is the same as the numerator of the previous case.

Example in Further Depth

In the "generic" case where shifting $(i,j) \mapsto (i-k, j-k)$ (straight down by $2k$ ranks) still gives a point in P, we get the following simpler formula

Corollary: For $k \leq \min\{i,j\}$, $\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k,j-k)}{\varphi_{k+1}(i-k,j-k)}$

Example 3: We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for $k = 0,1,2,3,4,5,6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

When $\mathbf{k = 6}$, we get $M = [6-2]_+ + [6-1]_+ = 9 > k$. Therefore, by part (b) of the main theorem, then part (a),

$$\rho_B^7(2,1) = 1/\rho_B^{6-2-1}(3-2, 2-1) = 1/\rho_B^3(1,1) = \mu^{(1,1)}\left[\frac{\varphi_1(1,1)}{\varphi_0(1,1)}\right]$$

$$= \mu^{(1,1)}\left[\frac{A_{12}A_{22} + A_{12}A_{31} + A_{21}A_{31}}{A_{11}A_{11}A_{21}A_{22}A_{31}A_{32}}\right] = \frac{A_{01}A_{11} + A_{01}A_{20} + A_{10}A_{20}}{A_{00}A_{01}A_{10}A_{11}A_{20}A_{21}} = x_{21}$$

Example in Further Depth

When k $= \mathbf{6}$, we get $M = [6-2]_+ + [6-1]_+ = 9 > k$. Therefore, by part (b) of the main theorem, then part (a),

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The lattice paths involved here are the same as for the $k = 4$ computation.

We can deduce this by $A_{00} = 1/x_{00}, A_{10} = x_{00}/x_{10}, A_{01} = x_{00}/x_{01}$, $A_{11} = (x_{10} + x_{01})/x_{11}, A_{20} = x_{10}/x_{20}$, and $A_{21} = (x_{20} + x_{11})/x_{21}$.

Periodicity also kicks in: $\rho_B^7(2,1) = \rho_B^0(2,1) = x_{21}$ using $(r + s + 2) = 7$.

Sketch of Proof

By the definition of birational rowmotion,

$$\rho_B^{k+1}(i,j) = \frac{\left(\rho_B^k(i,j-1) + \rho_B^k(i-1,j)\right) \cdot \left(\rho_B^{k+1}(i+1,j) \parallel \rho_B^{k+1}(i,j+1)\right)}{\rho_B^k(i,j)}$$

where

$$A \parallel B = \frac{1}{\frac{1}{A} + \frac{1}{B}}.$$

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where

$$A \mid\mid B = \frac{1}{\frac{1}{A} + \frac{1}{B}}.$$

By induction on k, and the fact that we apply birational rowmotion from top to bottom, we can rewrite this formula as

$$\frac{\left(\rho_B^k(i,j-1) + \rho_B^k(i-1,j)\right) \cdot \left(\frac{\varphi_k(i-k+1,j-k)}{\varphi_{k+1}(i-k+1,j-k)} \mid\mid \frac{\varphi_k(i-k,j-k+1)}{\varphi_{k+1}(i-k,j-k+1)}\right)}{\rho_B^k(i,j)}$$

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Lemma Given the definition of $A \parallel B$ given above,

$$\frac{A}{B} \parallel \frac{C}{D} = \frac{AC}{CB + AD}.$$

Sketch of Proof

Lemma Given the definition of $A \mid\mid B$ given above,

$$\frac{A}{B} \mid\mid \frac{C}{D} = \frac{AC}{CB + AD}.$$

Using the Lemma, we can further rewrite the above as

$$\frac{\left(\frac{\varphi_{k-1}(i-k+1,j-k)}{\varphi_k(i-k+1,j-k)} + \frac{\varphi_{k-1}(i-k,j-k+1)}{\varphi_k(i-k,j-k+1)}\right) \cdot \left(\frac{\varphi_k(i-k+1,j-k)\ \varphi_k(i-k,j-k+1)}{\varphi_k(i-k,j-k+1)\varphi_{k+1}(i-k+1,j-k) + \varphi_k(i-k+1,j-k)\varphi_{k+1}(i-k,j-k+1)}\right)}{\frac{\varphi_{k-1}(i-k+1,j-k+1)}{\varphi_k(i-k+1,j-k+1)}}.$$

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Which equals, after cross-multiplication:

$$\frac{\left(\frac{\varphi_k(i-k,j-k+1)\varphi_{k-1}(i-k+1,j-k)+\varphi_k(i-k+1,j-k)\varphi_{k-1}(i-k,j-k+1)}{\varphi_k(i-k,j-k+1)\varphi_{k+1}(i-k+1,j-k)\ +\ \varphi_k(i-k+1,j-k)\varphi_{k+1}(i-k,j-k+1)}\right)}{\frac{\varphi_{k-1}(i-k+1,j-k+1)}{\varphi_k(i-k+1,j-k+1)}}.$$

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Letting $\alpha_k(i,j) =$
$\varphi_k(i-k, j-k+1)\frac{\varphi_{k-1}(i-k+1,j-k)}{\varphi_{k-1}(i-k+1,j-k+1)} + \varphi_k(i-k+1, j-k)\frac{\varphi_{k-1}(i-k,j-k+1)}{\varphi_{k-1}(i-k+1,j-k+1)}$,
we can rewrite the above expression as

$$\frac{\alpha_k(i,j)}{\alpha_{k+1}(i,j)}.$$

Claim It is sufficient to prove $\alpha_k(i,j) = \varphi_k(i-k, j-k)$ for all $k \geq 0$ to prove our main theorem.

Sketch of Proof

Letting $\alpha_k(i,j) =$
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we can rewrite the above expression as

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Symbolically, we can rewrite the expression

$$\frac{\left(\frac{A}{B} + \frac{C}{D}\right) \cdot \left(\frac{B}{G} \parallel \frac{D}{H}\right)}{\frac{E}{F}} = \frac{\left(\frac{A}{B} + \frac{C}{D}\right) \cdot \left(\frac{BD}{DG + BH}\right)}{\frac{E}{F}}$$

as

$$\frac{ADF + BCF}{DEG + BEH} = \frac{A\frac{D}{E} + B\frac{C}{E}}{D\frac{G}{F} + B\frac{H}{F}}.$$

Sketch of Proof

We wish to prove $\alpha_k(i,j) = \varphi_k(i-k, j-k)$, hence it is sufficient to verify the following Plücker-like identity:

$$\varphi_k(i-k, j-k)\varphi_{k-1}(i-k+1, j-k+1) =$$
$$\varphi_k(i-k, j-k+1)\varphi_{k-1}(i-k+1, j-k) + \varphi_k(i-k+1, j-k)\varphi_{k-1}(i-k, j-k+1).$$

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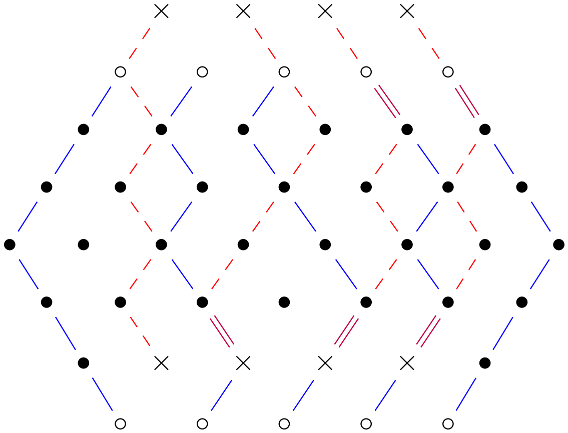
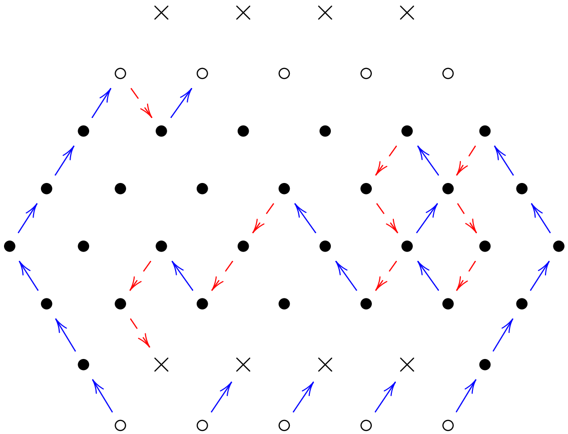
Example (k=4):



We build **bounce paths** and **twigs** (paths of length one from $\circ$ to $\times$) starting from the bottom row of $\circ$'s.

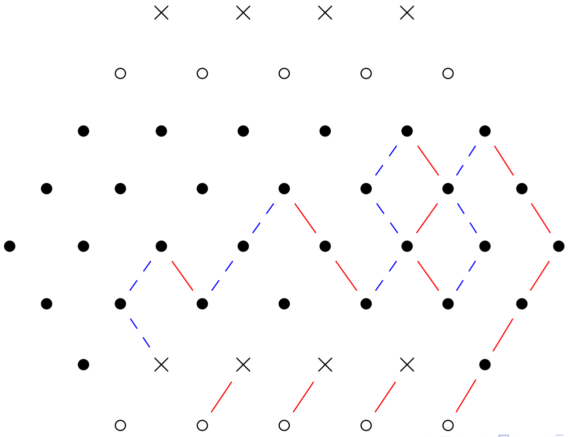
Example (k=4):



Sketch of Proof

We then reverse the colors along the $(k-2)$ **twigs** and the **one bounce path from** $\circ$ **to** $\times$ (rather than $\circ$ to $\circ$).

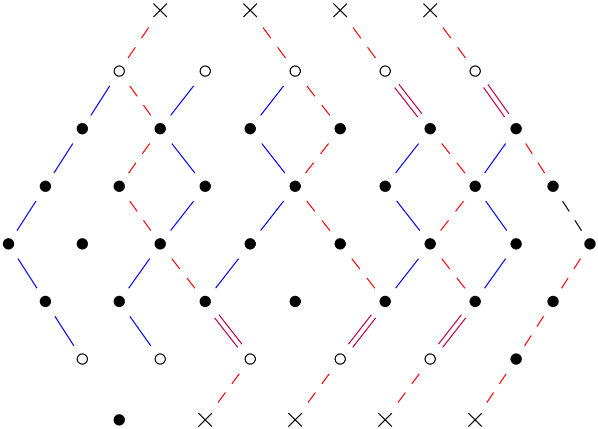
Example (k=4):



Sketch of Proof

Swap in the new colors and shift the $\circ$'s and $\times$'s in the bottom two rows.

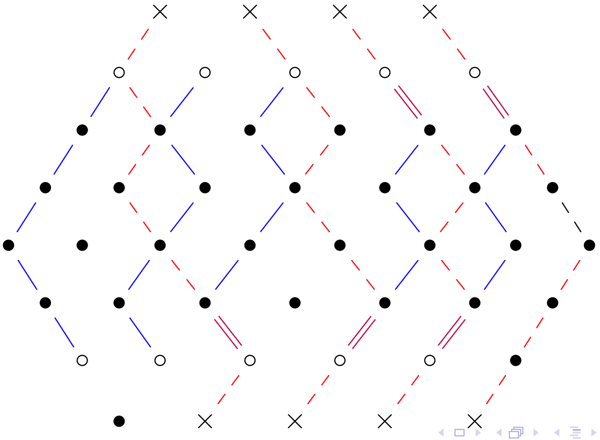
Example (k=4):



Sketch of Proof

$$\varphi_k(i-k,j-k)\varphi_{k-1}(i-k+1,j-k+1) = \varphi_k(i-k,j-k+1)\varphi_{k-1}(i-k+1,j-k) + \varphi_k(i-k+1,j-k)\varphi_{k-1}(i-k,j-k+1).$$

Example (k=4):



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Thanks for Listening

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