Applications of New F-polynomial Formulas in terms of C-Vectors

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Cluster Algebras and Related Topics, AMS Spring Eastern Sectional 2019

April 13, 2019

http://math.umn.edu/∼musiker/Fpoly19.pdf

Thanks to NSF Grant DMS-1745638. Part of this work done during 2018 REU in Combinatorics at University of Minnesota, Twin Cities.

arXiv:1812.01910 and forthcoming work

Quivers and Exchange Matrices with Principal Coefficients

Given a quiver Q (i.e. a directed graph) with n vertices, we build an n-by-n skew-symmetric matrix $B_Q = [b_{ij}]_{i=1,\ j=1}^{n}$ whose entries are

$$b_{ij} = (\#\text{arrows from } i \text{ to } j) - (\#\text{arrows from } j \text{ to } i).$$

Note: More generally, we can let B_Q be skew-symmetrizable, meaning there exists a diagonal matrix D with positive integer entries such that DB_Q is skew-symmetric, i.e. satisfies $(DB_Q)^T = -DB_Q$. However, for this talk we will focus on the quiver, i.e. the skew-symmetric, case.

We build the corresponding $2n$-by-n exchange matrix with principal coefficients via $\widetilde{B_Q} = \begin{bmatrix} B_Q \\ I_n \end{bmatrix}$, where I_n denotes the n-by-n identity matrix.

Equivalently, $\widetilde{B_Q}$ corresponds to the exchange matrix of the framed quiver $\widetilde{Q} = Q \cup \{1', 2', \ldots, n'\}$ with a single arrow from $i' \to i$ for each $1 \leq i \leq n$.

Quivers and Exchange Matrices with Principal Coefficients

If $Q = 1 \to 2$, then $B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\widetilde{Q} =$ $\begin{tikzcd} 1' \arrow[d] & 2' \arrow[d] \\ 1 \arrow[r] & 2 \end{tikzcd}$ and $\widetilde{B_Q} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

If $Q = 1 \Rightarrow 2$, then $B_Q = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$, $\widetilde{Q} =$ $\begin{tikzcd} 1' \arrow[d] & 2' \arrow[d] \\ 1 \arrow[r, Rightarrow] & 2 \end{tikzcd}$ and $\widetilde{B_Q} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

If $Q =$ $\begin{tikzcd} 1 \arrow[r, Rightarrow] & 2 \arrow[rr, bend right] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}$, then $\widetilde{Q} =$ $\begin{tikzcd} 1' \arrow[d] & 2' \arrow[d] & 3' \arrow[d] & 4' \arrow[d] \\ 1 \arrow[r, Rightarrow] & 2 \arrow[rr, bend right] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}$

$$B_Q = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix},$$

and

$$\widetilde{B_Q} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Quiver Mutation

Given a quiver Q and its vertex j, we can define $Q' = \mu_j Q$, the **mutation of Q at j**, by a 3 step process:

1) For any 2-path $i \to j \to k$, add a new arrow $\begin{tikzcd} i \arrow[rr, bend right] & j & k \end{tikzcd}$.

2) Reverse the direction of all arrows incident to j.

3) Delete any 2-cycle $\begin{tikzcd} i \arrow[rr, bend right] & j & k \arrow[ll, bend right] \end{tikzcd}$ created from the above two steps.

Examples: If $Q = \begin{tikzcd} 1 \arrow[r, shift left] \arrow[r, shift right] & 2 \arrow[rr, bend right] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}$, then

$$\mu_1 Q = \begin{tikzcd} 1 & 2 \arrow[l, shift left] \arrow[l, shift right] \arrow[rr, bend right] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad \mu_2 Q = \begin{tikzcd} 1 \arrow[rrr, bend left] \arrow[rrr, bend left=40] & 2 \arrow[l, shift left] \arrow[l, shift right] \arrow[r] & 3 & 4 \arrow[ll, bend left] \end{tikzcd}$$

$$\mu_3 Q = \begin{tikzcd} 1 \arrow[r, shift left] \arrow[r, shift right] & 2 \arrow[r] & 3 \arrow[r] & 4 \end{tikzcd}, \qquad \mu_4 Q = \begin{tikzcd} 1 \arrow[r, shift left] \arrow[r, shift right] & 2 & 3 \arrow[r] & 4 \arrow[ll, bend left] \end{tikzcd}$$

Note: Mutation is an **involution**, meaning that $\mu_j^2 Q = Q$ for any vertex j.

Exchange Matrix Mutation

Quiver mutation induces an analogous dynamic on exchange matrices B_Q. We define $[b'_{ij}] = B'_Q = \mu_k B_Q$, the **mutation of** $B_Q = [b_{ij}]$ **at k**, by

$$b'_{ij} = \begin{cases} -b_{ij} \text{ if } i = k \text{ or } j = k \\ b_{ij} + [b_{ik}]_+[b_{kj}]_+ - [-b_{ik}]_+[-b_{kj}]_+ \text{ otherwise} \end{cases}$$

using $[\alpha]_+ = \max(\alpha, 0)$.

Examples: If $Q = \begin{tikzcd} 1 \arrow[r, Rightarrow] & 2 \arrow[rr, bend right] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}$, $B_Q = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$, then

$$\mu_1 Q = \begin{tikzcd} 1 & 2 \arrow[l, Rightarrow] \arrow[rr, bend right] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}, \qquad \mu_1 B_Q = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix}.$$

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Examples: If $Q = \begin{tikzcd} 1 \arrow[r, Rightarrow] & 2 \arrow[rr, bend right] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}$, $B_Q = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$, then

$$\mu_2 Q = \begin{tikzcd} 1 \arrow[rrr, bend left] \arrow[rrr, bend left=50] & 2 \arrow[l, Rightarrow] \arrow[r] & 3 & 4 \arrow[ll, bend left] \end{tikzcd}, \qquad \mu_2 B_Q = \begin{bmatrix} 0 & -2 & 0 & 2 \\ 2 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ -2 & 1 & 0 & 0 \end{bmatrix}.$$

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Examples: If $Q = \begin{tikzcd} 1 \arrow[r, Rightarrow] & 2 \arrow[rr, bend right] & 3 \arrow[l] & 4 \arrow[l] \end{tikzcd}$, $B_Q = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$, then

$$\mu_4 Q = \begin{tikzcd} 1 \arrow[r, Rightarrow] & 2 & 3 \arrow[r] & 4 \arrow[ll, bend left] \end{tikzcd}, \qquad \mu_4 B_Q = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$

Examples of mutation with principal coefficients

As framed quivers (for the case of a type A_2 quiver):

$$
\begin{tikzcd}
1' \arrow[d] & 2' \arrow[d] & \to^{\mu_1} & 1' \arrow[dr] & 2' \arrow[d] & \to^{\mu_2} & 1' & 2' \arrow[dl] & \to^{\mu_1} & 1' & 2' & \to^{\mu_2} & 1' \arrow[dr] & 2' & \to^{\mu_1} & 1' \arrow[dr] & 2' \arrow[dl] \\
1 \arrow[r] & 2 & & 1 \arrow[u] & 2 \arrow[l] & & 1 \arrow[r] & 2 \arrow[ul] \arrow[u] & & 1 \arrow[ur] & 2 \arrow[ul] \arrow[l] & & 1 \arrow[ur] \arrow[r] & 2 & & 1 & 2 \arrow[l]
\end{tikzcd}
$$

As $2n$-by-n exchange matrices:

$$
\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \to^{\mu_1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \to^{\mu_2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{bmatrix} \to^{\mu_1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}
$$

$$
\to^{\mu_2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \to^{\mu_1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.
$$

Examples of mutation with principal coefficients

Starting with the framed quiver for the case of the Kronecker quiver

$$\begin{array}{ccc} 1' & & 2' \\ \downarrow & & \downarrow \\ 1 & \Rightarrow & 2 \end{array}$$

As $2n$-by-n exchange matrices:

$$\begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \to^{\mu_1} \begin{bmatrix} 0 & -2 \\ 2 & 0 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} \to^{\mu_2} \begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 3 & -2 \\ 2 & -1 \end{bmatrix} \to^{\mu_1} \begin{bmatrix} 0 & -2 \\ 2 & 0 \\ -3 & 4 \\ -2 & 3 \end{bmatrix}$$

$$\to^{\mu_2} \begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 5 & -4 \\ 4 & -3 \end{bmatrix} \to^{\mu_1} \begin{bmatrix} 0 & -2 \\ 2 & 0 \\ -5 & 6 \\ -4 & 5 \end{bmatrix} \to^{\mu_2} \begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 7 & -6 \\ 6 & -5 \end{bmatrix} \to^{\mu_1} \begin{bmatrix} 0 & 2 \\ -2 & 0 \\ -7 & 8 \\ -6 & 7 \end{bmatrix} \to \dots$$

Cluster seeds and their mutation

A **seed for a cluster algebra** is defined as a choice of a quiver (equivalently an exchange matrix) on N vertices and a choice of a **cluster** $\{x_1, x_2, \ldots, x_N\}$ where the x_i are formal variables, called **cluster variables**.

We define **cluster mutation** alongside quiver mutation yielding (a priori) rational functions in $\mathbb{Q}(x_1, x_2, \ldots, x_N)$ defined by

$$\{x_1, \ldots, x_N\} \to^{\mu_k} \{x_1, \ldots, x_N\} \cup \{x_k'\} \setminus \{x_k\} \text{ where}$$

$$x_k' = \frac{\prod_{i=1}^n x_i^{[b_{ik}]_+} + \prod_{k=1}^n x_i^{[-b_{ik}]_+}}{x_k} = \frac{\prod_{i \to k} x_i + \prod_{k \to i} x_i}{x_k}$$

using the exchange matrix B_Q, or equivalently the arrows in the quiver Q.

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using the exchange matrix B_Q, or equivalently the arrows in the quiver Q.

Theorem (Fomin-Zelevinsky 2001) The Laurent Phenomenon holds for all cluster variables, namely the rational functions resulting from iterating cluster mutation are in fact Laurent polynomials, i.e. $\frac{P(x_1,\ldots,x_N)}{x_1^{d_1}\cdots x_n^{d_n}}$ where P is a polynomial with integer coefficients and each d_i is a nonnegative integer.

F-polynomials

If we start with a framed quiver $\widetilde{Q} = Q \cup \{1', 2', \ldots, n'\}$ and the intial cluster $\{x_1, \ldots, x_N\} = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, we iterate cluster mutation with the extra restriction disallowing mutation at vertices i'.

Consequently, the binomial exchange relation for cluster mutation

$$x_k' = \frac{\prod_{i=1}^{n} x_i^{[b_{ik}]_+} + \prod_{k=1}^{n} x_i^{[-b_{ik}]_+}}{x_k} = \frac{\prod_{i \to k} x_i + \prod_{k \to i} x_i}{x_k}$$

will involve $y_1, y_2, \ldots, y_n$ in the numerator, but never in the denominator.

By letting $x_1 = x_2 = \cdots = x_n = 1$, and iterating cluster mutation, we replace cluster variables (which are Laurent polynomials in x_i's and y_i's) with polynomials in $y_1, y_2, \ldots, y_n$, which are called **F-polynomials**.

F-polynomials

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Example:

$$\begin{tikzcd}
1' \arrow[d] & 2' \arrow[d] & \rightarrow^{\mu_1} & 1' \arrow[dr] & 2' \arrow[d] & \rightarrow^{\mu_2} & 1' & 2' \arrow[dl] & \rightarrow^{\mu_1} & 1' & 2' & \rightarrow^{\mu_2} & 1' \arrow[dr] & 2' & \rightarrow^{\mu_1} & 1' \arrow[dr] & 2' \arrow[dl] \\
1 \arrow[r] & 2 & & 1 \arrow[u] & 2 \arrow[l] & & 1 \arrow[r] & 2 \arrow[ul] \arrow[u] & & 1 \arrow[ur] & 2 \arrow[ul] \arrow[l] & & 1 \arrow[ur] \arrow[r] & 2 & & 1 & 2 \arrow[l]
\end{tikzcd}$$

$$\{F_1, F_2\} = \{1, \;\; 1\} \rightarrow^{\mu_1} \{y_1 + 1, \;\; 1\} \rightarrow^{\mu_2} \{y_1 + 1, \;\; y_1 y_2 + y_1 + 1\}$$

$$\rightarrow^{\mu_1} \{y_2 + 1, \;\; y_1 y_2 + y_1 + 1\} \rightarrow^{\mu_2} \{y_2 + 1, \;\; 1\} \rightarrow^{\mu_1} \{1, \;\; 1\}$$

Given a framed quiver $\widetilde{Q}$ and its images under a sequence of mutations, we define the c-vectors associated to the seed t by

$$\mathbf{c_{j,t}} = [c_{1j}, c_{2j}, \ldots, c_{nj}]^T$$

where $c_{ij} = \#\text{arrows from } i' \to j$. Equivalently, $\mathbf{c_{j,t}}$ is the jth column of the bottom half of the $2n$-by-n exchange matrix associated to seed t.

In particular, the initial c-vectors, for seed t_0, equal unit vectors

$$\mathbf{c_{1,t_0}} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{c_{2,t_0}} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathbf{c_{n,t_0}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

and then recursively c-vectors mutate alongside quivers and exchange matrices. Letting $\mathbf{c_{j,\ \mu_k t}} = [c'_{1j}, c'_{2j}, \ldots, c'_{nj}]^T$ for each $1 \leq j \leq n$, we have

$$c'_{ij} = \begin{cases} -c_{ij} = -c_{ik} \text{ if } j = k \\ c_{ij} + [c_{ik}]_+[b_{kj}]_+ - [-c_{ik}]_+[-b_{kj}]_+ \text{ otherwise} \end{cases}.$$

c-vectors for $1 \to 2$

$$\begin{tikzcd}
1' \arrow[d] & 2' \arrow[d] & \to^{\mu_1} & 1' \arrow[dr] & 2' \arrow[d] & \to^{\mu_2} & 1' & 2' \arrow[dl] & \to^{\mu_1} & 1' & 2' & \to^{\mu_2} & 1' \arrow[dr] & 2' & \to^{\mu_1} & 1' \arrow[dr] & 2' \arrow[dl] \\
1 \arrow[r] & 2 & & 1 \arrow[u] & 2 \arrow[l] & & 1 \arrow[r] & 2 \arrow[u] \arrow[ul] & & 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] & & 1 \arrow[r] \arrow[ur] & 2 & & 1 & 2 \arrow[l]
\end{tikzcd}$$

$$t_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \to^{\mu_1} t_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \to^{\mu_2} t_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$\to^{\mu_1} t_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} \to^{\mu_2} t_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \to^{\mu_1} t_5 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{c}_{\mathbf{1},\mathbf{t_0}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c}_{\mathbf{2},\mathbf{t_0}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{c}_{\mathbf{1},\mathbf{t_1}} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c}_{\mathbf{2},\mathbf{t_1}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{c}_{\mathbf{1},\mathbf{t_2}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{c}_{\mathbf{2},\mathbf{t_2}} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\mathbf{c}_{\mathbf{1},\mathbf{t_3}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c}_{\mathbf{2},\mathbf{t_3}} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c}_{\mathbf{1},\mathbf{t_4}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c}_{\mathbf{2},\mathbf{t_4}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c}_{\mathbf{1},\mathbf{t_5}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{c}_{\mathbf{2},\mathbf{t_5}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

c-vectors for $1 \Rightarrow 2$

$$t_0 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \to^{\mu_1} t_1 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} \to^{\mu_2} t_2 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 3 & -2 \\ 2 & -1 \end{bmatrix}$$

$$\to^{\mu_1} t_3 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \\ -3 & 4 \\ -2 & 3 \end{bmatrix} \to^{\mu_2} t_4 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 5 & -4 \\ 4 & -3 \end{bmatrix} \to^{\mu_1} t_5 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \\ -5 & 6 \\ -4 & 5 \end{bmatrix} \to \dots$$

$$\mathbf{c_{1,t_1}} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_{2,t_2}} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \mathbf{c_{1,t_3}} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}, \mathbf{c_{2,t_4}} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}, \mathbf{c_{1,t_5}} = \begin{bmatrix} -5 \\ -4 \end{bmatrix}, \dots$$

c-vector Sign Coherence

For $1 \to 2$ and $\mu_1\mu_2\mu_1\mu_2\mu_1$,

$$\mathbf{c_{1,t_1}} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_{2,t_2}} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_{1,t_3}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_{2,t_4}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_{1,t_5}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For $1 \Rightarrow 2$ and $\mu_1\mu_2\mu_1\mu_2\mu_1\cdots$,

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Theorem (Derksen-Weyman-Zelevinsky 2010) Each c-vector consists exclusively of nonnegative entries or exclusively of nonpositive entries.

Sign Coherence implies we can assign a sign $\epsilon_{j,t_r} \in \{\pm 1\}$ to each $\mathbf{c_{j,t_r}}$.

Note: Conjectured by Fomin-Zelevinsky in *Cluster Algebras IV*, 2006, and proven in the skew-symmetrizable case by Gross-Hacking-Keel-Kontsevich.

Three definitions of g-vectors

1) For a framed quiver $\widetilde{Q}$ with exchange matrix $\begin{bmatrix} B_Q \\ I_n \end{bmatrix}$, define a $\mathbb{Z}^n$-grading by $\deg(x_i) = \mathbf{e}_i$ and $\deg(y_j) = -\mathbf{b_j}$, where $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ is the initial cluster, $\mathbf{e}_i$ is the ith unit vector and $\mathbf{b_j}$ is the jth column of B_Q.

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Then for any cluster variable x' written as a Laurent polynomial in $\mathbb{Q}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, y_2, \ldots, y_n]$, the $\mathbb{Z}^n$-grading of each such Laurent monomial of x' coincide. This common multidegree is defined to be the g-**vector** attached to x'. (See **Section 6 of Cluster Algebras IV**.)

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Then for any cluster variable x' written as a Laurent polynomial in $\mathbb{Q}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, y_2, \ldots, y_n]$, the $\mathbb{Z}^n$-grading of each such Laurent monomial of x' coincide. This common multidegree is defined to be the g-**vector** attached to x'. (See **Section 6 of Cluster Algebras IV**.)

2) As a consequence of sign coherence, any F-polynomial has a constant term of 1. Utilizing this, the g-vector of x' agrees with the exponent vector, in x_i's, of the unique Laurent monomial of x' containing no y_j's.

1) For a framed quiver $\widetilde{Q}$ with exchange matrix $\begin{bmatrix} B_Q \\ I_n \end{bmatrix}$, define a $\mathbb{Z}^n$-grading by $\deg(x_i) = \mathbf{e}_i$ and $\deg(y_j) = -\mathbf{b_j}$, where $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ is the initial cluster, $\mathbf{e}_i$ is the ith unit vector and $\mathbf{b_j}$ is the jth column of B_Q.

Then for any cluster variable x' written as a Laurent polynomial in $\mathbb{Q}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, y_2, \ldots, y_n]$, the $\mathbb{Z}^n$-grading of each such Laurent monomial of x' coincide. This common multidegree is defined to be the g-**vector** attached to x'. (See **Section 6 of Cluster Algebras IV**.)

2) As a consequence of sign coherence, any F-polynomial has a constant term of 1. Utilizing this, the g-vector of x' agrees with the exponent vector, in x_i's, of the unique Laurent monomial of x' containing no y_j's.

3) Let C_t (resp. G_t) denote the matrices whose columns are the c-vectors (resp. g-vectors) associated to seed t. **Theorem 4.1 of Nakanishi 2011:**

$$\text{As another consequence of sign coherence, } G_t = (C_t^T)^{-1}.$$

F-polynomials from C-Vectors and G-Vectors

Theorem (Gupta '18) as will be re-expressed in (Gupta-M '19+) :
Given a framed quiver $\widetilde{Q}$ and a mutation sequence $\overline{\mu} = \mu_{i_1}\mu_{i_2}\cdots\mu_{i_\ell}$, consider the sequence of cluster seeds $t_0 \to^{\mu_{i_1}} t_1 \to^{\mu_{i_2}} \dots t_{\ell-1} \to^{\mu_{i_\ell}} t_\ell$.

Then the F-polynomial resulting from the final mutation, i.e. $F_{i_\ell;t_\ell}$, is expressible as a product of recursively defined formulas, dependent only on c-vectors and g-vectors, followed by a monomial specilization:

F-polynomials from C-Vectors and G-Vectors

Theorem (Gupta '18) as will be re-expressed in (Gupta-M '19+) : Given a framed quiver $\widetilde{Q}$ and a mutation sequence $\overline{\mu} = \mu_{i_1}\mu_{i_2}\cdots\mu_{i_\ell}$, consider the sequence of cluster seeds $t_0 \to^{\mu_{i_1}} t_1 \to^{\mu_{i_2}} \dots t_{\ell-1} \to^{\mu_{i_\ell}} t_\ell$.

Then the F-polynomial resulting from the final mutation, i.e. $F_{i_\ell;t_\ell}$, is expressible as a product of recursively defined formulas, dependent only on c-vectors and g-vectors, followed by a monomial specilization:
Let $L_1 = 1 + z_1$ and $L_k = 1 + z_k L_1^{\mathbf{c_1}\cdot B_Q|\mathbf{c_k}|} L_2^{\mathbf{c_2}\cdot B_Q|\mathbf{c_k}|} \cdots L_{k-1}^{\mathbf{c_{k-1}}\cdot B_Q|\mathbf{c_k}|}$ for $k \geq 2$.

Then $F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c_j}\cdot\mathbf{g}_\ell}|_{z_1=y^{|\mathbf{c_1}|},\dots,z_\ell=y^{|\mathbf{c}_\ell|}}$. Also see [Nagao10] and [Keller12].

Note: Before the monomial specialization, the L_j's and F_{i_ℓ,t_ℓ}'s may be rational functions in the z_i's.

Here, $\mathbf{c_p}$ (resp. $|\mathbf{c_p}|$ or $\mathbf{g_p}$) denotes the pth c-vector (resp. the normalized c-vector $\epsilon_p\mathbf{c_p}$ or the g-vector) along the mutation sequence $\overline{\mu}$, B_Q denotes the exchange matrix associated to Q before any mutations, $\mathbf{a}\cdot\mathbf{b}$ denotes ordinary dot product, and $\mathbf{y}^{(d_1,d_2,\dots,d_n)}$ is shorthand for $y_1^{d_1}y_2^{d_2}\cdots y_n^{d_n}$.

Type A_2 Quiver Example

Let $L_1 = 1 + z_1$ and $L_k = 1 + z_k L_1^{\mathbf{c_1} \cdot B_Q |\mathbf{c_k}|} L_2^{\mathbf{c_2} \cdot B_Q |\mathbf{c_k}|} \cdots L_{k-1}^{\mathbf{c_{k-1}} \cdot B_Q |\mathbf{c_k}|}$ for $k \geq 2$.

Supoose $B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1 \mu_2 \mu_1 \mu_2 \mu_1$.

Type A_2 Quiver Example

Let $L_1 = 1+z_1$ and $L_k = 1+z_k L_1^{\mathbf{c_1}\cdot B_Q|\mathbf{c_k}|} L_2^{\mathbf{c_2}\cdot B_Q|\mathbf{c_k}|} \cdots L_{k-1}^{\mathbf{c_{k-1}}\cdot B_Q|\mathbf{c_k}|}$ for $k \geq 2$.

Supoose $B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\mu_1$. Then

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$B_Q|\mathbf{c_2}| = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B_Q|\mathbf{c_3}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_Q|\mathbf{c_4}| = \begin{bmatrix} 0 \\ -1 \end{bmatrix} B_Q|\mathbf{c_5}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$L_1 = 1 + z_1, \quad L_2 = 1 + z_2 L_1^{-1} = 1 + z_2(1+z_1)^{-1} = \frac{1+z_1+z_2}{1+z_1}$$

$$L_3 = 1 + z_3 L_1^{-1} L_2^{-1} = 1 + \frac{z_3}{1+z_1} \frac{1+z_1}{1+z_1+z_2} = \frac{1+z_1+z_2+z_3}{1+z_1+z_2}$$

Type A_2 Quiver Example (continued)

Let $L_1 = 1+z_1$ and $L_k = 1+z_k L_1^{\mathbf{c_1}\cdot B_Q|\mathbf{c_k}|} L_2^{\mathbf{c_2}\cdot B_Q|\mathbf{c_k}|} \cdots L_{k-1}^{\mathbf{c_{k-1}}\cdot B_Q|\mathbf{c_k}|}$ for $k \geq 2$.

Supoose $B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\mu_1$. Then

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$B_Q|\mathbf{c_2}| = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B_Q|\mathbf{c_3}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_Q|\mathbf{c_4}| = \begin{bmatrix} 0 \\ -1 \end{bmatrix} B_Q|\mathbf{c_5}| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$L_4 = 1 + z_4 L_1^0 L_2^1 L_3^1 = 1 + z_4 \frac{1+z_1+z_2}{1+z_1} \frac{1+z_1+z_2+z_3}{1+z_1+z_2} = \frac{1+z_1+z_4(1+z_1+z_2+z_3)}{1+z_1}$$

$$L_5 = 1 + z_5 L_1^{-1} L_2^{-1} L_3^0 L_4^1 = 1 + \frac{z_5}{1+z_1} \frac{1+z_1}{1+z_1+z_2} \frac{1+z_1+z_4(1+z_1+z_2+z_3)}{1+z_1}$$
$$= \frac{(1+z_1)(1+z_1+z_2) + z_5 + z_1 z_5 + z_4 z_5(1+z_1+z_2+z_3)}{(1+z_1+z_2)(1+z_1)}$$

Type A_2 Quiver Example (continued)

$$B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ \overline{\mu} = \mu_1\mu_2\mu_1\mu_2\mu_1. \qquad F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c_j}\cdot\mathbf{g}_\ell}\Big|_{z_1=y^{|\mathbf{c_1}|},\ldots,z_\ell=y^{|\mathbf{c}_\ell|}}$$

$$L_1 = 1+z_1, \quad L_2 = \frac{1+z_1+z_2}{1+z_1}, \quad L_3 = \frac{1+z_1+z_2+z_3}{1+z_1+z_2}, \quad L_4 = \frac{1+z_1+z_4(1+z_1+z_2+z_3)}{1+z_1},$$

$$L_5 = \frac{(1+z_1)(1+z_1+z_2)+z_5+z_1z_5+z_4z_5(1+z_1+z_2+z_3)}{(1+z_1+z_2)(1+z_1)},$$

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{g_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{g_2} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \mathbf{g_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{g_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{g_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$F_1 = L_1 = 1+z_1, \quad F_2 = L_1L_2 = 1+z_1+z_2,$$

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$$F_4 = L_1^{-1}L_2^{-1}L_4 = \frac{1+z_1+z_4(1+z_1+z_2+z_3)}{(1+z_1+z_2)(1+z_1)},$$

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Type A_2 Quiver Example (continued)

$B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\mu_1$. $\quad F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c_j}\cdot\mathbf{g}_\ell}|_{z_1=y^{|\mathbf{c_1}|},\ldots,z_\ell=y^{|\mathbf{c}_\ell|}}$

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$$F_5 = L_2^{-1}L_3^{-1}L_5 = \frac{(1 + z_1)(1 + z_1 + z_2) + z_5 + z_1z_5 + z_4z_5(1 + z_1 + z_2 + z_3)}{(1 + z_1 + z_2)(1 + z_1 + z_2 + z_3)}$$

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Based on $\epsilon_3 = -1$, $\epsilon_4 = +1$, $\epsilon_5 = +1$, and B_Q as above, we get

$$F_3F_1 = F_2 + z_3, \quad F_4F_2 = z_4F_3 + 1, \quad F_5F_3 = z_5F_4 + 1,$$

and these recurrences are valid for these expressions as rational functions.

Type A_2 Quiver Example (continued)

$B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\mu_1$. $\quad F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c_j}\cdot\mathbf{g}_\ell}\big|_{z_1=y^{|\mathbf{c_1}|},\ldots,z_\ell=y^{|\mathbf{c}_\ell|}}$

$$F_1 = L_1 = 1 + z_1, \quad F_2 = L_1L_2 = 1 + z_1 + z_2,$$

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$$F_5 = L_2^{-1}L_3^{-1}L_5 = \frac{(1 + z_1)(1 + z_1 + z_2) + z_5 + z_1z_5 + z_4z_5(1 + z_1 + z_2 + z_3)}{(1 + z_1 + z_2)(1 + z_1 + z_2 + z_3)}$$

$$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{c_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{c_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{c_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Letting $z_1 = y_1$, $z_2 = y_1y_2$, $z_3 = y_2$, $z_4 = y_1$, $z_5 = y_2$, we get polynomials

$$F_1 = y_1 + 1,\ F_2 = y_1y_2 + y_1 + 1,\ F_3 = y_2 + 1,\ F_4 = 1,\ F_5 = 1.$$

F-polynomials from C-Vectors and G-Vectors (2nd Version)

Theorem (Gupta '18) as will be re-expressed in (Gupta-M '19+) :
Given a framed quiver $\widetilde{Q}$ and a mutation sequence $\overline{\mu} = \mu_{i_1}\mu_{i_2}\cdots\mu_{i_\ell}$, consider the sequence of cluster seeds $t_0 \rightarrow^{\mu_{i_1}} t_1 \rightarrow^{\mu_{i_2}} \ldots t_{\ell-1} \rightarrow^{\mu_{i_\ell}} t_\ell$.

$$\text{Let } L_1 = 1 + z_1 \text{ and } L_k = 1 + z_k L_1^{\mathbf{c_1}\cdot B_Q|\mathbf{c_k}|} L_2^{\mathbf{c_2}\cdot B_Q|\mathbf{c_k}|} \cdots L_{k-1}^{\mathbf{c_{k-1}}\cdot B_Q|\mathbf{c_k}|} \text{ for } k \geq 2$$

$$\text{and } F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c_j}\cdot \mathbf{g}_\ell}\big|_{z_1=y^{|\mathbf{c_1}|},\ldots,z_\ell=y^{|\mathbf{c}_\ell|}}.$$

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Theorem (Gupta '18) as will be re-expressed in (Gupta-M '19+) : Given a framed quiver $\widetilde{Q}$ and a mutation sequence $\overline{\mu} = \mu_{i_1}\mu_{i_2}\cdots\mu_{i_\ell}$, consider the sequence of cluster seeds $t_0 \to^{\mu_{i_1}} t_1 \to^{\mu_{i_2}} \ldots t_{\ell-1} \to^{\mu_{i_\ell}} t_\ell$.

Let $L_1 = 1 + z_1$ and $L_k = 1 + z_k L_1^{\mathbf{c_1}\cdot B_Q|\mathbf{c_k}|} L_2^{\mathbf{c_2}\cdot B_Q|\mathbf{c_k}|} \cdots L_{k-1}^{\mathbf{c_{k-1}}\cdot B_Q|\mathbf{c_k}|}$ for $k \geq 2$

$$\text{and } F_{i_\ell;t_\ell} = \prod_{j=1}^{\ell} L_j^{\mathbf{c_j}\cdot\mathbf{g}_\ell}|_{z_1=y^{|\mathbf{c_1}|},\ldots,z_\ell=y^{|\mathbf{c}_\ell|}}.$$

Then the F-polynomial resulting from the final mutation, i.e. $F_{i_\ell;t_\ell}$, can also be expressed as a sum of a product of binomial coefficients:

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{j=1}^{\ell} \binom{\mathbf{c_j}\cdot\left(\mathbf{g}_\ell + \sum_{k=j+1}^{\ell} m_k B_Q|\mathbf{c_k}|\right)}{m_j} \mathbf{y}^{\sum_{j=1}^{\ell} m_j|\mathbf{c_j}|}.$$

Note: This expression as a power series leaves the polynomiality (finiteness of the sum) and positivity of the coefficients as surprising consequences.

Kronecker Quiver Example

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{j=1}^{\ell} \binom{\mathbf{c_j}\cdot\left(\mathbf{g}_\ell + \sum_{k=j+1}^{\ell} m_k B_Q |\mathbf{c_k}|\right)}{m_j} \mathbf{y}^{\sum_{j=1}^{\ell} m_j |\mathbf{c_j}|}.$$

Suppose $B_Q = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\cdots\mu_{i_\ell}$.

Kronecker Quiver Example

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{j=1}^{\ell} \binom{\mathbf{c_j} \cdot \left(\mathbf{g}_\ell + \sum_{k=j+1}^{\ell} m_k B_Q|\mathbf{c_k}|\right)}{m_j} \ \mathbf{y}^{\sum_{j=1}^{\ell} m_j|\mathbf{c_j}|}.$$

Suppose $B_Q = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\cdots\mu_{i_\ell}$. Then

$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\mathbf{c_2} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$, $\mathbf{c_3} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$, $\ldots$, $\mathbf{c_p} = \begin{bmatrix} -p \\ -p+1 \end{bmatrix}$, $|\mathbf{c_p}| = \begin{bmatrix} p \\ p+1 \end{bmatrix}$,

and $\mathbf{g_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{g_2} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $\mathbf{g_3} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, $\ldots$, $\mathbf{g_q} = \begin{bmatrix} -q \\ q+1 \end{bmatrix}$.

Kronecker Quiver Example

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\dots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{j=1}^{\ell} \binom{\mathbf{c_j}\cdot\left(\mathbf{g}_\ell + \sum_{k=j+1}^{\ell} m_k B_Q|\mathbf{c_k}|\right)}{m_j} \ \mathbf{y}^{\sum_{j=1}^{\ell} m_j|\mathbf{c_j}|}.$$

Suppose $B_Q = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\cdots\mu_{i_\ell}$. Then

$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\mathbf{c_2} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$, $\mathbf{c_3} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$, $\dots$, $\mathbf{c_p} = \begin{bmatrix} -p \\ -p+1 \end{bmatrix}$, $|\mathbf{c_p}| = \begin{bmatrix} p \\ p+1 \end{bmatrix}$,
and $\mathbf{g_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{g_2} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $\mathbf{g_3} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, $\dots$, $\mathbf{g_q} = \begin{bmatrix} -q \\ q+1 \end{bmatrix}$. Hence

$\mathbf{c_j}\cdot\mathbf{g}_\ell = \left[\begin{smallmatrix} -j \\ -j+1 \end{smallmatrix}\right]\cdot\left[\begin{smallmatrix} -\ell \\ \ell+1 \end{smallmatrix}\right] = \ell - j + 1,\ \mathbf{c_j}\cdot B_Q|\mathbf{c_k}| = \left[\begin{smallmatrix} -j \\ -j+1 \end{smallmatrix}\right]\cdot\left[\begin{smallmatrix} -2k+2 \\ -2k \end{smallmatrix}\right] = 2(j-k).$

Kronecker Quiver Example

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{j=1}^{\ell} \binom{\mathbf{c_j}\cdot\left(\mathbf{g}_\ell + \sum_{k=j+1}^{\ell} m_k B_Q|\mathbf{c_k}|\right)}{m_j} \ \mathbf{y}^{\sum_{j=1}^{\ell} m_j|\mathbf{c_j}|}.$$

Suppose $B_Q = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2\cdots\mu_{i_\ell}$. Then

$\mathbf{c_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\mathbf{c_2} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$, $\mathbf{c_3} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$, $\ldots$, $\mathbf{c_p} = \begin{bmatrix} -p \\ -p+1 \end{bmatrix}$, $|\mathbf{c_p}| = \begin{bmatrix} p \\ p+1 \end{bmatrix}$,

and $\mathbf{g_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{g_2} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $\mathbf{g_3} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, $\ldots$, $\mathbf{g_q} = \begin{bmatrix} -q \\ q+1 \end{bmatrix}$. Hence

$$\mathbf{c_j}\cdot\mathbf{g}_\ell = \left[\begin{smallmatrix} -j \\ -j+1 \end{smallmatrix}\right]\cdot\left[\begin{smallmatrix} -\ell \\ \ell+1 \end{smallmatrix}\right] = \ell - j + 1,\ \ \mathbf{c_j}\cdot B_Q|\mathbf{c_k}| = \left[\begin{smallmatrix} -j \\ -j+1 \end{smallmatrix}\right]\cdot\left[\begin{smallmatrix} -2k+2 \\ -2k \end{smallmatrix}\right] = 2(j-k).$$

Consequently, we simplify the formula in the Kronecker case to

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \binom{\ell - i + 1 - 2\sum_{j=i+1}^{\ell}(j-i)m_j}{m_i} \ y_1^{\sum_{i=1}^{\ell} i m_i} y_2^{\sum_{i=1}^{\ell}(i-1)m_i}.$$

Kronecker Quiver Example (continued)

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \binom{\ell - i + 1 - 2\sum_{j=i+1}^{\ell}(j-i)m_j}{m_i} \; y_1^{\sum_{i=1}^{\ell} i m_i} y_2^{\sum_{i=1}^{\ell}(i-1)m_i}.$$

$$F_{1;t_1} = \sum_{m_1=0}^{\infty} \binom{1}{m_1} y_1^{m_1} = 1 + y_1$$

$$F_{2;t_2} = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \binom{2-2m_2}{m_1}\binom{1}{m_2} y_1^{m_1+2m_2} y_2^{m_2} = 1 + 2y_1 + y_1^2 + y_1^2 y_2.$$

$$F_{1;t_3} = \sum_{m_1,m_2,m_3\in\mathbb{Z}_{\geq 0}} \binom{3-2m_2-4m_3}{m_1}\binom{2-2m_3}{m_2}\binom{1}{m_3} y_1^{m_1+2m_2+3m_3} y_2^{m_2+2m_3} =$$

$$1 + 3y_1 + 3y_1^2 + y_1^3 + 2y_1^2 y_2 + 2y_1^3 y_2 + y_1^3 y_2^2.$$

Kronecker Quiver Example (continued)

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \binom{\ell - i + 1 - 2\sum_{j=i+1}^{\ell}(j-i)m_j}{m_i} \; y_1^{\sum_{i=1}^{\ell} i m_i} y_2^{\sum_{i=1}^{\ell}(i-1)m_i}.$$

$$F_{1;t_1} = \sum_{m_1=0}^{\infty} \binom{1}{m_1} y_1^{m_1} = \underline{1} + \underline{y_1}$$

These two terms correspond to $m_1 = 0$ and $m_1 = 1$, respectively. There are no contributions for $m_1 \geq 2$.

Kronecker Quiver Example (continued)

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \binom{\ell - i + 1 - 2\sum_{j=i+1}^{\ell}(j-i)m_j}{m_i} \; y_1^{\sum_{i=1}^{\ell} i m_i} y_2^{\sum_{i=1}^{\ell}(i-1)m_i}.$$

$$F_{1;t_1} = \sum_{m_1=0}^{\infty} \binom{1}{m_1} y_1^{m_1} = \underline{1} + \underline{y_1}$$

These two terms correspond to $m_1 = 0$ and $m_1 = 1$, respectively. There are no contributions for $m_1 \geq 2$.

$$F_{2;t_2} = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \binom{2-2m_2}{m_1} \binom{1}{m_2} y_1^{m_1+2m_2} y_2^{m_2} = \underline{1 + 2y_1 + y_1^2} + \underline{y_1^2 y_2}.$$

The two underlined contributions correspond to $m_2 = 0$ and $m_2 = 1$, respectively. Analogously, there are no contributions for $m_2 \geq 2$.

Kronecker Quiver Example (continued)

$$F_{i_\ell;t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \binom{\ell - i + 1 - 2\sum_{j=i+1}^{\ell}(j-i)m_j}{m_i} \; y_1^{\sum_{i=1}^{\ell} i m_i} y_2^{\sum_{i=1}^{\ell}(i-1)m_i}.$$

$$F_{1;t_1} = \sum_{m_1=0}^{\infty} \binom{1}{m_1} y_1^{m_1} = \underline{1} + \underline{y_1}$$

These two terms correspond to $m_1 = 0$ and $m_1 = 1$, respectively. There are no contributions for $m_1 \geq 2$.

$$F_{2;t_2} = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \binom{2-2m_2}{m_1} \binom{1}{m_2} y_1^{m_1+2m_2} y_2^{m_2} = \underline{1 + 2y_1 + y_1^2} + \underline{y_1^2 y_2}.$$

The two underlined contributions correspond to $m_2 = 0$ and $m_2 = 1$, respectively. Analogously, there are no contributions for $m_2 \geq 2$.

The first three terms correspond to $m_1 = 0, m_1 = 1, m_1 = 2$, respectively, and there are no contributions for $m_1 \geq 2$.

Kronecker Quiver Example (continued)

$$F_{1;t_3} = \sum_{m_1,m_2,m_3 \in \mathbb{Z}_{\geq 0}} \binom{3-2m_2-4m_3}{m_1}\binom{2-2m_3}{m_2}\binom{1}{m_3} y_1^{m_1+2m_2+3m_3} y_2^{m_2+2m_3} =$$

$$\underline{1 + 3y_1 + 3y_1^2 + y_1^3 + 2y_1^2y_2 + 2y_1^3y_2} + \underline{y_1^3y_2^2}.$$

The two underlined contributions correspond to $m_3 = 0$ and $m_3 = 1$, respectively. Again, there are no contributions for $m_3 \geq 2$.

Kronecker Quiver Example (continued)

$$F_{1;t_3} = \sum_{m_1,m_2,m_3 \in \mathbb{Z}_{\geq 0}} \binom{3-2m_2-4m_3}{m_1}\binom{2-2m_3}{m_2}\binom{1}{m_3} y_1^{m_1+2m_2+3m_3} y_2^{m_2+2m_3} =$$

$$\underline{1+3y_1+3y_1^2+y_1^3+2y_1^2y_2+2y_1^3y_2} + \underline{y_1^3y_2^2}.$$

The two underlined contributions correspond to $m_3 = 0$ and $m_3 = 1$, respectively. Again, there are no contributions for $m_3 \geq 2$.

Further refinement of this sum by tracking $m_2 = 0$ and $m_2 = 1$, respectively, under the assumption $m_3 = 0$ yields

$$\underline{\underline{1+3y_1+3y_1^2+y_1^3}} + \underline{\underline{2y_1^2y_2+2y_1^3y_2}} + \underline{y_1^3y_2^2}.$$

Kronecker Quiver Example (continued)

$$F_{1;t_3} = \sum_{m_1,m_2,m_3 \in \mathbb{Z}_{\geq 0}} \binom{3-2m_2-4m_3}{m_1}\binom{2-2m_3}{m_2}\binom{1}{m_3} y_1^{m_1+2m_2+3m_3} y_2^{m_2+2m_3} =$$

$$\underline{1+3y_1+3y_1^2+y_1^3+2y_1^2y_2+2y_1^3y_2} + \underline{y_1^3y_2^2}.$$

The two underlined contributions correspond to $m_3 = 0$ and $m_3 = 1$, respectively. Again, there are no contributions for $m_3 \geq 2$.

Further refinement of this sum by tracking $m_2 = 0$ and $m_2 = 1$, respectively, under the assumption $m_3 = 0$ yields

$$\underline{\underline{1+3y_1+3y_1^2+y_1^3} + \underline{2y_1^2y_2+2y_1^3y_2}} + \underline{y_1^3y_2^2}.$$

However, in addition we get an **infinite** number of contributions

$$\sum_{m_1=0}^{\infty} \binom{-1}{m_1} y_1^{m_1+4} y_2^2 + \sum_{m_1=0}^{\infty} \binom{-1}{m_1} y_1^{m_1+3} y_2^2; \quad \text{recall} \quad \binom{-1}{m_1} = (-1)^{m_1}$$

arising when $m_2 = 2, m_3 = 0$ or $m_2 = 0, m_3 = 1$.

Kronecker Quiver Example (continued)

$$F_{1;t_3} = \sum_{m_1,m_2,m_3\in\mathbb{Z}_{\geq 0}} \binom{3-2m_2-4m_3}{m_1}\binom{2-2m_3}{m_2}\binom{1}{m_3} y_1^{m_1+2m_2+3m_3} y_2^{m_2+2m_3} =$$

$$\underline{1+3y_1+3y_1^2+y_1^3+2y_1^2y_2+2y_1^3y_2} + \underline{y_1^3y_2^2}.$$

The two underlined contributions correspond to $m_3 = 0$ and $m_3 = 1$, respectively. Again, there are no contributions for $m_3 \geq 2$.

Further refinement of this sum by tracking $m_2 = 0$ and $m_2 = 1$, respectively, under the assumption $m_3 = 0$ yields

$$\underline{\underline{1+3y_1+3y_1^2+y_1^3} + \underline{2y_1^2y_2+2y_1^3y_2}} + \underline{y_1^3y_2^2}.$$

However, in addition we get an **infinite** number of contributions

$$\sum_{m_1=0}^{\infty}\binom{-1}{m_1} y_1^{m_1+4}y_2^2 + \sum_{m_1=0}^{\infty}\binom{-1}{m_1} y_1^{m_1+3}y_2^2; \quad \text{recall} \quad \binom{-1}{m_1} = (-1)^{m_1}$$

arising when $m_2 = 2, m_3 = 0$ or $m_2 = 0, m_3 = 1$. This telescoping infinite sum vanishes except for the term of $y_1^3y_2^2$ for $m_1 = 0, m_2 = 0, m_3 = 1$.

Kronecker Quiver Example (continued)

The formulae continue as

$$F_{2;t_4} = \sum_{m_1,m_2,m_3,m_4 \in \mathbb{Z}_{\geq 0}} \binom{4-2m_2-4m_3-6m_4}{m_1}\binom{3-2m_3-4m_4}{m_2}$$

$$\times \quad \binom{2-2m_4}{m_3}\binom{1}{m_4} y_1^{m_1+2m_2+3m_3+4m_4} y_2^{m_2+2m_3+3m_4}$$

$$F_{1;t_5} = \sum_{m_1,m_2,m_3,m_4,m_5 \in \mathbb{Z}_{\geq 0}} \binom{5-2m_2-4m_3-6m_4-8m_5}{m_1}\binom{4-2m_3-4m_4-6m_5}{m_2} \times$$

$$\binom{3-2m_4-4m_5}{m_3}\binom{2-2m_5}{m_4}\binom{1}{m_5} y_1^{m_1+2m_2+3m_3+4m_4+5m_5} y_2^{m_2+2m_3+3m_4+4m_5}$$

$F_{1;t_5}$ includes terms such as $6y_1^5y_2^3 \; - \; 2y_1^5y_2^3 = 4y_1^5y_2^3$ in its expansion, corresponding to $(m_1, m_2, m_3, m_4, m_5) = (0,1,1,0,0)$ and $(1,0,0,1,0)$, respectively. In particular, the contributions from negative binomial coefficients yield a positive term, yet arises from a non-trivial difference.

Formula for general Rank Two, i.e. r-Kronecker Case

For the case of $B_Q = \begin{bmatrix} 0 & r \\ -r & 0 \end{bmatrix}$ and $\overline{\mu} = \mu_1\mu_2\mu_1\mu_2 \cdots \mu_{i_\ell}$,

$$F_{i_\ell, t_\ell} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \binom{s_{\ell-i} - r\sum_{j=i+1}^{\ell} s_{j-i-1} m_j}{m_i} \; y_1^{\sum_{i=1}^{\ell} s_{i-1} m_i} y_2^{\sum_{i=1}^{\ell} s_{i-2} m_i}$$

where $s_{-1} = 0, s_0 = 1$, $s_{k+1} = rs_k - s_{k-1}$ for $k \geq 0$.

Cluster Monomials (F-polys) from C-Vectors and G-Vectors

Theorem (Gupta '18) as will be re-expressed in (Gupta-M '19+) :
Given a framed quiver $\widetilde{Q}$ and a mutation sequence $\overline{\mu} = \mu_{i_1}\mu_{i_2}\cdots\mu_{i_\ell}$, consider the sequence of cluster seeds $t_0 \rightarrow^{\mu_{i_1}} t_1 \rightarrow^{\mu_{i_2}} \ldots t_{\ell-1} \rightarrow^{\mu_{i_\ell}} t_\ell$.

Let $\{F_{1;t_\ell}, F_{2;t_\ell}, \ldots, F_{n;t_\ell}\}$ be the F-polynomials associated to the cluster seed after the final mutation. Let $F_{t_\ell}^{(d_1,\ldots,d_n)} = F_{1;t_\ell}^{d_1} F_{2;t_\ell}^{d_2} \cdots F_{n;t_\ell}^{d_n}$ and $\mathbf{g}^{(\mathbf{d_1},\mathbf{d_2},\ldots,\mathbf{d_n})}$ be the associated $\mathbf{d}$-weighted linear combination of g-vectors.

Theorem (Gupta '18) as will be re-expressed in (Gupta-M '19+) : Given a framed quiver $\widetilde{Q}$ and a mutation sequence $\overline{\mu} = \mu_{i_1}\mu_{i_2}\cdots\mu_{i_\ell}$, consider the sequence of cluster seeds $t_0 \to^{\mu_{i_1}} t_1 \to^{\mu_{i_2}} \ldots t_{\ell-1} \to^{\mu_{i_\ell}} t_\ell$.

Let $\{F_{1;t_\ell}, F_{2;t_\ell}, \ldots, F_{n;t_\ell}\}$ be the F-polynomials associated to the cluster seed after the final mutation. Let $F_{t_\ell}^{(d_1,\ldots,d_n)} = F_{1;t_\ell}^{d_1} F_{2;t_\ell}^{d_2} \cdots F_{n;t_\ell}^{d_n}$ and $\mathbf{g}^{(\mathbf{d_1},\mathbf{d_2},\ldots,\mathbf{d_n})}$ be the associated $\mathbf{d}$-weighted linear combination of g-vectors. Then this F-polynomial analogue of a cluster monomial can be expressed as a sum of products of binomial coefficients:

$$F_{t_\ell}^{(d_1,\ldots,d_n)} = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}} \prod_{j=1}^{\ell} \binom{\mathbf{c_j} \cdot \left(\mathbf{g}^{(\mathbf{d_1},\mathbf{d_2},\ldots,\mathbf{d_n})} + \sum_{k=j+1}^{\ell} m_k B_Q |\mathbf{c_k}|\right)}{m_j} \mathbf{y}^{\sum_{j=1}^{\ell} m_j |\mathbf{c_j}|}.$$

Here, $\mathbf{c_p}$ (resp. $|\mathbf{c_p}|$) denotes the pth c-vector (resp. the normalized c-vector $\epsilon_p \mathbf{c_p}$) along the mutation sequence $\overline{\mu}$, B_Q denotes the exchange matrix associated to Q before any mutations, $\mathbf{a}\cdot\mathbf{b}$ denotes ordinary dot product, and $\mathbf{y}^{(d_1,d_2,\ldots,d_n)}$ is shorthand for $y_1^{d_1} y_2^{d_2} \cdots y_n^{d_n}$.

Thanks for Coming (`http://math.umn.edu/∼musiker/Fpoly19.pdf`)

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