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Linear Systems on Tropical Curves

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Abstract A tropical curve Γ is a metric graph with possibly unbounded edges, and tropical rational functions are continuous piecewise linear functions with integer slopes. We define the complete linear system $|D|$ of a divisor D on a tropical curve Γ analogously to the classical counterpart. We investigate the structure of $|D|$ as a cell complex and show that linear systems are quotients of tropical modules, finitely generated by vertices of the cell complex. Using a finite set of generators, $|D|$ defines a map from Γ to a tropical projective space, and the image can be modified to a tropical curve of degree equal to $\deg(D)$ when $|D|$ is base point free. The tropical convex hull of the image realizes the linear system $|D|$ as a polyhedral complex. We show that curves for which the canonical divisor is not very ample are hyperelliptic. We also show that the Picard group of a $\mathbb{Q}$-tropical curve is a direct limit of critical groups of finite graphs converging to the curve.

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1 Introduction

A tropical curve is a combinatorial or graph-theorectic analogue of an algebraic curve or a Riemann surface. Divisors, ranks of divisors, linear systems, and Jacobians on tropical curves have been studied recently by other authors [Bak08, BF06, BF09, BN07, CDPR10, GK08, MZ06]. A Riemann-Roch Theorem for tropical curves was proven independently by Gathmann and Kerber [GK08] and by Mikhalkin and Zharkov [MZ06], using the work of Baker and Norine for finite graphs [BN07]. The tropical Jacobian was studied by Mikhalkin and Zharkov [MZ06] and by Baker and Faber [BF09]. In [Bak08], Baker showed that the rank of a linear system cannot decrease when specializing from an arithmetic surface to a graph or a tropical curve. Using this result, Cools, Draisma, Payne, and Robeva gave a new proof the Brill-Noether Theorem [CDPR10]. In this paper, we study linear systems on tropical curves from an abstract combinatorial piont of view. We give combinatorial proofs of some results about linear systems on tropical curves that are analogous to those about algebraic curves, and we give examples where this analogy fails.

An abstract tropical curve Γ is a connected metric graph with possibly unbounded edges. A *divisor* D on Γ is a formal finite $\mathbb{Z}$-linear combination $D = \sum_{x \in \Gamma} D(x) \cdot x$ of points of Γ. The *degree* of a divisor is the sum of the coefficients, $\sum_x D(x)$. The divisor is *effective* if $D(x) \geq 0$ for all $x \in \Gamma$; in this case we write $D \geq 0$. We call $\text{supp}(D) = \{x \in \Gamma : D(x) \neq 0\}$ the support of the divisor D.

A *(tropical) rational function* f on Γ is a continuous function $f : \Gamma \to \mathbb{R}$ that is piecewise-linear on each edge with finitely many pieces and integral slopes. The *order* $\text{ord}_x(f)$ of f at a point $x \in \Gamma$ is the sum of outgoing slopes at x. The *principal divisor* associated to f is

$$(f) := \sum_{x \in \Gamma} \text{ord}_x(f) \cdot x.$$

A point $x \in \Gamma$ is called a *zero* of f if $\text{ord}_x(f) > 0$ and a *pole* of f if $\text{ord}_x(f) < 0$. We call two divisors D and D' linearly equivalent and write $D \sim D'$ if $D - D' = (f)$ for some f. For any divisor D on Γ, let $R(D)$ be the set of all rational functions f on Γ such that the divisor $D + (f)$ is effective, and $|D| = \{D' \geq 0 : D' \sim D\}$, the *linear system* of D. Let $\mathbb{1}$ denote the set of constant functions on Γ.

The set $R(D)$ is naturally embedded in the set $\mathbb{R}^{\Gamma}$ of all real-valued functions on Γ, and $|D|$ is a subset of the d^{th} symmetric product of Γ where $d = \deg(D)$. The map $R(D)/\mathbb{1} \to |D|$ given by $f \mapsto D + (f)$ is a homeomorphism from $R(D)/\mathbb{1}$ to $|D|$. It was shown in [GK08, MZ06] that $|D|$ is a cell complex, so is $R(D)/\mathbb{1}$. Our aim is to study the combinatorial and algebraic structure of this object $R(D)$.

In Section 2 we give definitions and state linear equivalence in terms of weighted chip firing moves, which are continuous analogues of the chip firing games on finite graphs. In Section 3 we show that $R(D)$ is a finitely generated tropical semi-module and describe a generating set. In Section 4, we study the cell complex structure of $|D|$. We show that the vertex set of $|D|$ coincides with the generating set of $R(D)$ described in Section 3. We give a triangulation of the link of each cell as the order complex of a poset of possible weighted chip firing moves.

Any finite set $\mathcal{F}$ of linearly equivalent divisors induces a map $\phi_{\mathcal{F}}$ from the abstract curve to a tropical projective space. This map is described in Section 5. If $\mathcal{F}$ generates $R(D)$, we show that the tropical convex hull of the image of this map is homeomorphic to $|D|$. The image of this map $\phi_{\mathcal{F}}$ can be naturally modified to an embedded tropical curve, and we show in Section 6 that the embedded curve has the same degree as the divisor that we start with if the linear system of the divisor is base point free. In Section 7, we show that every divisor of positive degree is ample. We also show that if the canonical divisor is not very ample, then the tropical curve is hyperelliptic (but not vice versa). Finally in Section 8, we show that the Picard group of a $\mathbb{Q}$-tropical curve is the direct limit of Picard groups for finite graphs obtained by subdividing the edges.

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2 Metric graphs, rational functions, and chip-firing

A *metric graph* Γ is a complete connected metric space such that each point $x \in \Gamma$ has a neighborhood U_x isometric to a star-shaped set of valence $\mathrm{val}(x) \geq 1$ endowed with the path metric. To be precise, a star-shaped set of valence v is a set of the form

$$S(v, r) = \{z \in \mathbb{C} : z = te^{2\pi ik/v} \text{ for some } 0 \leq t < r \text{ and } k \in \mathbb{Z}\}.$$

The points $x \in \Gamma$ with valence different from 2 are precisely those where Γ fails to look locally like an open interval. Accordingly, we refer to a point of valence 2 as a *smooth* point.

Let $V(\Gamma)$ be any finite nonempty subset of Γ such that $V(\Gamma)$ contains all of the points with $\mathrm{val}(x) \neq 2$. Then $\Gamma \setminus V(\Gamma)$ is a finite disjoint union of open intervals. For a metric graph Γ, we say that a choice of such $V(\Gamma)$ gives rise to a *model* $G(\Gamma)$ for Γ. Each edge has a nonzero length inherited from the metric space Γ.

Let $V_0(\Gamma) = \{x \in \Gamma : \mathrm{val}(x) \neq 2\}$, where val denotes the valence of a vertex of $V(\Gamma)$. Unless Γ is a circle, $V_0(\Gamma)$ gives a model. For some of our applications, we may choose a model whose vertex set is strictly bigger than $V_0(\Gamma)$. However unless otherwise specified, the reader may assume that $G(\Gamma)$ denotes the coarsest model and that a *vertex* is an element of $V_0(\Gamma)$.

A *tropical curve* is a metric graph in which the leaf edges may have length ∞. A leaf edge is an edge adjacent to a one-valent vertex. Note that we add a "point at infinity" for each unbounded edge. A tropical rational function on a tropical curve may attain values $\pm\infty$ at points at infinity.

We will use the term *subgraph* in a topological sense, that is, as a compact subset of a tropical curve Γ with a finite number of connected components. For a subgraph $\Gamma' \subset \Gamma$ and a positive real number l, the *chip firing move* $\mathrm{CF}(\Gamma', l)$ by a (not necessarily connected) subgraph is the tropical rational function $\mathrm{CF}(\Gamma', l)(x) = -\min(l, \mathrm{dist}(x, \Gamma'))$. It is constant 0 on Γ', has slope -1 in the l-neighborhood of Γ' directed away from Γ', and it is constant $-l$ on the rest of the graph. We will sometimes refer to an effective divisor D as a *chip configuration*. For example, for $D = c_1 \cdot x_1 + \cdots + c_n \cdot x_n$, we say that there are c_i chips at the point $x_i \in \Gamma$. The total number of chips is the *degree* of the divisor. We say that a subgraph $\Gamma' \subset \Gamma$ can *fire* if for each boundary point of Γ' there are at least as many chips as the number of edges pointing out of Γ'. In other words, Γ' can fire if the divisor $D + (\mathrm{CF}(\Gamma', l))$ is effective for some positive real number l. The chip configuration $D + (\mathrm{CF}(\Gamma', l))$ is then obtained from D by moving one chip from the boundary of Γ' along each edge out of Γ' by distance l. Here we assume that l was chosen to be small enough so that the chips do not pass through each other or pass through a non-smooth point.

We will now show that these chip firing moves are enough to move between linearly equivalent divisors (Proposition 3 below). To this end, call a tropical rational function f a *weighted chip firing move* if there are two disjoint (not necessarily connected) proper closed subgraphs Γ_1 and Γ_2 such that the complement $\Gamma \setminus (\Gamma_1 \cup \Gamma_2)$ consists only of open line segments and such that f is constant on Γ_1 and Γ_2 and linear (smooth) with integer slopes on the complement.

A weighted chip firing move f can also be thought of as a combinatorial transformation that acts on chip configurations. Such transformations move chips from the boundary of Γ_2 along the open line segments in the complement $\Gamma \setminus (\Gamma_1 \cup \Gamma_2)$. (Here we assume w.l.o.g. that $f(\Gamma_2) > f(\Gamma_1)$.) During this process, a law of conservation of momentum holds so that a stack of m chips that move together will only move a distance of l/m. The numbers l and m can be different on each component of the complement. Note that a (simple) chip firing move $\mathrm{CF}(\Gamma', l)$ with small l is a special case of a weighted chip firing move when all the slopes are 0 or ± 1.

Lemma 1 *A weighted chip firing move is an (ordinary) sum of chip firing moves (plus a constant).*

Proof Let f be a weighted chip firing move and Γ_1 and Γ_2 be as above. Then

$$\Gamma \setminus (\Gamma_1 \sqcup \Gamma_2) = L_1 \sqcup L_2 \sqcup \cdots \sqcup L_k$$

where the L_i's are open line segments between Γ_1 and Γ_2.

Suppose $f(\Gamma_2) - f(\Gamma_1) = \ell > 0$. Consequently, the slope of f on segment L_i (viewed in the direction from Γ_1 to Γ_2) must be $\mathfrak{s}_i = \ell / |L_i| \in \mathbb{Z}$ where $|L_i|$

is the length of edge L_i. Let $\mathfrak{s}$ be the least common multiple of all the $\mathfrak{s}_i$, and we let the k_i's be the integers such that $\mathfrak{s} = k_i\mathfrak{s}_\mathfrak{i}$. Then each L_i has length $k_i u_\Gamma$ where $u_\Gamma = \frac{\ell}{\mathfrak{s}}$. (In other words, by setting u_Γ to be a unit length, we obtain a rescaled version of Γ such that each rescaled L_i is of integral length.)

For $j = 0, \ldots, \mathfrak{s} - 1$ we build the subgraph Γ'_j from Γ_2 by attaching a subsegment of length $\lfloor j/\mathfrak{s}_i \rfloor u_\Gamma$ of each L_i. Then f is the sum of the chip firing moves $\mathrm{CF}(\Gamma'_j, u_\Gamma)$.

The following lemma makes the connection between $R(D)$ and chip firing games.

Lemma 2 *Every tropical rational function is an (ordinary) sum of chip firing moves (plus a constant).*

Proof Let f be a tropical rational function on Γ. Let S be the finite subset of Γ consisting of vertices of Γ and the corner locus (zeroes and poles) of f. Let $f(S) \subset \mathbb{R}$ be the set of values of f at points in S. We will proceed by induction on the size of $f(S)$. If $f(S)$ contains only one or two values, then f is either a constant function or already a weighted chip firing move. Suppose $f(S)$ contains at least three values. Let $c \in f(S)$ be a value that is neither the maximum nor the minimum in $f(S)$. Let f_1, f_2 be new tropical rational functions defined as $f_1(x) = \min(c, f(x))$ and $f_2(x) = \max(c, f(x))$ for all $x \in \Gamma$. Then $f = f_1 + f_2 - c$. Let S_1, S_2 be the sets consisting of the corner loci of f_1, f_2 respectively, together with the vertices of Γ. Both $f_1(S_1)$ and $f_2(S_2)$ have strictly fewer elements than $f(S)$, and the assertion follows by induction.

Note that even if we start with a tropical rational function $f \in R(D)$, the sequence of weighted chip firing moves $f_1, \ldots, f_n$ for which $f = f_1 + \cdots + f_n$ may not be in $R(D)$, i.e. the divisors $D + (f_i)$ may not be effective although $D + (f)$ is.

The following proposition follows easily from the two previous lemmas.

Proposition 3 *Two divisors are linearly equivalent if and only if one can be attained from the other using chip firing moves.*

3 Extremals and Generators of $R(D)$

The *tropical semiring* $(\mathbb{R}, \oplus, \odot)$ is the set of real numbers $\mathbb{R}$ with two tropical operations:

$$a \oplus b = \max(a, b), \text{ and } a \odot b = a + b.$$

The space $R(D)$ is naturally a subset of the space $\mathbb{R}^\Gamma$ of real-valued functions on Γ. For $f, g \in \mathbb{R}^\Gamma$, and $a \in \mathbb{R}$ the tropical sum $f \oplus g$ and the tropical scalar multiplication $a \odot f$ are defined by taking tropical sums and tropical products pointwise.

Lemma 4 *The space $R(D)$ is a tropical semi-module, i.e. it is closed under tropical addition and tropical scalar multiplication.*

Proof It is clear that $(c \odot f) = (f)$ for any $c \in \mathbb{R}$ and any tropical rational function f, so $R(D)$ is closed under tropical scalar multiplication. Let $f, g \in R(D)$ and $x \in \Gamma$. If $f(x) > g(x)$, then $\mathrm{ord}_x(f \oplus g) = \mathrm{ord}_x(f)$. If $f(x) < g(x)$, then $\mathrm{ord}_x(f \oplus g) = \mathrm{ord}_x(g)$. If $f(x) = g(x)$, then for each direction $\overrightarrow{v}$, the outgoing slope of $f \oplus g$ in the neighborhood of x in the direction $\overrightarrow{v}$ is the maximum of each outgoing slope in the direction $\overrightarrow{v}$ of f and g, so $\mathrm{ord}_x(f \oplus g) \geq \mathrm{ord}_x(f)$. Hence $\mathrm{ord}_x(f \oplus g) + D(x) \geq 0$ for all $x \in \Gamma$, so $f \oplus g \in R(D)$.

Tropical semi-modules in $\mathbb{R}^n$ are also called *tropically convex sets* [DS04]. Since $R(D + (f)) = R(D) + f$, the tropical algebraic structure of $R(D)$ does not depend on the choice of the representative D. An element $f \in R(D)$ is called *extremal* if for any $g_1, g_2 \in R(D)$, $f = g_1 \oplus g_2 \implies f = g_1$ or $f = g_2$. An element f is an extremal if and only if all its tropical scalar multiples $c \odot f$ also are extremals. Any generating set of $R(D)$ must contain all extremals up to tropical scalar multiplication.

Lemma 5 *A tropical rational function f is an extremal of $R(D)$ if and only if there are not two proper subgraphs Γ_1 and Γ_2 covering Γ (i.e. $\Gamma_1 \cup \Gamma_2 = \Gamma$) such that each can fire on $D + (f)$.*

Proof Suppose that there are two such graphs that can fire. The corresponding rational functions g_1, g_2 can be chosen so that g_i is zero on Γ_i, and they are non-positive. Since $\Gamma_1 \cup \Gamma_2 = \Gamma$, $g_1 \oplus g_2 = 0$, so $(f + g_1) \oplus (f + g_2) = f$ and f is not an extremal.

Now suppose $f = g_1 \oplus g_2$ for some $g_1, g_2 \neq f$ in $R(D)$. Let Γ_i be the loci where $f = g_i$. Then $\Gamma_1 \cup \Gamma_2 = \Gamma$. Let $\varepsilon_i > 0$ be such that g_i is smooth in the ε_i-neighborhood of Γ_i, outside of Γ_i. Then each Γ_i can fire distance ε_i.

A *cut set* of a graph Γ is a set of points $A \subset \Gamma$ such that $\Gamma \backslash A$ is not connected. A *smooth cut set* is a cut set consisting of smooth points (2-valent points). Note that being a smooth cut set depends only on the topology of Γ and is not affected by the choice of model $G(\Gamma)$.

Theorem 6 *Let $\mathcal{S}$ be the set of rational functions $f \in R(D)$ such that the support of $D + (f)$ does not contain a smooth cut set. Then*

(a) $\mathcal{S}$ contains all the extremals of $R(D)$,
(b) $\mathcal{S}$ is finite modulo tropical scaling, and
(c) $\mathcal{S}$ generates $R(D)$ as a tropical semi-module.

For the proof of (b) we need a boundedness lemma.

Lemma 7 *For $D \geq 0$ every slope of $f \in R(D)$ is bounded by* $\deg D$.

This improves the bound in [GK08, Lemma 1.8].

Proof Let $f \in R(D)$ and suppose the slope of f at $x \in \Gamma$ is $m > 0$. Consider the connected component $\overline{\Gamma}$ containing x in the tropical curve $f^{-1}([f(x), \infty])$.

Let $\bar{f}$ denote the restriction of f to $\overline{\Gamma}$. The order of the restricted principal divisor at x satisfies $\mathrm{ord}_x(\bar{f}) \geq m$, and because $\deg(\bar{f}) = 0$, the degree of the negative part of $(\bar{f})$ is at least m. Now, $D \geq 0$ and $D + (f) \geq 0$ imply that $\deg D \geq m$.

Proof (Proof of Theorem 6) (a) Suppose $f \notin \mathcal{S}$, then $D + (f)$ splits Γ into two subgraphs Γ_1 and Γ_2. Both of these graphs can fire, and the union of their closures is the entire Γ, so by Lemma 5, f is not an extremal.

(b) Let $f \in \mathcal{S}$. The support of $D + (f)$ meets the interior of each edge in at most one point, because two points on the same edge form a smooth cut set. Removing the set of edges meeting the support of $D + (f)$ does not disconnect Γ, and so the remaining edges contain a spanning tree of Γ.

There are finitely many spanning trees in a graph, and there are finitely many possible slopes for each edge in this spanning tree because of Lemma 7. Therefore, the number of possible values of f on vertices of Γ is finite modulo tropical scaling. (Here, vertices are non-smooth points. If Γ is a circle, then fix any point as a vertex.)

On each non-tree edge, knowing the values and the slopes of f at the two end points uniquely determines f, given the fact that all the chips of $D + (f)$ must fall on the same point of a given edge. We conclude that $\mathcal{S}$ is finite modulo tropical scaling.

(c) Let f be an arbitrary function in $R(D)$. We need to show that f can be written as a finite tropical sum of elements of $\mathcal{S}$. Let $N(f)$ be the number of smooth points in $\mathrm{supp}(D+(f))$. If f is not already in $\mathcal{S}$, then there is a smooth cut set A and two components Γ_1 and Γ_2. Let g_1 and g_2 be the weighted chip firing moves that fire all chips on their boundaries as far as possible. Then $f = (f + g_1) \oplus (f + g_2)$. Repeating this decomposition terminates after a finite number of steps because $0 \leq N(f + g_i) < N(f)$ for each $i = 1, 2$.

Proposition 8 *Any finitely generated tropical sub-semimodule M of $\mathbb{R}^\Gamma$ is generated by the extremals.*

Proof Let $f_1, f_2, \ldots, f_n$ be a generating set of $M \subset \mathbb{R}^\Gamma$. Suppose f_n is not an extremal. Then $f_n = g \oplus h$ for some $g, h \in M$ such that $f_n \neq g$ and $f_n \neq h$. Since $f_1, \ldots, f_n$ generate M, we have

$$\begin{aligned} g &= (a_1 \odot f_1) \oplus \cdots \oplus (a_{n-1} \odot f_{n-1}) \oplus (a_n \odot f_n) \quad \text{and} \\ h &= (b_1 \odot f_1) \oplus \cdots \oplus (b_{n-1} \odot f_{n-1}) \oplus (b_n \odot f_n) \end{aligned}$$

for some $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$. Since $g \leq f_n,\ h \leq f_n$ pointwise, and $g \neq f_n,\ h \neq f_n$, we must have $a_n < 0$ and $b_n < 0$. Then

$$f_n = g \oplus f = (a_1 \odot f_1) \oplus \cdots \oplus (a_{n-1} \odot f_{n-1}) \oplus (b_1 \odot f_1) \oplus \cdots \oplus (b_{n-1} \odot f_{n-1}),$$

so f_n is in the tropical semi-module generated by $f_1, \ldots, f_{n-1}$. We can remove non-extremals from any finite generating set this way, so M is generated by the extremals.

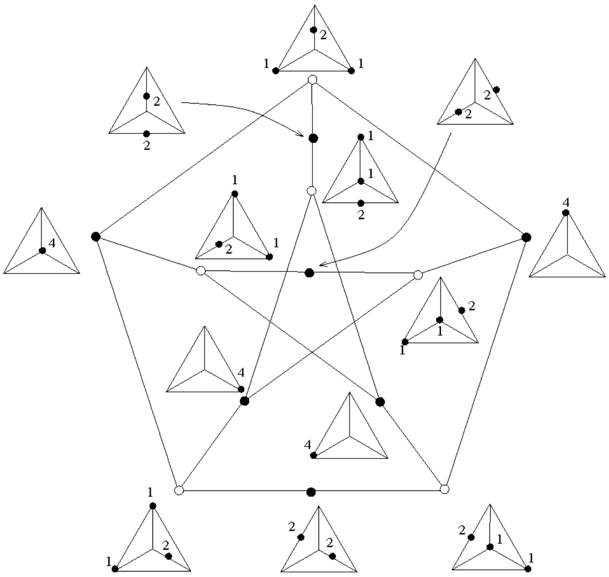


Fig. 1 Let the tropical curve Γ be the complete graph on four vertices with edges of equal length. Let K be the canonical divisor. The 13 divisors shown here, together with K, correspond to the elements of $\mathcal{S}$ that generate $R(K)$, from Theorem 6. The seven black dots in the Petersen graph correspond to the extremals.

Corollary 9 *The tropical semimodule $R(D)$ is generated by the extremals. This generating set is minimal and unique up to tropical scalar multiplication.*

The set of extremals can be obtained from $\mathcal{S}$ by removing the elements that do not satisfy the condition in Lemma 5.

Example 10 Let Γ be a tropical curve with the complete graph on 4 vertices with equal edge lengths as a model. Consider the canonical divisor K, that is the divisor with value 1 on the four vertices and zero elsewhere. The canonical divisor is defined in general in Section 4.2. Then the set $\mathcal{S}$ from Theorem 6 consists of 14 elements, 7 of which are extremals. See Figure 1.

If the edge lengths of the complete graph are not all equal, then the set $\mathcal{S}$ may be different from this. We will describe the local cell complex structure of $R(K)$ near K in the next section, in Example 20.

4 Cell complex structure of $|D|$

As seen in the previous section, $R(D) \subset \mathbb{R}^{\Gamma}$ is finitely generated as a tropical semi-module or a tropical polytope. However, it is not a polyhedral complex in the ordinary sense. For example, let Γ be the line segment $[0, 1]$, and D be the point 1. Then $R(D)$ is the tropical convex hull of $f, g \in \mathbb{R}^{\Gamma}$ where $f(x) = x$ and $g(x) = 0$. Although $R(D)$ is one-dimensional, it does not contain the usual line segment between any two points in it. Letting $\mathbb{1}$ denote the constant function taking the value 1 at all points, we consider functions in $R(D)$ modulo addition of $\mathbb{1}$, i.e. translation.

Lemma 11 *The set $R(D)/\mathbb{1}$ does not contain any nontrivial ordinary convex sets.*

Proof Let $f, g \in R(D)$ be two tropical rational functions that are not translates of each other. Then there is a smooth point $x \in \Gamma$ at which the slopes of f and g differ. Let $0 < \lambda < 1$ be such that the convex combination $\lambda f + (1-\lambda)g$ has a non-integer slope at x. Then $\lambda f + (1-\lambda)g$ is not a tropical rational function, so it is not in $R(D)$.

Recall that $R(D)/\mathbb{1}$, i.e. $R(D)$ modulo tropical scaling can be identified with the linear system $|D| := \{D+(f) : f \in R(D)\}$ via the map $f \mapsto D+(f)$. In what follows, elements of $|D|$ and elements of projectivized $R(D)$, i.e. $R(D)/\mathbb{1}$, will be used interchangeably.

A choice of model $G(\Gamma)$ induces a polytopal cell decomposition of $\mathrm{Sym}^d\,\Gamma$, the d^{th} symmetric product of Γ. Andreas Gathmann and Michael Kerber [GK08] as well as Grigory Mikhalkin and Ilia Zharkov [MZ06] describe $|D|$ as a cell complex $|D|_{G(\Gamma)} \subset \mathrm{Sym}^d\,\Gamma$. Let us coordinatize this construction.

We identify each open edge $e \in E$ with the interval $(0, \ell(e))$ thereby giving the edge a direction, and we identify $\mathrm{Sym}^k\,e$ with the open simplex $\{x \in \mathbb{R}^k : 0 < x_1 < \ldots < x_k < \ell(e)\}$. A cell of $|D|$ is indexed by the following discrete data:

- $d_v \in \mathbb{Z}$ for every vertex $v \in V$,
- a composition (i.e. an ordered partition) $d_e = d_e^{(1)} + \cdots + d_e^{(r_e)}$ for every edge e of Γ, and
- an integer m_e for every edge e of Γ.

Then, a divisor D' belongs to that cell if

- $d_v = D'(v)$ for all $v \in V$,
- D' is given on e by $\sum_i d_e^{(i)} x_i$ for $0 < x_1 < \ldots < x_{r_e} < \ell(e)$, and
- the slope of f at the start of edge e is m_e, where f is such that $(f)+D = D'$.

The intersection of $|D|$ with an open cell of $\mathrm{Sym}^d\,\Gamma$ is a union of cells of $|D|$.

This cell complex structure depends on the choice of the model $G(\Gamma)$, but not on the choice of representative divisor D in the linear system $|D|$. In particular, choosing a finer model amounts to subdividing the cell complex $|D|$, and choosing a different divisor $D' = D + (g)$ amounts to changing the integer slopes at the starting points on the edges by the slopes of g, but this does not change the cells. Whenever we talk about a cell complex structure of $|D|$, we are impliciting assuming a model $G(\Gamma)$. Unless Γ is a circle, there is a unique coarsest model with the least number of vertices.

Example 12 Let Γ be a circle (for example a single vertex v with a loop edge e attached). Consider D to be the divisor $3v$. As we analyze in Example 17,

$|D|$ contains two 2-cells in this case. The elements of both cells are divisors $D' = x + y + z$ with distinct points x, y, and z on the interior of e. However the two 2-cells differ from one another by the slope of the function f (defined by $D' = D + (f)$) at v. The outgoing slopes of f at v are given by $[-2, -1]$ for one 2-cell and by $[-1, -2]$ for the other.

This example shows that the combinatorial type of the divisor D' – the cell of $\operatorname{Sym}^d \Gamma$ containing D' – does not determine the cell of $|D|$ containing D'. The different cells of $|D|$ in one cell of $\operatorname{Sym}^d \Gamma$ are indexed by the slopes of f.

Proposition 13 *For $D' \in |D|$ let $I_{D'}$ be the set of points in the support of D' that lie in the interior of edges. Then the dimension of the carrier of D' is one less than the number of connected components of $\Gamma \backslash I_{D'}$.*

Here, the carrier of D' is the cell containing D' in its interior. Recall that Γ is connected, and note that being in the interior of an edge depends on the model $G(\Gamma)$.

Proof First, suppose $\Gamma \backslash I_{D'}$ is connected, that is, the chips on the interior of edges do not disconnect Γ. As in the proof of Theorem 6(b), the combinatorial type of D' together with the slope data which specifies the cell of $|D|$ determines the function f with $D' = D + (f)$ up to tropical scaling. Hence, D' is a vertex and its carrier cell has dimension zero.

Next, suppose $\overline{\Gamma \backslash I_{D'}}$ has k connected components $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$, where $k \geq 2$. Let $C_i := \overline{\Gamma_i} \cap \overline{\Gamma \backslash \Gamma_i}$ be the set of points in the boundary of Γ_i for each $i = 1, \ldots, k$. Points in C_i are in $I_{D'}$, so they lie in the interiors of edges. Then Γ_i can fire: there is a weighted chip firing move f_i moving the $D'(x)$ chips at $x \in C_i$ a distance of $\varepsilon / D'(x)$ away from Γ_i while maintaining the combinatorial structure of D'. That is, $D' + (f_i)$ stays in the same cell as D'. In our cell coordinates, the f_i yield k segments ξ_i with components $\xi_{i,x} = \pm\varepsilon / D'(x)$ for $x \in C_i$. We claim that those segments span a $(k-1)$-dimensional space and thus the cell in question has dimension at least $k - 1$. So suppose $\sum \lambda_i \xi_i = 0$. Whenever $x \in \Gamma_i \cap \Gamma_j$ $(i \neq j)$, the corresponding cell coordinate tells us that $\lambda_i = \lambda_j$. Because Γ is connected, all λ_i's must be the same, and the space of dependencies among the ξ_i is one-dimensional as claimed.

On the other hand, the only subgraphs that can fire D' without changing the combinatorial type are unions of Γ_i's. Whenever $D'' = D' + (f)$ is a divisor in the cell of D' near D', the function f is necessarily a weighted chip firing move. So the cell has dimension at most $k - 1$.

Theorem 14 *Let G be a model for Γ, and let $\mathcal{S}_G$ be the set of functions $f \in R(D)$ such that the support of $D + (f)$ does not contain an interior cut set (i.e. a cut set consisting of points in interior of edges in the model G). Then*

(a) $\mathcal{S}_G$ contains the set $\mathcal{S}$ from Theorem 6,
(b) $\mathcal{S}_G$ is finite modulo tropical scaling, and
(c) $\mathcal{S}_G = \{f \in R(D) : D + (f) \text{ is a vertex of } |D|\}$.

Proof The statement (a) follows from definitions since points in the interior of edges (for any model) are smooth, and the statement (b) can be shown in the exact same way as Theorem 6(b). By the previous proposition, any element of $\mathcal{S}_G$ has dimension 0. This shows (c).

This shows in particular that the cell complex $|D|$ has finitely many vertices. If the model G is the coarsest one, i.e. the vertices of G are non-smooth points of Γ, then $\mathcal{S}_G = \mathcal{S}$. If Γ is a circle, then there is no unique coarsest model.

Proposition 15 *Each closed cell in the cell complex is finitely-generated as a tropical semi-module by its vertices. In particular, it is tropically convex.*

Proof We proceed as in the proof of Theorem 6(c). Let f be a tropical rational function in $R(D)$. If $\text{supp}(D + (f))$ contains an interior cut set, then we can fire two complementary components in each direction as far as possible. This amounts to writing f as the tropical sum of two other functions g_1, g_2 in the boundary of the same closed cell. By induction on the dimension of the carrier, g_1 and g_2 can be written as a tropical linear combination of the vertices of the closed cell containing them.

Example 16 (Line Segment) Any tree is a genus zero tropical curve. Like genus zero algebraic curves, two divisors on a tree are linearly equivalent if and only if they have the same degree d. The simplest tree is a line segment consisting of an edge e between two vertices, say v_1 and v_2. In this case, $|D|$ is a d-simplex. The vertices of $|D|$ correspond to ordered pairs $[d_1, d_2]$ summing to d associated to the chip configuration at v_1 and v_2.

Example 17 (Circle) A circle is the only tropical curve where the canonical divisor K is 0. Let Γ be homeomorphic to a circle and let D be of degree 3. Then $D \sim 3x$ for some point $x \in \Gamma$. The coarsest cell structure of $R(D)$ is a triangle, but it is not realized by any model on Γ because Γ does not have a unique coarsest model. If the model contains only one vertex v and $D \sim 3v$, then $R(D)$ is a triangle subdivided by a median; see Figure 2. In particular $|D|$ contains four 0-cells, five 1-cells, and two 2-cells. If the model $G(\Gamma)$ consists of a vertex u such that $D \not\sim 3u$, then the cell complex structure would be different. If the model $G(\Gamma)$ consists of 3 equally spaced vertices v_1, v_2, v_3, and $D \sim 3v_1$, then $R(D)$ is isomorphic as a polyhedral complex to the barycentric subdivision of a triangle.

Example 18 (Circle with higher degree divisor) Let Γ be a circle graph with only a single vertex v and a single edge e, a loop based at v. Let $D = dv$; then the linear system $|D|$ is a cone over a cell complex, which we denote as $P_d(circle)$, which has an f-vector given by the following:

$$\text{The number of } i\text{-cells of } P_d(circle) = f_i = (i+1)\binom{d}{i+2}.$$

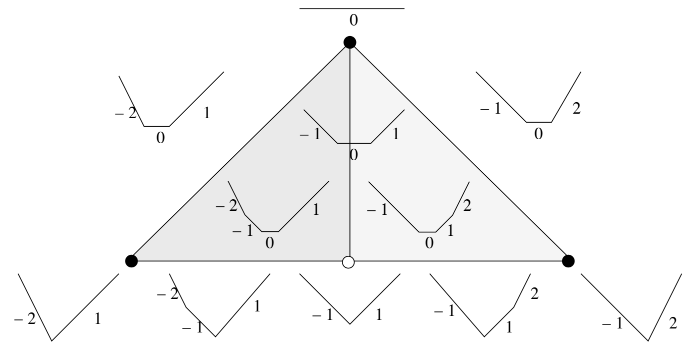


Fig. 2 The polyhedral cell complex $R(3v)/\mathbb{1}$ on $\Gamma = S^1$. The three black vertices are the extremals, and they correspond to the three divisors which are linearly equivalent to $3v$ and have the form $3w$. We have presented S^1 as the line segment $[0, 1]$ with points 0 and 1 identified.

Consequently, the f-vector for $|D|$ is given by

$$\begin{cases} \binom{d}{2} + 1 \text{ if } i = 0 \\ (i+1)\binom{d}{i+2} + i\binom{d}{i+1} \text{ if } i \geq 1. \end{cases}$$

To see how to get these f-vectors, we note that a divisor $D' \sim dv$ corresponds to a tropical rational function f such that $dv + (f) = D'$. One such f is the zero function, this corresponds to the cone point. Each other tropical rational function is parameterized by an increasing sequence of integer slopes $(a_1, \ldots, a_{i+2})$ such that $a_1 < 0$, $a_{i+2} > 0$, and $a_{i+2} - a_1 \leq d$. The first slope must be negative and the last slope must be positive so that the values of f at the two ends of the loop e agree. The cells not incident to the cone point yield the cell complex $P_d(circle)$, and are given by sequences $(a_1, \ldots, a_{i+2})$ such that all $a_i \neq 0$. To finish the computation of the f-vector for $P_d(circle)$, we pick an ordered pair $[j, k]$ with $j, k \geq 1$ and $j + k = i + 2$ to denote the number of negative and positive a_k's, respectively. After setting $a_1 = -\ell$, we note that the number of ways to pick the remaining negative a_k's is given by $\binom{\ell-1}{j-1}$, and the number of ways to pick a subset of positive a_k's such that $a_{i+2} - a_1 \leq d$ is given by $\binom{d-\ell}{k}$. Summing over possible ℓ, and using a standard identity involving binomial coefficients (for instance see [BQ03, Identity 136]), we obtain $\binom{d}{i+2}$ such tropical rational functions for each $[j, k]$. Since there are $i + 2$ such $[j, k]$'s, we get the above number of i-cells not incident to the cone point. For the case of $d = 4$, see Figure 3.

Example 19 (Circle. Cell structure of $|D|$ as a simplex) In Examples 17 and 18, we saw that having to choose a model, even one with only one vertex, gives $|D|$ a cell structure of a subdivided simplex. Moreover, different choices of models, even if they contain only one vertex each, may give combinatorially different cell complex structures for $|D|$. We wish to describe $|D|$ as a simplex.

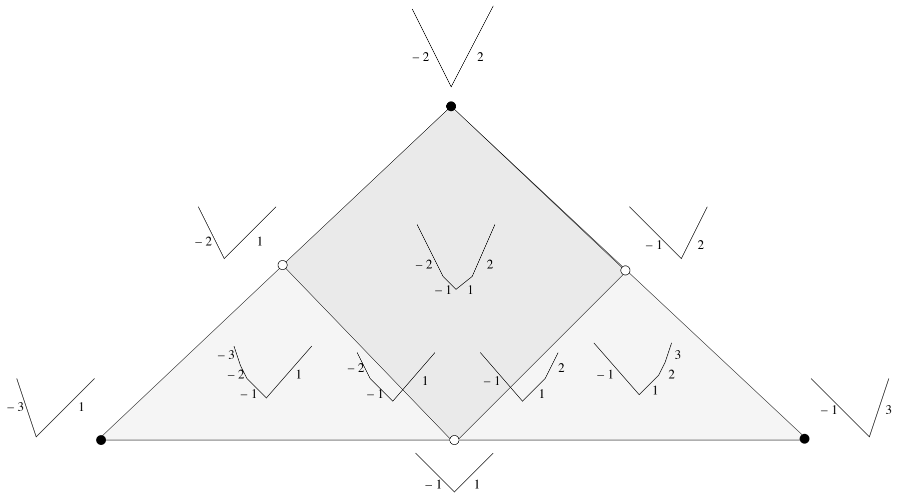


Fig. 3 The polyhedral cell complex $R(4v)/\mathbb{1}$ on $\Gamma = S^1$ is a subdivided tetrahedron, a cone over this subdivided triangle with the cone-point corresponding to the constant function. (The labels of most 1-cells are suppressed, but may be read off from the incident vertices or 2-cells.) The cone-point plus the three black vertices are the extremals.

First, let us look at the embedding of $|D|$ in the symmetric product of the tropical curve. Let Γ be the circle $\mathbb{R}/\mathbb{Z}$, and $D = d \cdot [0]$ be a divisor of degree d. The embedding of $|D|$ in $\mathrm{Sym}^d \Gamma = \mathrm{Sym}^d(\mathbb{R}/\mathbb{Z})$ is given by

$$\{x \in (0,1]^d \,:\, 0 < x_1 \le x_2 \le \ldots x_d \le 1, \quad x_1 + x_2 + \cdots + x_d \in \mathbb{Z}\}.$$

To see this, first consider a tropical rational function g on the line segment $[0,1]$ with $(g) = x_1 + x_2 + \cdots + x_d - d \cdot 0$ and $g(1) = 0$. Then $g(0) = x_1 + x_2 + \cdots + x_d$. If $g(0) \in \mathbb{Z}$, then adding g and a function l with constant slope $g(0)$ on $[0,1]$ gives a tropical rational function $f = g + l$ on the circle with $(f) + D = x_1 + x_2 + \cdots + x_d$. It is easy to check that any $f \in R(D)$ can be obtained this way. Although this description gives $|D|$ a uniform coordinate system, this does not give us a cell complex structure.

In fact, $|D|$ can be realized as a $(d-1)$-dimensional simplex, on d vertices. There is a unique set of d points $v_1, v_1, \ldots, v_d$ in Γ such that $D \sim dv_i$ for all $i = 1, \ldots, d$. These d points are equally spaced along Γ. The extremals of $R(D)$ are

$$\mathcal{E} = \{f \in R(D) : (f) + D = d \cdot v_i \text{ for some } i = 1, 2, \ldots, d\}.$$

Consider the $(d-1)$-dimensional simplex on vertices $V = \{dv_1, dv_2, \ldots, dv_d\}$, that is, the simplicial complex containing a $(k-1)$-dimensional cell for any k subset of V. We would like to stratify $|D|$ into these cells. For any divisor $D' \in |D|$, elements in the same cell as D' are obtained from D' by weighted chip firing moves that do not change the cyclically-ordered composition $d = a_1 + a_2 + \cdots + a_k$ associated to divisor $a_1x_1 + a_2x_2 + \cdots + a_kx_k$ where $x_1, x_2, \ldots, x_k$ are distinct and cyclically ordered along the circle (with a fixed orientation).

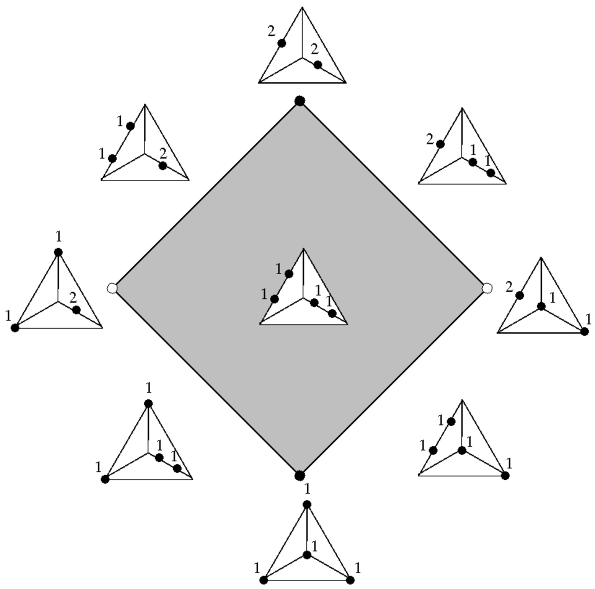


Fig. 4 A non-simplicial cell in the canonical linear system $|K|$ where Γ is the complete graph on four vertices with edges of equal length.

The complement of the support of $D' = a_1x_1 + a_2x_2 + \cdots + a_kx_k$ consists of k segments. For each of these segments, there is a unique extremal in $R(D')$ that is maximal and constant on it. These k extremals of $R(D')$, which are naturally identified with extremals of $R(D)$, are precisely the vertices of the cell of D' and their convex hull is the cell of D'.

Example 20 (K_4 continued) As in Example 10, consider the graph K_4 with equal edge lengths and the canonical divisor K. The coarsest cell structure of $|K|$ consists of 14 vertices and topologically is the cone over the Petersen graph shown in Figure 1. The cone point is the canonical divisor K. The "cones" over the 3 subdivided edges of the Petersen graph are quadrangles. The maximal cells of $|K|$ consist of 12 triangles and 3 quadrangles. In particular, $|K|$ is not simplicial. The quadrangle obtained from "coning" over the bottom edge of the Petersen graph is shown in Figure 4.

4.1 Local structure of a cell complex

If B is a cell complex and x is a point in B, then the link(x, B) denotes the cell complex obtained by intersecting B with a sufficiently small sphere centered at x. We will define a triangulation of link$(D, |D|)$ which is finer than the cell structure. Note that $|D|$ and $|D'|$ are isomorphic as cell complexes, so link$(D, |D|) \cong$ link$(D, |D'|)$ for any $D' \sim D$.

Let $D' \in \mathrm{link}(D, |D|)$ and f be a rational function such that $D' = D+(f)$. Let $h_0 > h_1 > \cdots > h_n$ be the values taken on by f on the set of points that are either vertices of Γ or where f is not smooth. Notice that h_0 and h_n are maximum and minimum values of f, respectively. Since $D+(f) \in \mathrm{link}(D, |D|)$, we may assume that $h_0 - h_n$ is sufficiently small. Let $G = (\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_n = \Gamma)$ be a chain of subgraphs of Γ where $\Gamma_i = \{x \in \Gamma : f(x) \geq h_i\}$.

Let $G' = (\Gamma_1' \subset \Gamma_2' \subset \cdots \subset \Gamma_n' = \Gamma)$ be the chain of *compactified* graphs, where Γ_i' is the union of edges of Γ_i that are between two vertices of Γ. Each cell can be subdivided by specifying more combinatorial data: the chain G' obtained this way and the slopes at the non-smooth points. We call this the *fine subdivision.*

For an effective divisor D, we can naturally associate the *firing poset* $\mathcal{P}_D$ as follows. An element of $\mathcal{P}_D$ is a weighted chip firing move without the information about the length, i.e. it is a closed subgraph $\Gamma' \subset \Gamma$ together with an integer c_e for each out-going direction e of Γ' such that for each point $x \in \Gamma'$ we have $\sum c_e \leq D(x)$ where the sum on the left is taken over the all outgoing directions e from x and $D(x)$ denotes the coefficient of x in D. We say that $(\Gamma', c') \leq (\Gamma'', c'')$ if $\Gamma' \subset \Gamma''$ and $c_e' \geq c_e''$ for each common outgoing direction e of Γ' and Γ''.

Theorem 21 *The fine subdivision of the link of a divisor D in its linear system $|D|$ is a geometric realization of the order complex of the firing poset $\mathcal{P}_D$.*

Proof By the discussion above, a cell in a fine subdivision $\mathrm{link}(D, |D|)$ corresponds to a unique chain in the firing poset. For any chain in the firing poset, we can construct an element in $\mathrm{link}(D, |D|)$ by performing the weighted chip firing moves in the order given by the chain, starting from the smallest element. The element constructed this way defines a cell in the fine subdivision.

Note that the link of an element in $|D|$ does not depend on the precise location of the chips, but on the combinatorial data of the location. In other words, changing the edge lengths, without changing which edges the chips are on, does not affect the combinatorial structure of the link.

This Theorem, along with Proposition 13 allows us to explicitly describe the 1-cells incident to a 0-cell D' of $|D|$. For this, we need to define a specific subset of the weighted chip-firing moves. In particular, we call a weighted chip-firing move f (which is constant on Γ_1 and Γ_2) to be *doubly-connected* if Γ_1 and Γ_2 are both connected subgraphs.

Proposition 22 *Given $D' \in |D|$, and a model G such that* $\mathrm{supp}(D') \subset V(G)$ *(so that D' is a 0-cell in $|D|$), the 1-cells incident to D' correspond to the set of doubly-connected weighted chip-firing moves that are legal on chip configuration D' (up to combinatorial type).*

Proof Let f be a weighted chip-firing move which is legal at D' that is constant on Γ_1 and Γ_2 such that $f(\Gamma_2) = f(\Gamma_1) - \epsilon$ for small $\epsilon > 0$. Then D'', defined as $D' + (f)$ has a chip on each of the line segments L_i connecting Γ_1 and Γ_2. Then the dimension of the corresponding cell of D'' is one if and only if Γ_1 and Γ_2 are both connected.

4.2 Bergman subcomplex of $|K|$

Now we analyze the linear systems of an important family of divisors. The *canonical divisor* K on Γ is

$$K := \sum_{x \in \Gamma} (\mathrm{val}(x) - 2) \cdot x.$$

Notice that since vertices of valence two do not contribute to the divisor K, the definition of K does not depend on the choice of model.

Let M be a matroid on a ground set E. The *Bergman fan* of M is the set of $w \in \mathbb{R}^E$ such that w attains its maximum at least twice on each circuit C of M. The only matroids considered here are *cographic matroids* of graphs. For a graph G with edge set E, the cographic matroid is the matroid on the ground set E whose dependent sets are cuts of G, i.e. the sets of edges whose complement is disconnected. The *Bergman complex* is the cell complex obtained by intersecting the Bergman fan with a sphere centered at the origin. The following result will be useful to us later.

Theorem 23 *[AK06]*

1. *The Bergman complex (with its fine subdivision) is a geometric realization of the order complex of the lattice of flats of M.*
2. *The Bergman fan is pure of codimension rank(M).*

Note that adding or removing parallel elements does not change the simplicial complex structure of the Bergman complex because the lattice of flats remains unchanged up to isomorphism. In particular, if G_1 and G_2 are two graphs, forming two models of the same tropical curve, then the corresponding cographic matroids have isomorphic Bergman complexes.

Lemma 24 *A subset of edges of a graph forms a flat of the cographic matroid if and only if its complement is a union of circuits of the graph.*

Proof The rank of a set A of edges is one more than the size of A minus the number of connected components of the complement of A. Hence A forms a flat if and only if there is no other edge outside of A such that removing the edge increases the number of connected components of the complement. This happens if and only if no connected component of the complement contain a one element cut set, i.e. the connected components are unions of circuits. Loops are considered circuits.

Suppose Γ has genus at least one but K_Γ is not effective. Let Γ' be the subgraph of Γ obtained by removing all the leaf edges recursively. Then the canonical divisor K' of Γ' is effective, and we can apply the following arguments for K' in Γ' or Γ.

Theorem 25 *The fine subdivision of* $\mathrm{link}(K, |K|)$ *contains the fine subdivision of the Bergman complex* $B(M^*(\Gamma))$ *as a subcomplex.*

Proof The complement of a flat is a union of cocircuits, so the lattice of flats is isomorphic to the lattice of unions of cocircuits, ordered by reverse-inclusion. The cocircuits of the cographic matroid are the circuits of the graph. For the canonical divisor K, the proper union of circuits can always fire. Hence the proper part of the poset of union of circuits is a subposet of the firing poset, and so is the proper part of the lattice of flats.

The Bergman complex may be a proper subcomplex of the link because there may be subgraphs that can fire on the canonical divisor but that are not union of circuits, e.g. two triangles connected by an edge in the graph of a triangular prism. Moreover, if Γ is not trivalent, there may be vertices that can fire more than one chip on each edge, so the firing poset may be strictly larger and so can the dimension of the order complex.

Example 26 (K_4 continued)

Let Γ be a tropical curve with the complete graph on four vertices as a model, with arbitrary edge lengths. Consider the canonical divisor K. In this case, the firing poset coincides with the lattice of unions of circuits, which is anti-isomorphic to the lattice of flats. Hence the link of the canonical divisor is isomorphic to the Bergman complex of the cographic matroid on the complete graph. Since the complete graph on four vertices is self-dual, its co-Bergman complex is the space of trees on five taxa, which is the Petersen graph [AK06]. See Figure 5.

In the case when all edge lengths are equal, the quadrangles of $|K|$ described in Example 20 are subdivided in this fine subdivision of the $\mathrm{link}(K, |K|)$. Note that the link of the canonical divisor stays the same when we vary the edge lengths, while the generators and cell structure of $R(K)$ in Figure 1 may change.

5 The induced map and projective embedding of a tropical curve

A finite set $\mathcal{F} = (f_1, \ldots, f_r) \subset R(D)$ induces a map $\phi_{\mathcal{F}} \colon \Gamma \to \mathbb{TP}^{r-1}$, defined as $\phi_{\mathcal{F}}(x) = (f_1(x), \ldots, f_r(x))$ for each $x \in \Gamma$,. This is a map into $\mathbb{TP}^{r-1}$ rather than $\mathbb{R}^r$ as we consider $\mathcal{F}$ to be defined up to translation by $\mathbb{1}$.

Theorem 27 *Let* $\langle \mathcal{F} \rangle \subset R(D)$ *be the tropical sub-semimodule of* $R(D)$ *generated by* $\mathcal{F}$*. Then* $\langle \mathcal{F} \rangle / \mathbb{1}$ *is homeomorphic to the tropical convex hull of the image of* $\phi_{\mathcal{F}}$*. In particular, if* $\mathcal{F}$ *generates* $R(D)$*, then* $|D|$ *is homeomorphic to the tropical convex hull of* $\phi_{\mathcal{F}}(\Gamma)$*.*

The *tropical convex hull* of a set is the tropical semi-module generated by the set.

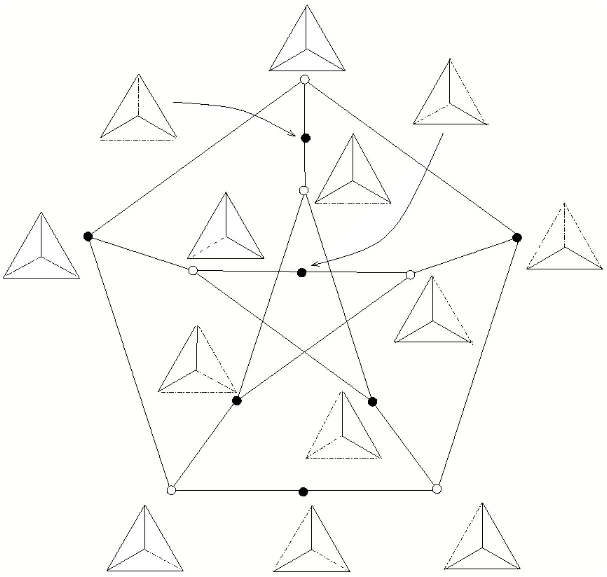


Fig. 5 Link of the canonical divisor in the canonical class, where Γ is the complete graph on four vertices, with arbitrary edge lengths. Order complex of the firing poset. The firing subgraphs in Γ are shown by solid lines. See Example 26. Compare with Figure 2 in [AK06].

Proof The intuition behind this theorem is the result from [DS04] that the tropical convex hull of the rows of a matrix is isomorphic to the tropical convex hull of the columns. Here, the matrix $M_{\mathcal{F}}$ in question has entry $f_i(x)$ in row i and column x. As in [DS04], we define a convex set

$$P_{\mathcal{F}} = \{(y,z) \in (\mathbb{R}^r \times \mathbb{R}^\Gamma)/(\mathbb{1}, -\mathbb{1}) \,:\, y_i + z(x) \geq f_i(x)\}.$$

Let $B_{\mathcal{F}}$ be the union of bounded faces of $P_{\mathcal{F}}$, i.e. $B_{\mathcal{F}}$ contains points in the boundary of $P_{\mathcal{F}}$ that do not lie in the relative interior of an unbounded face of $P_{\mathcal{F}}$ in $(\mathbb{R}^r \times \mathbb{R}^\Gamma)/(\mathbb{1}, -\mathbb{1})$. We will show that $B_{\mathcal{F}}$ projects bijectively onto $\langle \mathcal{F} \rangle / \mathbb{1} \subset \mathbb{R}^\Gamma/\mathbb{1}$ on the one hand, and to $\operatorname{tconv} \phi_{\mathcal{F}}(\Gamma) \subset \mathbb{TP}^{r-1}$ on the other, establishing a homeomorphism.

As in [DS04], we associate a *type* to $(y,z) \in P_{\mathcal{F}}$ as follows:

$$\operatorname{type}(y,z) := \{(i,x) \in [r] \times \Gamma : y_i + z(x) = f_i(x)\}.$$

In other words, a type is a collection of defining hyperplanes that contains (y,z), so elements in the relative interior of the same face have the same type. The recession cone of $P_{\mathcal{F}}$ is $\{(y,z) \in (\mathbb{R}^r \times \mathbb{R}^\Gamma)/(\mathbb{1}, -\mathbb{1}) \,:\, y_i + z(x) \geq 0\}$, which is the quotient of the positive orthant in $(\mathbb{R}^r \times \mathbb{R}^\Gamma)$ by $(\mathbb{1}, -\mathbb{1})$. Hence, a point $(y,z) \in P_{\mathcal{F}}$ lies in $B_{\mathcal{F}}$ if and only if we cannot add arbitrary positive multiples of any coordinate direction to it while staying in the same face of $P_{\mathcal{F}}$, which means keeping the same type. This holds if and only if

(1) The projection of $\operatorname{type}(y,z)$ onto $[r]$ is surjective, and

(2) The projection of type(y, z) onto Γ is surjective.

For $(y, z) \in P_{\mathcal{F}}$, these two conditions are equivalent respectively to

(1′) $y_i = \max\{f_i(x) - z(x) : x \in \Gamma\}$ for all $i \in [r]$, i.e. $y = M_{\mathcal{F}} \odot -z$, and
(2′) $z(x) = \max\{f_i(x) - y_i : i \in [r]\}$ for all $x \in \Gamma$, i.e. $z = -y \odot M_{\mathcal{F}}$.

where $M_{\mathcal{F}}$ is the $[r] \times \Gamma$ matrix with entry $f_i(x)$ in row i and column x, and $\odot$ is tropical matrix multiplication. These two conditions respectively imply that the projections of $B_{\mathcal{F}}$ onto $\mathbb{R}^{\Gamma}/\mathbb{1}$ and $\mathbb{R}^r/\mathbb{1}$ are one-to-one.

On the other hand, let $z \in \langle \mathcal{F} \rangle$, then $z = (u_1 \odot f_1) \oplus \cdots \oplus (u_r \odot f_r) = u \odot M_{\mathcal{F}}$ for some $u \in \mathbb{R}^r$ such that $z \geq u_i \odot f_i$ for each $i = 1, 2, \ldots, r$. Let $y \in \mathbb{R}^r$ such that $y_i = \min\{c : z \geq -c \odot f_i\}$ for $i = 1, 2, \ldots, r$; then $z = -y \odot M_{\mathcal{F}}$, so (y, z) satisfies (2′). Moreover, by construction, $-y_i \odot f_i(x) = z(x)$ for some x, so (y, z) satisfies (1). Thus $(y, z) \in B_{\mathcal{F}}$, and the set $B_{\mathcal{F}}$ projects surjectively onto $\langle \mathcal{F} \rangle/\mathbb{1} \subset \mathbb{R}^{\Gamma}/\mathbb{1}$. Similarly, the image under the projection onto $\mathbb{R}^r/\mathbb{1}$ is the tropical convex hull of image$(\phi_{\mathcal{F}})$, and the homeomorphism follows.

Remark 28 All of the bounded faces of the convex set $P_{\mathcal{F}}$ are in fact vertices. If the union of bounded faces $B_{\mathcal{F}}$ contained a non-trivial line segment, then its projection $\langle \mathcal{F} \rangle/\mathbb{1}$ would as well, contradicting Lemma 11.

Example 29 (Circle, degree 3 divisor) Let Γ be a circle of circumference 3, identified with $\mathbb{R}/3\mathbb{Z}$ and let D be the degree 3 divisor $[0] + [1] + [2]$. Let $f_0, f_1, f_2 \in R(D)$ be the extremals corresponding to divisors $3 \cdot [0], 3 \cdot [1]$, and $3 \cdot [2]$ respectively, and suppose $f_i([i]) = -1$ for each $i = 0, 1, 2$. Then the image of Γ under $\phi_{\mathcal{F}}$, for $\mathcal{F} = (f_0, f_1, f_2)$ is a union of three line segments between the points

$$\phi_{\mathcal{F}}([0]) = (-1, 0, 0), \quad \phi_{\mathcal{F}}([1]) = (0, -1, 0), \quad \phi_{\mathcal{F}}([2]) = (0, 0, -1) \quad \text{in } \mathbb{TP}^3.$$

In this case, the (max-) tropical convex hull of the image of $\phi_{\mathcal{F}}$ coincides with the usual convex hull and is a triangle. However, it is not the tropical convex hull of any proper subset of image$(\phi_{\mathcal{F}})$. In particular, $|D|$ is not a tropical polytope, i.e. it is not the tropical convex hull of a finite set of points.

Example 30 (Dumbbell graph, canonical divisor) Let Γ be the tropical curve obtained from two cycles connecting them with an edge, as in [GK08, Example 1.11]. Let K be the canonical divisor on it. Then the functions $\mathcal{F} = f_1, f_2, f_3$ corresponding on the three divisors in Figure 6 generate $R(K)$. The image of Γ under $\phi_{\mathcal{F}}$ and its tropical convex hull is also shown in the figure, where $\mathbb{TP}^2$ is identified with $\{(0, x, y) : x, y \in \mathbb{R}\}$ and the coordinates x and y are graphed.

We know from [DS04] that tropically convex sets are contractible.

Corollary 31 *The sets $|D|$ and $R(D)$ are contractible.*

Tropical linear spaces are tropically convex [Spe08], so any tropical linear space containing the image $\phi_{\mathcal{F}}(\Gamma)$ must also contain its tropical convex hull.

Corollary 32 *Any tropical linear space in $\mathbb{TP}^{r-1}$ containing $\phi_{\mathcal{F}}(\Gamma)$ has dimension at least dim$(\langle \mathcal{F} \rangle)$.*

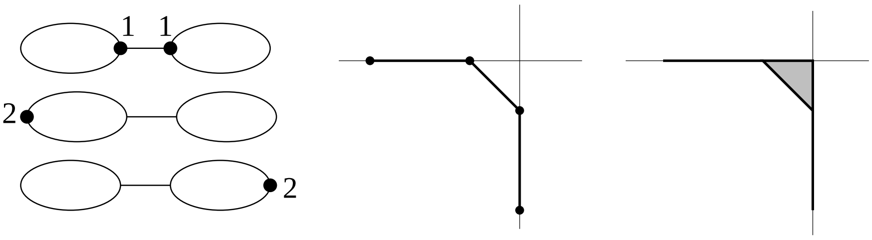


Fig. 6 Example 30. The functions corresponding to the three divisors on the left are used to map the tropical curve Γ to $\mathbb{TP}^2$. The image $\phi_{\mathcal{F}}(\Gamma)$ is the middle figure, and its tropical convex hull is the figure on the right. For example, the image of the leftmost point of Γ in $\mathbb{TP}^2$ is $(0,-2,0)$, and the image of the rightmost point of Γ is $(0,0,-2)$ if the loops have length 2 and the middle edge has lengh 1 in Γ.

6 Embedded and Parameterized Tropical Curves

An *embedded tropical curve* C is a one-dimensional polyhedral complex in $\mathbb{TP}^{n-1} = \mathbb{R}^n/\mathbb{1}$ with rational slopes, together with a *multiplicity* $m_e \in \mathbb{Z}_{>0}$ for each edge e such that the following *balancing condition* holds: For any point $x \in C$, we have $\sum m_e v_e = 0$ in $\mathbb{TP}^{n-1}$ where the sum is taken over all edges e containing x and $v_e \in \mathbb{TP}^{n-1}$ is the *primitive* integral vector in direction e pointing away from x. A vector $v \in \mathbb{TP}^{n-1}$ is called *primitive* if it has an integral representative in $\mathbb{Z}^n$ and generates the semigroup $(\mathbb{Z}^n \cap (\mathbb{R}_+ v + \mathbb{R}[\mathbb{1}]))/\mathbb{Z}[\mathbb{1}]$.

Suppose the embedded tropical curve C has k unbounded rays in primitive directions $v_1, \ldots, v_k \in \mathbb{TP}^{n-1}$, with multiplicities $m_1, \ldots, m_k$ respectively. For each v_i, choose the unique representative in $\mathbb{Z}^n$ such that all the coordinates are non-positive, and the maximum coordinate is zero. Because of the balancing condition, the sum $\sum_{i=1}^k m_i v_i = -d \cdot \mathbb{1}$ for a positive integer d. This integer d is called the *degree* of the embedded tropical curve C.

A *parameterized tropical curve* is a tuple (Γ, h) where Γ is an abstract tropical curve and $h : \Gamma \to \mathbb{TP}^{n-1}$ is a continuous function such that

1. the map h is piecewise-linear with integer slopes: for each point $x \in \Gamma$ and edge e adjacent to x, a neighborhood of x along e can be identified with a real interval $[0, l)$ (and x with 0) while preserving lengths, so that in this neighborhood h has the form $h : [0, l) \ni t \mapsto h(x) + t\, m_{e,x}\, v_{e,x} \in \mathbb{TP}^{n-1}$ where $v_{e,x}$ is a primitive integer vector and $m_{e,x}$ is a non-negative integer; moreover, h is linear except at finitely many places.
2. the *balancing condition* holds: For any finite point $x \in \Gamma$, $\sum m_{e,x} v_{e,x} = 0$ in $\mathbb{TP}^{n-1}$ where the sum is taken over all out-going directions e at x in Γ.

A point $x \in h(\Gamma)$ is called *smooth* if the fiber $h^{-1}(x)$ is finite and consists of points at which h is smooth. The smooth points are dense in $h(\Gamma)$. Let e be an edge of $h(\Gamma)$ and let x be a smooth point on it. Let the *multiplicity* of e in $h(\Gamma)$ be $\sum_{y \in h^{-1}(x)} m_y$ where h is of the form $t \mapsto t\, m_y\, v + x$ locally along either of the two outgoing directions from y that do not get contracted, and v is primitive. With this definition of multiplicity, the image $h(\Gamma)$ is an embedded tropical curve as defined earlier. The *degree* of a parameterized tropical curve (Γ, h) is the degree of the embedded tropical curve $h(\Gamma)$. The statement that (Γ, h) is a parameterized tropical curve is stronger than the statement that $h(\Gamma)$ is an embedded tropical curve.

Let Γ be an abstract tropical curve and D be a divisor such that $R(D)$ is not empty and not equal to $\mathbb{1}$. Choose a nonempty finite set of sections $\mathcal{F} = \{f_1, \ldots, f_n\} \subset R(D)$; then we get a map $\phi_{\mathcal{F}} : \Gamma \to \mathbb{TP}^{n-1}$. In general, the tuple $(\Gamma, \phi_{\mathcal{F}})$ is not a parameterized tropical curve because it does not satisfy the balancing condition.

For $x \in \Gamma$, let $u(x) = \sum m_{e,x} v_{e,x} \in \mathbb{TP}^{n-1}$ where the sum is as in the balancing condition for parameterized tropical curves stated above. Choose a representative of $u(x)$ in $\mathbb{Z}^n$ such that all the coordinates $u(x)_i$ are nonnegative and the minimum coordinate is 0. A point x satisfies the balancing condition if and only if $u(x) = 0$.

For the next results we need the following definition, inherited from classical algebraic geometry. We say that a set of divisors $\mathcal{D} = \{D_i\}_{i \in \mathcal{I}}$ has *no base points* if for every $x \in \Gamma$, there is a $D' \in \mathcal{D}$ with $D'(x) = 0$. For a fixed divisor D, we may say that a set of functions $\mathcal{F} \subset R(D)$ has no base points if $\mathcal{D}(\mathcal{F}) := \{D + (f) : f \in \mathcal{F}\}$, has none.

Lemma 33 *If $\mathcal{F} = \{f_1, \ldots, f_n\}$ has no base points, then the i^{th} coordinate $u(x)_i$ of $u(x)$ is the coefficient of x in the divisor $D + (f_i)$.*

Proof Since each f_i is in $R(D)$, we have $(D + (f_i))(x) \geq 0$. Since $\mathcal{F}$ has no base points, for every x, $(D + (f_i))(x) = 0$ for some $f_i \in \mathcal{F}$. By construction of $\phi_{\mathcal{F}}$, the i^{th} coordinate of the vector $m_{e,x} v_{e,x}$ is the outgoing slope of f_i at x along e. Hence $u(x)_i = (f_i)(x)$, and

$$u(x)_i - u(x)_j = (f_i)(x) - (f_j)(x) = (D + (f_i))(x) - (D + (f_j))(x).$$

This proves the assertion.

Corollary 34 *If $\mathcal{F} = \{f_1, \ldots, f_n\}$ has no base points, then the tuple $(\Gamma, \phi_{\mathcal{F}})$ satisfies the balancing condition as a parameterized tropical curve if and only if for each $i = 1, \ldots, n$, the support of the divisor $D + (f_i)$ contains only points at infinity.*

Now we describe a natural way to modify $(\Gamma, \phi_{\mathcal{F}})$ to obtain a parameterized tropical curve $(\tilde{\Gamma}, \tilde{\phi_{\mathcal{F}}})$. Let $\tilde{\Gamma}$ be a tropical curve obtained from Γ by attaching a leaf edge with infinite length at every finite point $x \in \Gamma$ at which the balancing condition is not satisfied. By Lemma 33, these points are precisely the union of supports of f_i's. Let $\tilde{f}_i$ be a tropical rational function $\tilde{\Gamma}$ such that $\tilde{f}_i|_{\Gamma} = f_i$ and the support of divisor $D + (\tilde{f}_i)$ contains only points at infinity. Let $\tilde{\mathcal{F}} = (\tilde{f}_1, \ldots, \tilde{f}_r)$. By Corollary 34, $(\tilde{\Gamma}, \phi_{\tilde{\mathcal{F}}})$ satisfies the balancing condition and is a parameterized tropical curve. The points at infinity are considered balanced and we do not attach new leaf edges to them. Note that this *modification* depends not only on the image $\phi_{\mathcal{F}}(\Gamma)$ but on Γ and $\mathcal{F}$ themselves. In particular, modification may not yield the "minimal" embedded

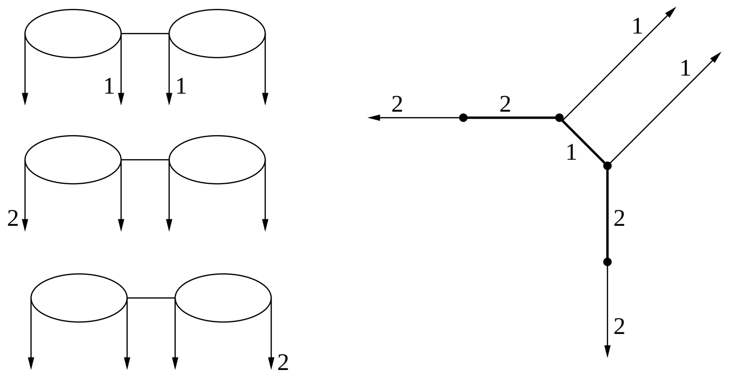


Fig. 7 Example 37. Modified dumbbell graph with its canonical morphism to $\mathbb{TP}^2$. Compare with Figure 6.

tropical curve containing $\phi_{\mathcal{F}}(\Gamma)$. See Example 40. However, it respects the structure of Γ and D.

Theorem 35 *The degree of the embedded tropical curve* $\phi_{\tilde{\mathcal{F}}}(\tilde{\Gamma})$ *equals* $\deg(D)$, *for any base-point-free* $\mathcal{F} \subset R(D)$.

Proof First consider the case when the image $\phi_{\mathcal{F}}(\Gamma)$ is bounded in $\mathbb{TP}^{n-1}$. Then all the unbounded rays of $\phi_{\tilde{\mathcal{F}}}(\tilde{\Gamma})$ are images of the new leaf edges in $\tilde{\Gamma}$. The new leaf edges are attached to the points $x \in \Gamma$ where $u(x) \neq 0$. Such a point x contributes $-u(x)$ to the computation of the degree of the embedded tropical curve $\phi_{\tilde{\mathcal{F}}}(\tilde{\Gamma})$. By Lemma 33, the sum of such $-u(x)$ over all $x \in \Gamma$ is equal to $-\deg(D) \cdot \mathbb{1}$. So $\phi_{\tilde{\mathcal{F}}}(\tilde{\Gamma})$ has degree $\deg(D)$ by definition.

Now suppose the image $\phi_{\mathcal{F}}(\Gamma)$ is not bounded. Without loss of generality, we may assume that the support of D contains no points at infinity. Consider a new abstract tropical curve Γ_0 obtained from Γ by truncating the unbounded leaf edges so that every f_i is linear on $\Gamma \backslash \Gamma_0$ and $\operatorname{supp}(D)$ does not intersect $\Gamma \backslash \Gamma_0$. Then $\mathcal{F}_0 = \{f_1|_{\Gamma_0}, \ldots, f_n|_{\Gamma_0}\}$ is a basepoint free subset of $R_{\Gamma_0}(D)$. Moreover, $\phi_{\tilde{\mathcal{F}}}(\tilde{\Gamma}) = \phi_{\tilde{\mathcal{F}}_0}(\tilde{\Gamma}_0)$ and we are back to the bounded case.

Example 36 (Tropical lines) Suppose Γ is a tree and $deg(D) = 1$. Let the leaf vertices be $v_1, v_2, \ldots, v_n$. Let $f_i \in R(D)$ be the function such that $D+(f_i) = v_i$. Then $\mathcal{F} = \{f_1, \ldots, f_n\}$ gives an embedding $\phi_{\mathcal{F}} : \Gamma \hookrightarrow \mathbb{TP}^{n-1}$, and modification produces a tropical line. The image $\phi_{\mathcal{F}}(v_i)$ of the leaf vertex v_i lies on the unbounded ray in $\phi_{\tilde{\mathcal{F}}}(\tilde{\Gamma})$ pointing in direction $-e_i$.

Example 37 (Dumbbell graph, canonical embedding) Consider the dumbbell graph and its canonical divisor class from Example 30. The modified tropical curve and its embedding as a degree 2 curve in $\mathbb{TP}^2$ is depicted in Figure 7.

Example 38 (Circle, degree 3 divisor) Consider the circle graph with a degree 3 divisor D as in Example 29. The linear system $R(D)$ has three extremals as depicted on the left of Figure 8. The modified tropical curve is embedded as a degree 3 tropical curve.

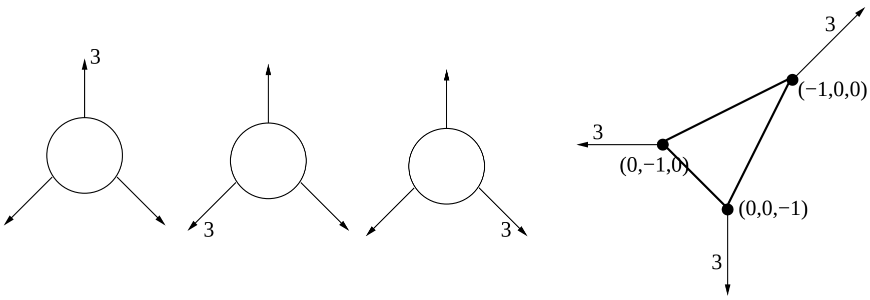


Fig. 8 Examples 29, 38. Modified circle graph with a degree 3 embedding $\mathbb{TP}^2$. The three divisors on the left corresponding to three extremals of $R(D)$.

Example 39 (Embedding a complete graph as the graph of a simplex) Let Γ be the tropical curve obtained from the complete graph on m vertices $v_1, v_2, \ldots, v_m$ with all edge lengths 1, and consider the canonical divisor K on Γ. Let $f_i \in R(K)$ be the function such that $K + (f_i) = m \cdot v_i$. For example, let $f_i = -1$ at v_i and $f_i = 0$ on the edges not adjacent to v_i, and linearly interpolate. Although $\mathcal{F} = \{f_1, f_2, \ldots, f_m\}$ does not generate $R(K)$, the map $\phi_{\mathcal{F}} : \Gamma \to \mathbb{TP}^{m-1}$ is an embedding. We have $\phi_{\mathcal{F}}(v_i) = -e_i$, and all f_i are linear on the interior of all edges, so the image $\phi_{\mathcal{F}}(\Gamma)$ consists of the edges of the simplex with vertices $\{-e_1, -e_2, \cdots, -e_m\}$. Modification attaches an unbounded ray in direction $-e_i$ with multiplicity m at the point $\phi_{\mathcal{F}}(v_i)$ for each $i \in \{1, 2, \ldots, m\}$. The tropical convex hull of $\phi_{\mathcal{F}}$ is a 2-dimensional polyhedral complex and lies in a 2-dimensional tropical linear space in $\mathbb{TP}^{m-1}$ corresponding to the uniform matroid. For $m = 4$, see [RGST, Figure 5a].

Example 40 ($\Gamma = K_4$, $D = K$, $\phi_{\mathcal{F}}$ not injective)

Let Γ be as in the previous example. Let $u_i \in \Gamma$ be the midpoints between v_i and v_4, for $i = 1, 2, 3$. See Figure 9. Let $f_1, f_2, f_3 \in R(K)$ be the functions such that $K + (f_1) = 2u_1 + v_2 + v_3$, $K + (f_2) = 2u_2 + v_1 + v_3$, and $K + (f_3) = 2u_3 + v_1 + v_2$. Then $\mathcal{F} = \{f_1, f_2, f_3\}$ is basepoint free, but $\phi_{\mathcal{F}}$ is not injective. All of v_1, v_2, v_3, v_4 and the edges between v_1, v_2, v_3 are mapped to the same point under $\phi_{\mathcal{F}}$. Their image is the point in the middle in the figure. The image $\phi_{\mathcal{F}}(\Gamma)$ consists of the three solid line segments in the figure. For a point in the interior of a solid line segment, the fiber has cardinality 2. So each of three line segments has multiplicity 2.

The tuple $(\Gamma, \phi_{\mathcal{F}})$ is not yet a parameterized tropical curve because it does not satisfy the balancing condition at the points $u_1, u_2, u_3, v_1, v_2, v_3 \in \Gamma$, which are in the supports of $K + (f_i)$. The point v_4 satisfies the balancing condition. Modification produces a parameterized tropical curve with six unbounded rays as shown in Figure 9. Each of these has multiplicity 2. From this, we see that the tropical curve has degree 4.

The embedded tropical curve $\phi_{\tilde{\mathcal{F}}}(\tilde{\Gamma})$ we obtained this way from the modification is not a "minimal" embedded tropical curve containing $\phi_{\mathcal{F}}(\Gamma)$. If we

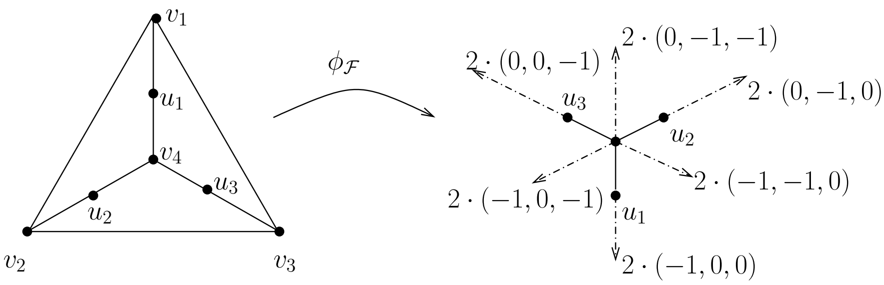


Fig. 9 Example 40. K_4 with a non-injective map $\phi_{\mathcal{F}}$ to $\mathbb{TP}^2$. The dotted rays in the image result from modification.

attached the unbounded rays only to u_1, u_2, u_3, we would obtain a smaller embedded tropical curve of degree two containing $\phi_{\mathcal{F}}(\Gamma)$.

7 Canonical embeddings

In this section we repeat the first steps in the classical theory of ample divisors and canonical embeddings in the tropical setting (compare [Har83, IV.3]). It turns out that for a few classical equivalences, only one implication survives tropical conditions.

Let Γ be a tropical curve and g its genus. The *rank* $r(D)$ of a divisor D is the maximum integer r such that $|D - E| \neq \emptyset$ for all degree-r divisors E. The Riemann-Roch Theorem [GK08, MZ06] (based on work of [BN07]) , which is the same for classical and tropical geometry, says the following:

$$r(D) - r(K - D) = \deg D + 1 - g. \qquad \text{(RR)}$$

We say that $D \geq 0$ is *very ample* if $R(D)$ separates points, that is, if for every $x \neq x' \in \Gamma$ there are $f, f' \in R(D)$ with $f(x) - f(x') \neq f'(x) - f'(x')$. We call D *ample* if some positive multiple kD is very ample.

Lemma 41 *A divisor D is very ample if and only if $\phi_{\mathcal{F}}$ is injective for any set $\mathcal{F}$ that generates $R(D)$.*

Proof The "if" direction is clear. To see the "only if" direction, suppose there exist $x, x' \in \Gamma$ such that $f_i(x) - f_i(x') = f_j(x) - f_j(x')$ for all pairs $f_i, f_j \in \mathcal{F}$; then the same is true for any pair of tropical linear combinations of the f_i, and thus for all pairs $f, f' \in R(D)$.

Lemma 42 *A divisor D is very ample if and only if for all $x \neq x' \in \Gamma$ there is a $D' \in |D|$ whose support contains a smooth cut set separating x and x'.*

Proof Suppose D is very ample with witnesses $f, f' \in R(D)$ for x, x'. Up to relabeling $c := f(x) - f'(x) - (f(x') - f'(x')) > 0$. Then for a generic $\epsilon \in (0, c)$, the support of the divisor $D' := D + (f \oplus \epsilon \odot f')$ contains the smooth cut set $\{x : f(x) - f'(x) = \epsilon\}$.

Conversely, if the support of $D' = D + (f) \in |D|$ separates x from x', we can use a cut function g with $g(x) = f(x) < g(x')$ to construct $f' := f \odot g$.

Lemma 43 *If $r(D) \geq 1$ then $|D|$ has no base points.*

Proof Let $x \in \Gamma$. Choose x' on an edge incident to x. By assumption, there is a $D' \in |D|$ with $D'(x') > 0$. If $D'(x) > 0$ then we can use x' to pull D' away from x, to get $D'' \in |D|$ with $D''(x) = 0$. Thus, x is not a base point.

The converse is false. Consider, for example, a curve Γ of positive genus with a *bridge* edge e (i.e. an edge whose removal disconnects Γ). Then for $x \in e$, $|x|$ has no base points, yet $r(x) = 0$.

Lemma 44 *If D is very ample, then $r(D) \geq 1$. In particular, very ample divisors have no base points.*

Proof Let $x \in \Gamma$. Choose a sequence $x_n \in \Gamma \setminus \{x\}$ converging to x. There are divisors $D_n \in |D|$ so that D_n contains a smooth cut set separating x_n and x. Because $|D| \subset \mathrm{Sym}^{\deg D} \Gamma$ is compact, there is a converging subsequence. Its limit $D' \in |D|$ has $D'(x) > 0$.

Theorem 45 *If $\deg D \geq 2g + 1$, then D is very ample.*

Corollary 46 *Every divisor of positive degree is ample.*

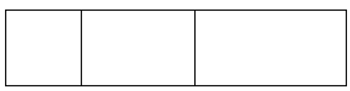
Proof (Proof of Theorem 45) As $\deg K = 2g - 2$, $\deg(K - D) \leq -1$, and $r(K - D) = -1$. By RR we have $r(D) \geq g + 1$.

Now, let $x \neq x' \in \Gamma$, and let $C = \{y_1, \ldots, y_r\}$ be a minimal smooth cut set separating x and x'. Then $\Gamma \setminus C$ has two connected components, U, U', and the corresponding cut divisors agree: $D_U = D_{U'} = y_1 + \ldots + y_r$. Furthermore, the genus of Γ can be computed as $g = g(\overline{U}) + g(\overline{U'}) + r - 1$. In particular, $r(D) \geq g + 1 \geq r$, and $|D - y_1 - \ldots - y_r| \neq \emptyset$.

Let $f \in R(D - y_1 - \ldots - y_r)$, and set $D_0 := D + (f)$. By construction, $D_0 \geq D_U$, and we can use cut functions to separate x from x'.

A tropical curve is *hyperelliptic* if there is a linear system $|D|$ with $\deg D = 2$ and $r(D) = 1$. In particular, it is not a tree.

Example 47 The curve with 2 vertices joined by $g + 1$ edges is hyperelliptic. The following genus 3 curve is hyperelliptic.



Proposition 48 *If $\deg D = 2$, then $\phi_D(\Gamma)$ is a tree. If in addition $r(D) = 1$, then the fiber $\phi_D^{-1}(x) = \{y \in \Gamma : \phi_D(y) = x\}$ has size 1 or 2 for all x in the image.*

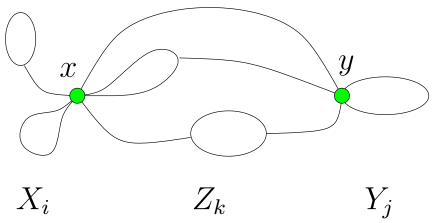
Proof A degree 2 tropical curve in $\mathbb{TP}^{n-1}$ cannot have cycles. For $n = 3$, this follows from the facts that the polytope $2\Delta_2 = 2 \cdot \mathrm{conv}(e_1, e_2, e_3)$ in $\mathbb{R}^3$ does not have any interior lattice points and that any degree 2 plane tropical curve is dual to a regular subdivision of a subset of $2\Delta_2 \cap \mathbb{Z}^3$. For higher n, this follows by induction and looking at projections. By Theorem 35, the image $\phi_D(\Gamma)$ can be modified to an embedded tropical curve of degree 2, so it must be a tree.

Now suppose $r(D) = 1$ and $x \in \Gamma$. We have $\phi_D^{-1}(\phi_D(x)) = \{x\}$ if $\Gamma \backslash x$ is disconnected, i.e. x lies on a bridge. Suppose x lies in a cycle in Γ. Since $r(D) = 1$ and $\deg(D) = 2$, there is a divisor $x + x' \in |D|$. The point $x' \in \Gamma$ is the unique element of $|D-x|$ because x lies in a cycle. Moreover x' lies in every cycle that contains x; otherwise we would have $|D| = \{x + x'\}$, contradicting $r(D) = 1$. Hence every cycle contains either both or none of $\{x, x'\}$. Let Z be a connected component of $\Gamma \backslash \{x, x'\}$. Then $\mathrm{val}_Z(x) = \mathrm{val}_Z(x') = 1$; otherwise there would be a cycle that contains only x or x'. So there is a function $f \in R(x + x')$ such that $f(x) = f(x') = 0$ but $f(q) < 0$ for any q in the relative interior Z°. Together with the constant function $\mathbb{1} \in R(x, x')$, f separates $\{x, x'\}$ from Z°. The maps ϕ_D and $\phi_{x+x'}$ differ only by a translation, so $\phi_D^{-1}(\phi_D(x)) \subset \{x, x'\}$.

Theorem 49 *If K is not very ample, then Γ is hyperelliptic.*

Proof Contracting a bridge edge does not affect either property. So it is safe to assume that Γ has no leaf nodes.

Suppose K is not very ample, and let x, y be two points that cannot be separated by $R(K)$. We claim that $r(x + y) = 1$. The complement $\Gamma \setminus \{x, y\}$ splits into connected components $(X_i^\circ)_{i=1,\ldots,r}$, $(Y_j^\circ)_{j=1,\ldots,s}$, and $(Z_k^\circ)_{k=1,\ldots,t}$, whose closures satisfy $X_i \cap \{x, y\} = \{x\}$, $Y_j \cap \{x, y\} = \{y\}$, and $Z_k \cap \{x, y\} = \{x, y\}$.



Because $r(K) = g - 1$, the linear systems $|K - (g-1)x|$ and $|K - (g-1)y|$ are non-empty. Thus we must have

$$\sum_k \mathrm{val}_{Z_k}(x) \geq g, \text{ and } \sum_k \mathrm{val}_{Z_k}(y) \geq g.$$

Otherwise we can separate x and y, contradicting our assumption. Because Z_k° is connected, the genus $g(Z_k)$ of Z_k satisfies

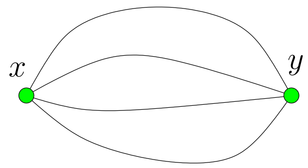
$$g(Z_k) \geq \mathrm{val}_{Z_k}(x) + \mathrm{val}_{Z_k}(y) - 2\,.$$

On the other hand,

$$g = \sum_i g(X_i) + \sum_j g(Y_j) + \sum_k g(Z_k) + t - 1\,.$$

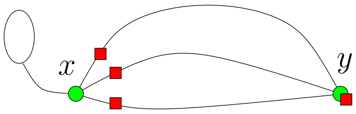
These relations imply $g - 1 \leq t \leq g + 1$. Moreover, at most one of the $g(X_i), g(Y_j), g(Z_k)$ can be positive as we will see below. There are (up to symmetry) three cases.

- $t = g + 1$: We must have $r = s = 0$, and all $g(Z_k) = 0$. Then $r(x+y) = 1$.

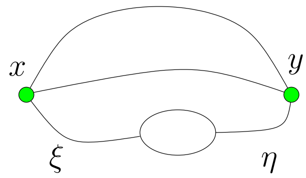


- $t = g$: There are two subcases.

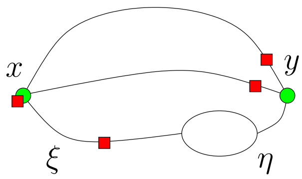
  ∘ $r = 1,\ s = 0$: (or $r = 0,\ s = 1$.) In this case, $|K|$ separates x and y as illustrated in the figure.



  ∘ $r = s = 0$: this is the only subtle case. One Z_k has genus 1. Its cycle has distance ξ from x and distance η from y. At least one of the distances must be non-zero because Z_k° is connected.

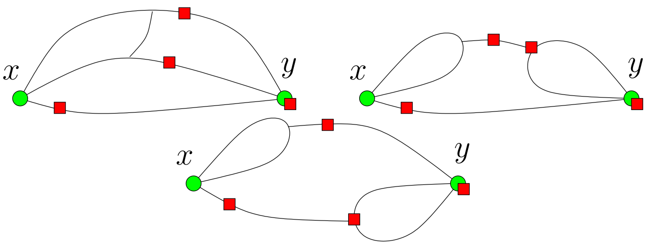


If $\xi \neq \eta$, then $|K|$ separates x and y.



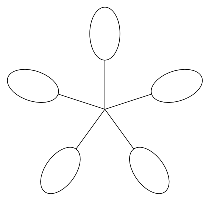
However, if $\xi = \eta$, then $r(x + y) = 1$.

- $t = g - 1$: In this case, all inequalities in our chain have to be sharp. In particular, $r = s = 0$. There is either one Z_k of genus 2 or two Z_k's of genus 1. Either way, $|K|$ separates x and y.



We have proved more than what the theorem states. In fact, we can choose a divisor $x + y$ with $r(x + y) = 1$ and $(g - 1)(x + y) \sim K$.

Sadly, the converse is false – even if we take the stronger statement above into account. For example, a flower with ≥ 3 petals satisfies the stronger condition, yet the canonical divisor is very ample.



The proof of Theorem 49 also yields the following. Here, a tropical curve is *generic* if maximal valency is three after contracting bridge edges, and the edge lengths are generic.

Theorem 50 *The canonical divisor of curves of genus $g \leq 2$ is not very ample. In particular, such curves are hyperelliptic. Generic curves of genus $g \geq 3$ are not hyperelliptic. In particular, their canonical divisor is very ample.*

Problem 51 Give a characterization of curves with K not very ample.

8 Tropical Picard Group and Continuous Chip-Firing

Following the classical definition of the Picard group in algebraic geometry, we define the *tropical Picard group* of a tropical curve as a quotient group. In particular, we take the group of degree zero Γ-divisors modulo the group of principal divisors, i.e. those of the form (f) where f is a tropical rational function. Before describing these groups further, we consider a finite graph analogue.

Given a finite undirected graph $G = (V, E)$ with $V = \{v_0, v_1, \ldots, v_{n-1}\}$, we define the Laplacian matrix of G to be

$$L(G) = D(G) - A(G)$$

where $A(G)$ is the adjacency matrix of G (A_{ij} = the number of edges between v_i and v_j) and $D(G)$ is the diagonal matrix

$$D(G) = diag(\mathrm{val}(v_0), \mathrm{val}(v_1), \ldots, \mathrm{val}(v_{n-1})).$$

We let $L_0(G)$ denote the reduced Laplacian matrix obtained by deleting the row and column corresponding to vertex v_0.

With the above notation, we define the *critical group* $K(G, v_0)$ (following [Big99] or [Dha90]) to be the cokernel

$$K(G, v_0) = \mathbb{Z}^{n-1}/(L_0(G)\ \mathbb{Z}^{n-1}).$$

Choosing a different v_0 gives an isomorphic critical group. For our calculations, we will use a set of explicit coset representatives of $K(G, v_0)$, known as *superstable chip configurations*, following [HLMPPW, Section 4].

A divisor D is a *chip configuration* on graph G *with sink* v_0 if the degree of D is zero and $D(v) \geq 0$ for $v \neq v_0$. We can also think of D as a vector $[D(v_0), \ldots, D(v_{n-1})]^T \in \mathbb{Z}^n$. Given a chip configuration D, we say that vertex $v \in V(G) \setminus \{v_0\}$ is *ready to fire* if $D(v) \geq \mathrm{val}(v)$. The *chip firing move* given by the vertex v is the map $\mathbb{Z}^n \to \mathbb{Z}^n$ given by $D \mapsto D - L_v$ where L_v is the column of the Laplacian matrix $L(G)$ corresponding to the vertex v. In other words, a chip firing move on a chip configuration moves a chip from v to each of its neighbors.

A divisor D is *stable* if for all $v \in V(G) \setminus \{v_0\}$, $0 \leq D(v) < \mathrm{val}(v)$, i.e. no vertex in $V(G) \setminus \{v_0\}$ is ready to fire. The distinguished vertex v_0 is ready to fire in D if and only if D is stable, and a sequence $[v_{\alpha_1}, v_{\alpha_2}, \ldots, v_{\alpha_\ell}]$ is *legal* if $v_{\alpha_{i+1}}$ is ready to fire in each $D^{(i)}$, the chip configuration resulting from firing the sequence $[v_{\alpha_1}, v_{\alpha_2}, \ldots, v_{\alpha_i}]$ on D. A *cluster* $A \subset V(G) \setminus \{v_0\}$ can fire if the result of all vertices $v \in A$ firing simultaneously results in a chip configuration with all coefficients nonnegative. (Note that it is possible for a cluster A to fire even if no ordering of the elements of A gives rise to a legal firing sequence.) Finally, a configuration is *superstable* if no cluster can fire.

We now use these explicit descriptions of elements of critical groups $K(G, v_0)$ to describe the tropical Picard group. We get an explicit group structure on the set of superstable chip configurations by defining the sum of D_1 and D_2 to be $\overline{D_1 + D_2}$, the unique superstable configuration linearly equivalent to $D_1 + D_2$.

Inspired by the terminology of [GK08], a $\mathbb{Q}$-tropical curve is a metric graph Γ having a model G, for which each of the edges have a rational length. An ordinary finite graph can be thought of as a $\mathbb{Q}$-tropical curve where all edge lengths are 1. Given a general metric graph (or tropical curve) Γ, we let $\Gamma_{\mathbb{Q}}$ denote the set of points of Γ whose distance from every vertex is rational. We let $\mathrm{Div}_{\mathbb{Q}}(\Gamma)$ denote the set of divisors on $\Gamma_{\mathbb{Q}}$, also referring to these as the $\mathbb{Q}$-divisor on Γ. Further, $\mathrm{Div}^0_{\mathbb{Q}}(\Gamma)$ will define the set of degree 0 $\mathbb{Q}$-divisors.

If Γ is a $\mathbb{Q}$-tropical curve, we let $\mathrm{Prin}_{\mathbb{Q}}(\Gamma)$ denote the principal $\mathbb{Q}$-divisors, i.e. the subset of $\mathbb{Q}$-divisors which are of the form (f) for f a tropical rational function. We define the $\mathbb{Q}$-tropical Picard group of a $\mathbb{Q}$-tropical curve to be the quotient group $\mathrm{Div}^0_{\mathbb{Q}}(\Gamma)/\,\mathrm{Prin}_{\mathbb{Q}}(\Gamma)$, which we denote as $\mathrm{Pic}_{\mathbb{Q}}(\Gamma)$.

Theorem 52 *The $\mathbb{Q}$-tropical Picard group of a $\mathbb{Q}$-tropical curve Γ is the direct limit of the critical groups corresponding to the subdivisions of Γ.*

A more general version of this Theorem was independently proven by Baker and Faber [BF09, Theorem 2.9 and 2.10]. Given a $\mathbb{Q}$-tropical curve Γ we uniformly scale all edges to get Γ' such that all edge lengths are integers. We define G_0 to be the finite graph obtained by taking the coarsest finite graph structure on Γ', with vertices given by points of valence one or at least three. In the case of the cycle graph, all points are of valence two, so up to symmetry we define G_0 to be the graph with one vertex and one edge which is a loop at that vertex. This is the unique example of Γ with no points of valence one or at least 3.

Without loss of generality, pick v_0 to be any vertex of G_0. Let G_k to be the model obtained by choosing more points as vertices such that all edges of G_k have length $1/k$. Since any vertex of G_0 is a vertex of G_k, it follows that v_0 is also a vertex of G_k for all $k \geq 1$. Let $K(G_k, v_0)$ denote the critical group on the subdivided graph G_k, using superstable configurations as coset representatives.

When k_1 divides k_2, we note that all vertices of G_{k_1} are in G_{k_2}, and thus we can define ψ_{k_1,k_2} to be the map

$$\psi_{k_1,k_2} : K(G_{k_1}, v_0) \to K(G_{k_2}, v_0)$$

sending the superstable configuration $D = \sum_{v \in G_{k_1}} D(v) \cdot v$ to $D' = \sum_{v \in G_{k_2}} D(v) \cdot v$. Note here that the image D' has a coefficient of zero attached to any vertex $v \in G_{k_2}$ which is not in G_{k_1}.

Since the valence of a vertex $v \in G_{k_1} \cap G_{k_2}$ is the same in both of these graphs, it follows that D is a *stable* configuration with respect to G_{k_1} if and only if $\psi_{k_1,k_2}(D)$ is stable with respect to G_{k_2}. We now wish to show that the superstability of D with respect to G_{k_1} implies the superstability of $\psi_{k_1,k_2}(D)$ with respect to G_{k_2}. To see this equivalence, we note the following.

Lemma 53 *If D is superstable with respect to graph G_{k_1} and vertex v_0, then $\psi_{k_1,k_2}(D)$ is superstable with respect to graph G_{k_2} and vertex v_0. Furthermore, if $\overline{D_1 + D_2} = D_3$ in $K(G_{k_1}, v_0)$, then $\overline{\psi_{k_1,k_2}(D_1) + \psi_{k_1,k_2}(D_2)} = \psi_{k_1,k_2}(D_3)$.*

Proof If D is *superstable* with respect to G_{k_1}, then it follows that any cluster of vertices of G_{k_1} outside v_0 cannot fire. In particular, for all subsets $S \subset V(G_{k_1})$, it follows that there exists at least one $v \in S$ such that, $\mathrm{outdeg}_S(v) > D(v)$.

The configuration $\psi_{k_1,k_2}(D)$ has the same support as D by construction. However, to check superstability of $\psi_{k_1,k_2}(D)$, we must consider all clusters of vertices in $G_{k_2} \setminus \{v_0\}$. Consider any subset $S \subset V(G_{k_1}) \setminus \{v_0\}$, such a subset is also a subset of $V(G_{k_2}) \setminus \{v_0\}$. For any $v \in V(G_{k_1})$, the outdegree of v, with respect to S is the same in G_{k_2} as it was in G_{k_1} so such subsets S cannot fire in $V(G_{k_2})$. Any subset $S \in V(G_{k_2})$ formed by adjoining all vertices along a subdivided edge similarly cannot fire. Finally, any subset S which contains a new degree 2 vertex v' in $V(G_{k_2}) \setminus V(G_{k_1})$ but not the two neighbors of v' cannot fire since $\mathrm{outdeg}_S(v') > 0 = D(v')$ for such subsets.

Lastly, if $\overline{D_1 + D_2} = D_3$, then $\psi_{k_1,k_2}(D_1 + D_2) = \psi_{k_1,k_2}(D_1) + \psi_{k_1,k_2}(D_2)$. We also have $\psi_{k_1,k_2}(\overline{D_1 + D_2}) = \overline{\psi_{k_1,k_2}(D_1 + D_2)}$ since for any D_3, $\psi_{k_1,k_2}(\overline{D_3})$ is superstable and is linearly equivalent to $\psi_{k_1,k_2}(D_3)$ so $\psi_{k_1,k_2}(\overline{D_3})$ must equal $\overline{\psi_{k_1,k_2}(D_3)}$.

Proof (Proof of Theorem 52) From Lemma 53, we see that the transition maps preserve superstability and are compatible with addition. Furthermore, the transition maps are injective and satisfy $\psi_{k_1,k_3} = \psi_{k_2,k_3} \circ \psi_{k_1,k_2}$ when $k_1|k_2$ and $k_2|k_3$. We use these transition maps to define the direct limit

$$\overline{K}(\Gamma, v_0) := \varinjlim_{k \geq 1} \{K(G_k, v_0)\} = \bigcup_{k=1}^{\infty} K(G_k, v_0) \Big/ \sim$$

where $D \in K(G_{k_1}, v_0)$ and $D' \in K(G_{k_2}, v_0)$ are equivalent if and only if $\psi_{k_1,k_3}(D) = \psi_{k_2,k_3}(D')$ in $K(G_{k_3}, v_0)$ for $k_3 = \mathrm{lcm}(k_1, k_2)$. Since the ψ_{k_i,k_j}'s are injective group homomorphisms, for all $k \geq 1$, $\overline{K}(\Gamma, v_0)$ contains a subgroup isomorphic to $K(G_k, v_0)$.

Thus the direct limit group is well-defined and it is straightforward to verify that its definition agrees with that of the $\mathbb{Q}$-tropical Picard group. In particular, we can think of this direct limit as the critical group on the infinite graph with countably infinitely many vertices, one vertex for each (rational) point of $\Gamma_{\mathbb{Q}}$. Since the divisors in the image of the Laplacian matrix are exactly those divisors which are of the form (f), it follows that the tropical Picard group of a $\mathbb{Q}$-tropical curve Γ is the direct limit of the critical groups as claimed.

One advantage of this description of the $\mathbb{Q}$-tropical Picard group on Γ as a direct limit of finite critical groups is that we can simulate (weighted-) chip-firing moves on Γ by looking at the limit of firing sequences on the finite subdivisions.

For all vertices $v \in G_{k_1}$, we define $H_0(v) = \{v\}$, let $N(H)$ denote the neighbors of graph H, and inductively define $H_i(v)$ as

$$H_i(v) = H_{i-1}(v) \cup N(H_{i-1}(v)).$$

In other words, letting $m = \frac{k_2}{k_1}$, $H_m(v)$ is the unique induced subgraph of G_{k_2} which is a tree with root v and leaves given by the neighbors of v in the unsubdivided graph G_{k_1}, and $\{v_0\} = H_0(v) \subset H_1(v) \subset H_2(v) \subset \cdots \subset H_{m-1}(v)$ is a chain of subgraphs; see Figure 10.

Using this notation, we get a method for emulating firing sequences of vertices in $V(G_{k_1} \setminus \{v_0\})$.

Lemma 54 *Assume for all $0 \leq i \leq m-1$ that we fire subset $H_i(v)$ by firing vertex v first and then radiating outwards towards the leaves. If $D_{(k_1)}$ is a chip configuration on G_{k_1} such that vertex v is ready, then the firing of vertex v in graph G_{k_1} can be emulated on chip configuration $D_{(k_2)}$ in graph G_{k_2} by firing the sequence*

$$[H_0(v), H_1(v), H_2(v), \ldots, H_{m-2}(v), H_{m-1}(v)].$$

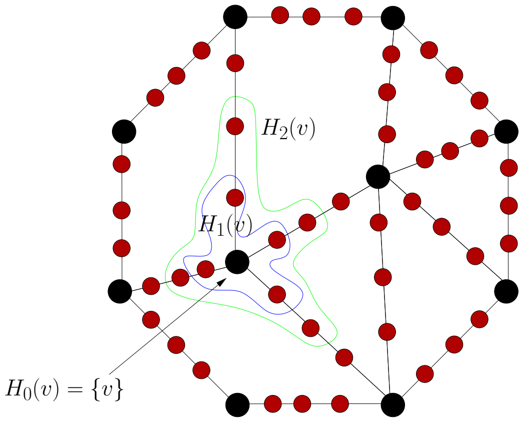


Fig. 10 The nested subgraphs $H_0(v) = \{v\} \subset H_1(v) \subset H_2(v) \subset \cdots \subset H_{m-1}(v)$.

Proof We assume $m = k_2/k_1 \geq 2$ since the statement is trivial otherwise. If vertex v can fire in the graph G_{k_1}, it follows that the coefficients $D^*_{(k_1)}(v) = D^*_{(k_2)}(v) \geq \mathrm{val}(v)$ where this valence is the same in both graphs G_{k_1} and G_{k_2}. Thus $v = H_0(v)$ can fire in G_{k_2}. Afterwards, vertex v may no longer be able to fire but we fire all vertices of $H_1(v)$ simultaneously. Only the leaves of $H_1(v)$, i.e. the neighbors of v, are on the boundary of the subgraph $H_1(v)$ and since v just fired, they each have one chip on them and one outgoing edge. Thus $H_1(v)$ may fire. By analogous logic we can then fire $H_2(v), H_3(v), \ldots, H_{m-1}(v)$ in turn, which results in a configuration identical to the one we obtain by firing vertex v in graph G_{k_1}. See Figure 11.

Example 55 (Circle)
The Picard group of a circle $\Gamma = S^1$ with length ℓ is the circle group $\mathbb{R}/\ell\,\mathbb{Z} \cong S^1$. In particular, if we look at the finite model $G = (\{v\}, \{e\})$, where e is a loop edge, and then subdivide e into k equal segments, we obtain $G_k = C_k$, the k-cycle graph. It is direct to verify that $\mathrm{Pic}(G_k) \cong \mathbb{Z}/k\mathbb{Z}$ with superstable representatives given by divisors of the form (w) or 0, where w is any vertex of $\mathrm{Pic}(G_k)$. The group law then exactly matches the group law on S^1 so we get the appropriate direct limit.

Example 56 (Genus 2 Banana Graph) Let Γ be the metric graph with two vertices (v_1 and v_2) connected by three parallel edges. We can use finite models of this graph to gain intuition for the structure of Γ's tropical Picard group. Let G_k be the finite graph on $3k+2$ vertices defined as the union of the three path graphs $[v_1, x_1, x_2, \ldots, x_k, v_2]$, $[v_1, y_1, y_2, \ldots, y_k, v_2]$, and $[v_1, z_1, z_2, \ldots, z_k, v_2]$, see Figure 12. By direct verification we see that $\mathrm{Pic}(G_k) \cong \mathbb{Z}/(k+1)\mathbb{Z} \times \mathbb{Z}/(3k+3)\mathbb{Z}$ using the generators $D_1 = x_1 - v_1$ and $D_2 = y_2 + z_1 - 2v_1$. (If $k = 1$, we use $D_2 = v_2 + z_1 - 2v_1$ instead.)

In particular, multiples of D_1 have the forms (for $1 \leq m \leq k$)

$$x_m - v_1, \quad v_2 - v_1, \quad x_m + v_2 - 2v_1, \quad 2v_2 - 2v_1, \quad \text{or} \quad y_{k+1-m} + z_{k+1-m} - 2v_1,$$

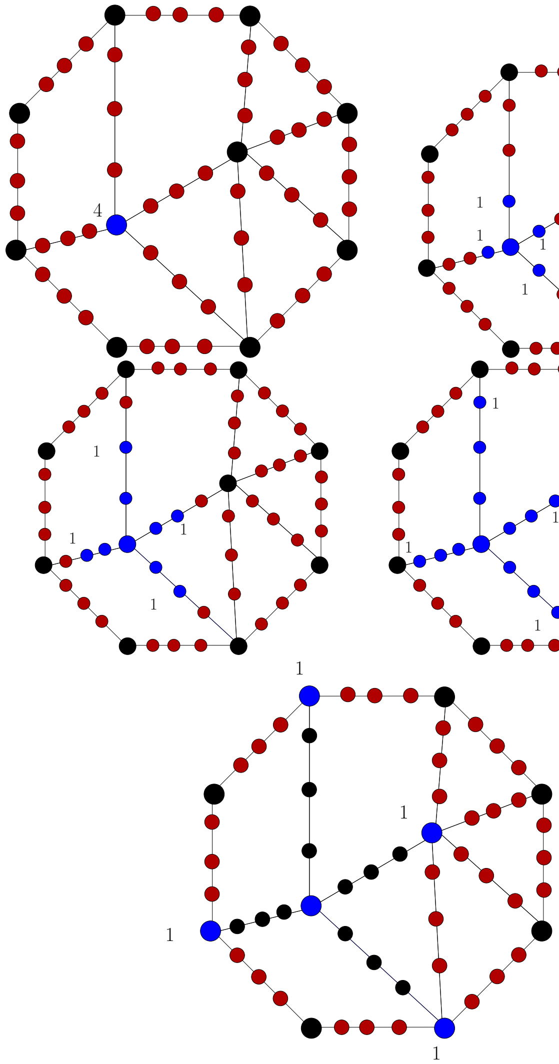


Fig. 11 Simulating chip-firing at v using a sequence of firing nested subgraphs on subdivided graph.

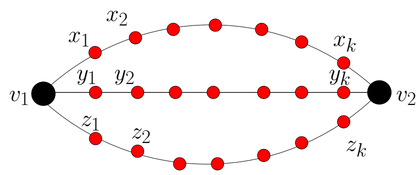


Fig. 12 Subdivided genus 2 Banana graph.

in order. Multiples of D_2 look like

$$y_{2m} + z_m - 2v_1, \quad v_2 + z_{\frac{k+1}{2}} - 2v_1, \quad \text{or} \quad x_{k+1-2m} + z_{\frac{k+1}{2}-m} - 2v_1,$$

in order, if k is odd, and the case where k is even is analogous. Taking the direct limit, we thus see that $\mathrm{Pic}_{\mathbb{Q}}(\Gamma) \cong \mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}$ generated by divisor classes $\overline{D_1}$ and $\overline{D_2}$ where sample representatives are of the forms

$$D_1 = x_\alpha - v_1 \text{ and } D_2 = y_{2\beta} + z_\beta - 2v_1.$$

Here x_α denotes the point on the top edge distance α from vertex v_1. The points y_α and z_α are defined analogously. We can also see that extending from rational points of Γ to real points can be accomplished in this case simply by letting distances α and β be real.

9 Conclusions and Open Questions

In this paper, we presented a number of properties of $|D|$ including verification that it is finitely generated as a tropical semi-module. We also provided some tools for explicitly understanding $|D|$ as a polyhedral cell complex such as a formula for the dimension of the face containing a given point, as well as applications such as using $|D|$ to embed an abstract tropical curve into tropical projective space.

There are many ways to continue this research for the future. It is quite tantalizing to investigate how the Baker-Norine rank of a divisor compares with the geometry and combinatorics of the associated linear system as a polyhedral cell complex. Also, is there any relation between $r(D)$ and the minimal number of generators of $R(D)$? Can we easily identify out of our finite generating set $\mathcal{S}$ which 0-cells correspond to extremals? How does the structure of $|D|$ change as we continuously move one point in the support of D or if we change the edge lengths of our metric graph while keeping the combinatorial type of the graph fixed?

In the case of finite graphs, i.e. divisors whose support lies within the set of vertices of the graph, can we combinatorially describe the associated linear systems? For example, is there a stabilization or an associated Ehrhart theory that one can use to count the sizes of such linear systems?

Lastly, what other results from classical algebraic geometry carry over to the theory of metric graphs (or tropical curves) and vice-versa?

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