Cluster Algebras, Invariants, Jacobians, and Tropical Curves
Gregg Musiker's Research Statment

My research focus is in the field of algebraic combinatorics, and I am most interested in topics that highlight connections between combinatorics and other mathematical fields, such as representation theory, number theory, and algebraic geometry. Four subfields of my current research are **(1)** Cluster algebras, **(2)** Arithmetic geometry, **(3)** Tropical geometry, and **(4)** Invariant theory. A common theme throughout my research has been the use of combinatorial objects and methods to provide new approaches for solving problems in other areas of mathematics.

Cluster algebras were defined by Fomin and Zelevinsky, and there are a number of motivational problems in this field. One such research avenue is given by the positivity conjecture, which has been open since the founding of this field in 2000. In [MSW], we obtain combinatorial formulas for certain cluster variables, which proves this conjecture for a large class of cluster algebras. I plan to continue this research on explicit combinatorial formulas and the positivity conjecture for even more cases. My other work in this area includes [IMPV], [Mus02], [Mus2], [MP07], and [MS].

Jacobians of graphs, also known as critical groups or sandpile groups were independently introduced by researchers in diverse fields such as graph theory, dynamical systems, electrical networks, and arithmetic geometry. As the name indicates, there are intriguing parallels with Jacobians of algebraic curves over finite fields, and I have explored these in my work [Mus07, Mus1, Mus09].

I also have studied tropical analogues of various objects coming from algebraic geometry. These include Jacobians and linear systems of tropical curves [HMY]. Tropical geometry is a piecewise linear version of classical algebraic geometry, where algebraic varieties become polyhedral cell complexes. Tropical curves also appear in the literature as metric graphs or quantum graphs.

Several of my research projects have also been in the area of invariant theory. One such project discussed connections between Kronecker coefficients and solutions to a certain system of Diophantine equations [GMWX09]. I also have studied quasi-invariants of the symmetric group, which interpolate between the ring of all polynomials and the ring of symmetric functions [BM05, BM08].

1. Combinatorial interpretations for cluster variables

Cluster algebras [FZ02a] are a certain class of commutative algebras, each of which is isomorphic to a subalgebra of the field of rational functions. Each cluster algebra has a distinguished set of generators, which are called cluster variables. These generators break up into a union of overlapping algebraically independent finite subsets called clusters, which together have the structure of a simplicial complex. Neighboring clusters are related to each other by binomial exchange relations. Such algebras have been found to be related to a number of other mathematical topics such as quiver representations, tropical geometry, Lie groups, Poisson geometry, and Teichmüller theory. A major result in the theory of cluster algebras is the Laurent Phenomenon [FZ02a, FZ02b] which states that cluster variables are not simply rational functions, as one might expect from iterative divisions, but can in fact be expressed as Laurent polynomials with integer coefficients in terms of any initial cluster $\{x_1, x_2, \ldots, x_n\}$. Fomin and Zelevinsky went on to make the following conjecture.

Conjecture 1 (Positivity Conjecture [FZ02a, FZ02b]). *The coefficients in the Laurent expansion of a cluster variable are all nonnegative integers, for any cluster algebra and for any initial seed.*

Positivity of the coefficients is significant, as it is conjecturally related to total-positivity properties of dual canonical bases [FZ99, FZ00]. Nonetheless, this conjecture is still open despite nearly a decade of work by many researchers, including Di Francesco-Kedem [DiFK], Nakajima [Nak], myself, and others, proving it for certain families of cluster algebras.

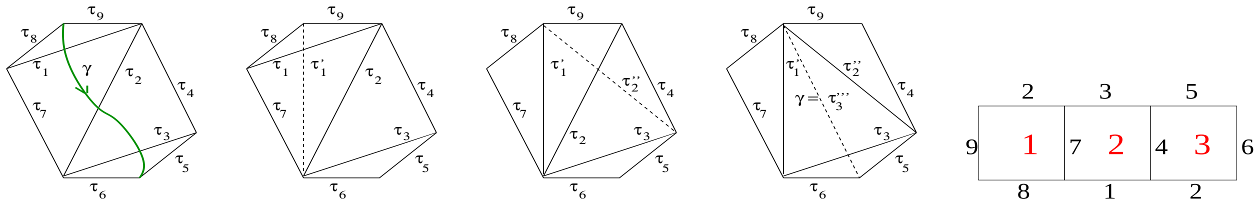

FIGURE 1. (Left:) A triangulated hexagon with a sequence of quadrilateral flips. (Right): The graph $G_{T,\gamma}$ constructed from arc γ and triangulation $T = \{\tau_1, \tau_2, \tau_3\}$.

An important class of cluster algebras are those arising from surfaces. These were defined by Fomin, Shapiro, and Thurston in [FST08], based on work of Fock-Goncharov [FG06, FG07, FG] and Gekhtman-Shapiro-Vainshtein [GSV05]. They comprise a very concrete family of cluster algebras where the cluster variables correspond to arcs in a given marked surface, and clusters correspond to triangulations (i.e. maximal collections of non-crossing arcs). Exchanges between two clusters correspond to flipping a diagonal in a quadrilateral. See Figure 1 for an example of such exchanges.

1.1. **Graph theoretic formulas for cluster variables.** I first explored the theory of cluster algebras in 2001 while a member of the NSF funded REACH group, led by Propp, as well as during the writing of my senior thesis under Stanley [Mus02]. My participation led to a paper with Propp [MP07] and a joint project with Itsara, Propp, and Viana [IMPV] that is described in [Pro]. Both projects provided graph theoretic interpretations of cluster variables, thereby showing positivity of their coefficients as Laurent polynomials. The cluster algebras studied in [MP07] were those of rank two and affine type, and [Pro] deals with one corresponding to the tree of Markoff numbers. Extending these methods, I provide formulas for certain cluster algebras of classical type in [Mus2].

More recently, I have worked with Schiffler and Williams to prove positivity for cluster algebras arising from surfaces. This class of cluster algebras was described above. Building on the work of Schiffler and Thomas [ST09, Sch], Schiffler and I obtained combinatorial formulas for any cluster algebra arising from an *unpunctured* surface, one with no marked points in the interior [MS]. Cluster algebras from surfaces with punctures could not be directly handled by the same method because of technical difficulties. For example, when there are punctures, there are triangulations in which not all arcs may be flipped, leading to *tagged arcs*. Nonetheless, in [MSW], Schiffler, Williams, and I obtained graph theoretic formulas for cluster algebras from all surfaces (with or without punctures) by using topological arguments, as well as algebraic and combinatorial identities.

Theorem 1 (Cluster Expansion Formula for Cluster Algebras from Surfaces [MSW])**.** *For every triangulation T of a surface (with or without interior marked points) and an* ordinary *arc γ, we construct a planar bipartite (snake) graph $G_{\gamma,T}$ such that the associated cluster variable is given as*

$$x_\gamma = \frac{\sum_{\text{perfect matching } P \text{ of } G_{\gamma,T}} x(P)y(P)}{x_1^{e_1(T,\gamma)} x_2^{e_2(T,\gamma)} \cdots x_n^{e_n(T,\gamma)}}.$$

Here, $e_i(T,\gamma)$ is the crossing number of τ_i and γ, $x(P)$ is the product of the weights *of the edges in set P, and $y(P)$ is the* height *of P. An analogous expansion formula holds for tagged arcs.*

Example 1. *See Figure 1. By applying successive Ptolemy exchange relations, one obtains the Laurent expansion* $x_\gamma = \frac{1}{x_1x_2x_3}\left(x_2^2x_5x_8+y_1x_2x_5x_7x_9+y_3x_2x_4x_6x_8+y_1y_3x_4x_6x_7x_9+y_1y_2y_3x_1x_3x_6x_9\right)$. *In* [MSW], *we give a construction of graph $G_{T,\gamma}$ which provides an alternative combinatorial expansion formula such that the* 5 *perfect matchings of $G_{T,\gamma}$ correspond to the* 5 *monomials of x_γ.*

In general, finding a sequence of binomial exchanges so that a specific choice of cluster variable is obtained is difficult, and though there are unexpected cancellations that preserve Laurentness, positivity, on the other hand, is not guaranteed. However, our theorem provides a *direct* and *postive* expansion formula, thus proving the positivity conjecture for cluster algebras from surfaces, among other corollaries. This class comprises *almost all* cluster algebras of finite-mutation type, due to work of Felikson-Shapiro-Tumarkin [FeShTu]. I have several plans of how to continue this research.

Problem 1 (Positivity for more cluster algebras)**.** *Extend these results to other cluster algebras thereby proving Fomin and Zelevinsky's positivity conjecture for further examples. Some specific examples I have in mind include cluster algebras of finite or affine type that are not simply-laced. I also have work in progress for graph theoretic formulas related to certain cluster variables arising in cluster algebras which are of* infinite mutation type*.*

The above theorem also has applications for computing Euler characteristics of the quiver Grassmannian and representations of quivers with potentials, due to the work of Caldero-Chapoton [CC06] and Derksen-Weyman-Zelevinsky [DWZ].

Problem 2 (Comparisons of formulas for Laurent expansions)**.** *Compare the combinatorial formula of Theorem 1 with various formulas arising from geometry and representation theory.*

2. Combinatorial techniques for studying curves over finite fields

The theory of algebraic varieties over finite fields has a rich history, with beautiful combinatorial problems such as point enumeration, which can be approached using the zeta function. Even though this function involves an exponential, i.e. $Z(V,t) = \exp\left(\sum_{k=1}^{\infty} \frac{N_k}{k} t^k\right)$ where N_k enumerates points on variety V over field extension $\mathbb{F}_{q^k}$, Dwork showed that $Z(V,t)$ is a rational function with integer coefficients. Consequently, there are symmetries and extra structure among the points. Some of these were codified by the Weil conjectures including the Riemann hypothesis for algebraic varieties. The latter was proven by Deligne in work that earned him a Fields medal. Examining specifically the case of algebraic curves, the Riemann hypothesis can be phrased in terms of bounds on the number of points on the curve. The work of Bombieri [Bom74] and Stark [Sta72] illustrate more elementary methods to prove such bounds, with their proofs relying on the Riemann-Roch theorem and properties of the Frobenius map acting on a curve over the algebraic closure of a finite field $\overline{\mathbb{F}_q}$.

Problem 3 (Combinatorially study zeta functions)**.** *Use combinatorial techniques to obtain alternate constructions for the zeta functions of curves and other abelian varieties.*

In graduate school [MusPhD], I explored this problem in the case of genus one curves, i.e. elliptic curves [Mus07, Mus1], and I plan to examine higher genus curves and other abelian varieties more in the future. One reason that elliptic curves are an interesting object to study is that they have a group structure using the divisor class group [Sil86]. This property allows them to be used for cryptographic purposes [Was03]. For an elliptic curve E, the zeta function $Z(E,t)$ equals $\frac{1-(1+q-N_1)t+qt^2}{(1-t)(1-qt)}$, and thus $N_k = \#E(\mathbb{F}_{q^k})$ will depend solely on q and N_1. In fact, Garsia observed that these expressions are polynomials which alternate in sign with respect to the power of N_1.

Question 1 (Combinatorial Interpretation of N_k?)**.** *Writing N_k as $\sum_{i=1}^{k}(-1)^{i+1}P_{i,k}(q)N_1^i$, the $P_{i,k}$'s are polynomials with* positive *integer coefficients. What is their combinatorial interpretation?*

We answer this query by using a three-parameter family of directed graphs (digraphs) with multiple edges. These $W_k(q,t)$'s are defined by deforming the wheel graph with k rim vertices. We let $\mathcal{W}_k(q,t)$ denote the number of oriented spanning trees of $W_k(q,t)$ with the center as the root.

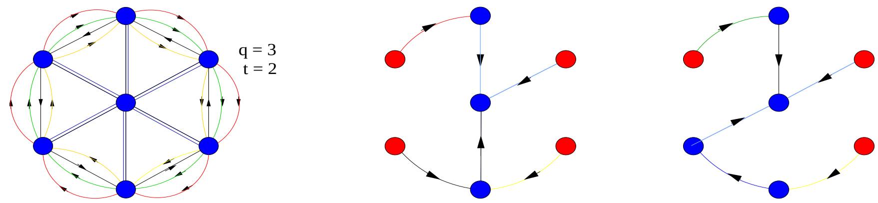

FIGURE 2. The (q,t)-wheel graph $W_6(3,2)$ and two of its directed spanning trees [Mus09].

Theorem 2 (Bivariate Polynomial Formulas for N_k versus $\mathcal{W}_k(q,t)$ [Mus09])**.** *For any integer $k \geq 1$, $\mathcal{W}_k(q,t)$, viewed as a function of q and t, is a polynomial with integer coefficients such that*

$$N_k = \sum_{i=1}^{k} (-1)^{i+1} P_{i,k}(q) N_1^i = -\mathcal{W}_k(Q,T)|_{Q=q, T=-N_1}.$$

In particular, the $P_{i,k}(q)$'s are polynomials with positive integer coefficients.

This result motivated a closer examination of the relationship between points on an elliptic curve E over $\mathbb{F}_{q^k}$ and spanning trees on the wheel graph $W_k(q,t)$. For instance, we can use the Matrix-Tree theorem to derive a formula for N_k in terms of the determinant of a three-line circulant matrix.

Theorem 3 (Determinantal Formulas for N_k and $\mathcal{W}_k(q,t)$ in terms of Circulant Matrices [Mus07])**.** *Let $M_1 = [-N_1]$, $M_2 = \begin{bmatrix} 1+q-N_1 & -1-q \\ -1-q & 1+q-N_1 \end{bmatrix}$, and for $k \geq 3$, let M_k be the k-by-k matrix*

$$\begin{bmatrix} 1+q-N_1 & -1 & 0 & \dots & 0 & -q \\ -q & 1+q-N_1 & -1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & -q & 1+q-N_1 & -1 & 0 \\ 0 & \dots & 0 & -q & 1+q-N_1 & -1 \\ -1 & 0 & \dots & 0 & -q & 1+q-N_1 \end{bmatrix}.$$

Then the sequence of integers $N_k = \#E(\mathbb{F}_{q^k})$ satisfies $N_k = -\det M_k$ for all $k \geq 1$.

Such matrices are related to orthogonal polynomials, and were also studied in [LWW04], where they obtained a different combinatorial interpretation in terms of permutation enumeration.

2.1. **Comparisons of elliptic curve groups with critical groups.** The critical group $K(G)$ associated to a graph G is a finite abelian group whose cardinality equals the number of spanning trees of G. They have appeared numerous times in the mathematical literature, and have been studied in connection to chip-firing games [BLS91, Big99], are also known as abelian sandpile groups [Dha90, Gab93], and have also been studied in arithmetic geometry [BN07, Lor91]. Simply put, $K(G)$ is a quotient group of the image of the reduced Laplacian matrix of the graph (or digraph) G. Here, the Laplacian $L(G)$ is a square matrix with a row and column corresponding to each of the vertices of G. The diagonal entries of $L(G)$ are the degrees of the vertices, and the (i,j)th entry is the negative of the number of directed edges between v_i and v_j. We obtain the reduced Laplacian matrix from $L(G)$ by deleting the row and column corresponding to a choice of root v_0.

In the case of the digraphs $W_k(q,t)$'s, the reduced Laplacian matrices are the family of matrices $\{M_k|_{N_1=-t}\}$ defined in Theorem 3, where we take the central vertex as the root. We abbreviate $K(W_k(q,t))$ as $K(q,k,t)$, and write the elements of $K(q,k,t)$ as vectors in $\mathbb{Z}^k$ such that the entries correspond to the rim vertices v_1 through v_k in circular order. Let ρ_k denote the map that cyclicly rotates the entries of a vector of $\mathbb{Z}^k$. In [Mus09], I show that $K(q,k_1,t)$ is a subgroup of $K(q,k_2,t)$ whenever $k_1|k_2$, hence I can define a direct limit $\overline{K}(q,t) := \lim_{k\geq 1}\{K(q,k,t)\}$. Furthermore, by the universal property, there is a unique map ρ which restricts to ρ_k for any $k \geq 1$.

Theorem 4 (Comparing the rotation map ρ to the Frobenius map π [Mus09])**.**

(1) *For any integers $k \geq 1$, $q \geq 0$, and $t \geq 1$, we have* $K(q,k,t) \cong \mathrm{Ker}(1-\rho^k) : \overline{K}(q,t) \to \overline{K}(q,t)$. *Compare this to* $E(\mathbb{F}_{q^k}) \cong \mathrm{Ker}(1-\pi^k) : E(\overline{\mathbb{F}_q}) \to E(\overline{\mathbb{F}_q})$.

(2) *As a map from $\overline{K}(q,t)$ to itself, we obtain $\rho^2 - (1+q+t)\rho + q = 0$, a simple analogue of the characteristic equation $\pi^2 - (1+q-N_1)\pi + q = 0$ on $E(\overline{\mathbb{F}_q})$.*

(3) *The abelian group $K(q,k,t)$ has a group structure that can be written as the product of at most two cyclic groups. (The analogous statement holds for the $E(\mathbb{F}_{q^k})$'s.)*

(4) *For any $n \geq 1$, $\overline{K}(q,t)$ contains the subgroup $\mathbb{Z}/n\mathbb{Z}$. Furthermore, $\overline{K}(q,t)$ contains $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ as a subgroup if and only if n and q are coprime. (The analogous statement holds when q is a power of a prime and E is an* ordinary *elliptic curve, as opposed to supersingular.)*

From these analogies, it appears that the direct limit group $\overline{K}(q,t)$ behaves like an *ordinary elliptic curve* over the algebraic closure of a finite field. Under this correspondence, rotation ρ corresponds to the Frobenius map π. In the case of wheel graphs, the groups $\overline{K}(q,t)$ and $K(q,k,t)$ may be defined for any nonnegative integer q, and the above results still hold. Also, in the case of elliptic curves, results such as (2), (3), and (4) are proven using high-powered machinery such as the Tate module, p-adic analysis, or the Weil pairing, but for wheel graphs, linear algebra suffices.

Problem 4 (Critical groups for other abelian varieties)**.** *Find other families of digraphs which correspond to Jacobians of algebraic curves of higher genus, or other abelian varieties.*

3. Chip-firing, tropical geometry and tropical curves

My research in the theory of critical groups has led me to study tropical curves. This is partially due to the work of Baker and Norine [BN07] who studied Abel-Jacobi theory on finite graphs. Under their correspondence, critical groups correspond to Jacobians of algebraic curves. This work was later extended from graphs to tropical curves. Tropical geometry is a version of algebraic geometry for the *tropical semiring* $(\mathbb{R}, \oplus, \odot)$, that is, the real numbers with the tropical operations

$$a \oplus b = \max(a,b) \text{ and } a \odot b = a + b.$$

This subject has important connections to topics of applied mathematics such as optimization and mathematical biology, as well as algebraic geometry, symplectic geometry, several complex variables, and geometric combinatorics [SS09]. Over the tropical semiring, a polynomial is a piece-wise linear function with integer slopes, and a *tropical curve* is the corner locus of such a tropical polynomial, meaning that it is the set in $\mathbb{R}^n$ where the maximum is achieved twice. There are results, such as Mikhalkin's Correspondence Theorem [Mik05], which state that Gromov-Witten invariants of the plane, enumerating curves through a set of generic marked points, are the same classically and tropically. On the other hand, some definitions, like the rank of a matrix, can be defined multiple ways classically, but for tropical linear algebra, these notions differ [DSS05].

Motivational Question 1 (Which theorems can survive tropical conditions?)**.** *What theorems from classical algebraic geometry are analogous in tropical geometry, and which need refinement?*

3.1. **Linear systems and Jacobians of tropical curves.**

In addition to my work on critical groups, described in Section 2.1, I have a recent paper with Haase and Yu [HMY] in which we analyze linear systems of tropical curves Γ and their structures as polyhedral cell complexes. This cell structure for tropical curves was introduced by Gathmann-Kerber [GK08] and Mikhalkin-Zharkov [MZ06]. The *linear system* $|D|$ of a divisor is the set (or space) of all effective divisors that are linearly equivalent to D. Here, a *divisor* is just a formal linear combination of points, and two divisors are *linearly equivalent* if their difference corresponds to a *principal divisor*. For graphs, principal divisors correspond to column vectors in the image of the Laplacian matrix.

Such linear systems are naturally related to the tropical semi-module $R(D)$, which denotes the set of tropical rational functions f such that $(f) + D$ is effective. Here, (f) denotes a divisor obtained by taking the sum of the outgoing slopes at every point of Γ. The modules $R(D)$ are not linear spaces, and instead have a rich structure as a polyhedral cell complex. Despite its similarity in definition to $L(D)$ in algebraic curve theory, $R(D)$ fails to satisfy a number of properties of $L(D)$. In fact, $R(D)$ is not even necessarily pure dimensional. In spite of this difficulty, we show among other results that $R(D)$ is finitely generated by vertices of the cell complex corresponding to $|D|$.

There is also a notion of *rank* of a divisor, due to Baker-Norine [BN07], which satisfies a graph theoretic (and tropical) analogue of Riemann-Roch. However, this rank is quite stranger than the dimension of a vector space, as it is in ordinary algebraic geometry. For example, the rank of D does not equal the minimal number of generators of $R(D)$, motivating the following questions.

Problem 5 (Better understanding of extremals)**.** *Is there any relation between the rank of D and the minimal number of generators of the tropical semi-module of piece-wise linear functions $R(D)$, associated to $|D|$? Which vertices of the polyhedral cell complex of $|D|$ correspond to the extremals?*

3.2. **Proposal for Future Work.** This fall, I am a postdoctoral fellow at MSRI in the program on tropical geometry, and have already had productive conversations with researchers such as E. Cotterill, E. Katz, S. Payne, and A. Stapledon. I look forward to more collaborative interactions and more avenues for research as my semester at MSRI continues. One overarching question that I am pursuing is the following: *Which theorems on algebraic curves carry over into the setting of graphs or metric graphs?* I am also investigating several aspects of Baker-Norine rank.

Problem 6 (Better understanding of Baker-Norine rank)**.** *Are there more combinatorial methods for computing the Baker-Norine rank of a linear system? How does the rank change as the edge lengths of a graph are deformed, or if we continuously move one point in the support of D?*

4. Explicit bases for quasi-invariants of the symmetric group

One of my projects in invariant theory is the study of quasi-invariants. The m-quasi-invariants $QI_m(G)$ are generalizations and variants of symmetric functions which were defined by Feigin and Veselov [FV02] for any Coxeter group G. Each $QI_m(G)$ is a polynomial ring, as well as a G-module. For example, a polynomial $P = P(x_1, x_2, \ldots, x_n)$ is said to be *m-quasi-invariant* with respect to S_n if and only if $(x_i - x_j)^{2m+1}$ divides the polynomial $(1 - s_{i,j})P$. In this way, $QI_0(S_n)$ is the standard polynomial ring, and the rest of the $QI_m(S_n)$'s interpolate between the polynomial ring and the ring of symmetric functions. Etingof and Ginzburg [EG02] showed that the $QI_m(S_n)$'s have nice combinatorial properties such as being Cohen-Macaulay and Gorenstein rings that afford the left-regular representation. Felder and Veselov give formulas for their Hilbert series in [FeVe02].

Using these results, Bandlow and I worked towards the construction of an explicit basis for m-quasi-invariants of the symmetric group, S_n [BM05, BM08]. We attacked this problem by breaking up the basis into isotypic components, that is subspaces that are left invariant under different representations of S_n. We thereby obtained basis elements defined using variants of Selberg integrals.

Theorem 5 (Basis for two row hook shapes [BM08])**.** *For the partition $[n-1, 1]$, a basis of m-quasi-invariants of S_n for the corresponding isotypic component is given as the following set.*

$$\left\{ \int_{x_1}^{x_j} t^k \prod_{i=1}^{n} (t - x_i)^m dt \right\}_{j=2,\ k=0}^{n,\ n-2}$$

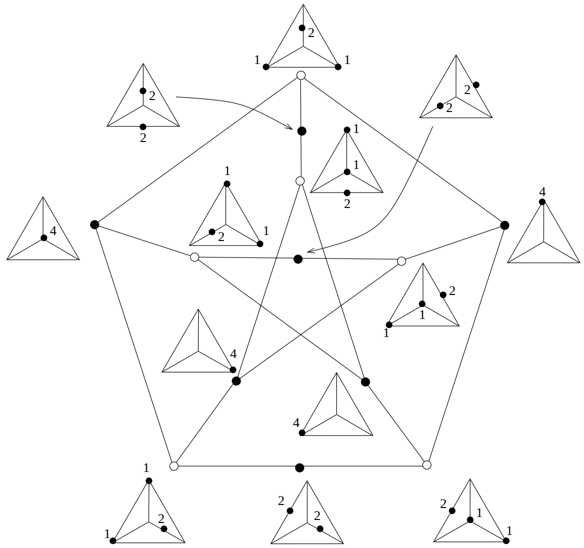

FIGURE 3. Let Γ be the metric graph over K_4 with edges of equal length. Let K be the canonical divisor, with a 1 at each vertex. The linear system $|K|$ is a cone over the Petersen graph shown here. The 13 divisors appearing, together with K, generate $R(K)$. The seven black vertices correspond to extremal generators [HMY].

This result inspired work by Tsuchida [Tsu], who was able to use determinants, whose entries consist of our basis elements for the $[n-1,1]$ case, to obtain bases, for a much larger set of isotypic components; namely those corresponding to hooks, i.e. partition $[n-k,1^k]$.

Problem 7 (Bases for non-hooks). *Obtain explicit bases for quasi-inviariants of S_n for isotypic components corresponding to other partitions. Alternatively, obtain bases for other Coxeter groups.*

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