ON BROUWER'S FIXED POINT THEOREM

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ABSTRACT. We give a short new proof of Brouwer's fixed point theorem based on one time application of the change of variables formula (no Stokes' formula or differential form calculus).

There are very many different proofs of the celebrated Brouwer fixed point theorem. Here we present one more, as we hope, a new one.

By $\mathbb{R}^d$ we denote the Euclidean space of points $x = (x^1, ..., x^d)$. When it makes sense, for real-valued $u(x)$ on $\mathbb{R}^d$ we denote

$$D_i u = \frac{\partial u}{\partial x^i}.$$

If $F = (F^i)$ is a smooth mapping of $\mathbb{R}^d$ to $\mathbb{R}^d$, we set

$$DF = (a^{ij})_{i,j=1}^d, \quad a^{ij} = D_j F^i$$

Theorem 1 (Brouwer's fixed-point theorem, 1910)**.** *Let K be a convex closed bounded subset of $\mathbb{R}^d$ and let $f : K \to K$ be a continuous mapping. Then f has fixed points in K (where $f(x) = x$).*

We prove this theorem after some preparation. The case that K, actually, belongs to a linear subspace of lower dimension is considered by concentrating on this lower dimensional subspace. If K has nonempty interior, upon mapping K onto $\bar{B} := \{x : |x| \le 1\}$, we easily reduce the situation to the one when $K = \bar{B}$.

Then we start with a lemma the proof of which have some intentional gaps, closed later in Remark 1.

Lemma 2. *Let Ω be a connected bounded domain in $\mathbb{R}^d$ with C^1 boundary and let $F, G : \bar{\Omega} \to \mathbb{R}^d$ be $C^1(\bar{\Omega})$ mappings such that*

$$F = G \quad on \quad \partial\Omega.$$

Then

$$\int_\Omega \det DF \, dx = \int_\Omega \det DG \, dx.$$

Proof. Observe that for small t the mappings $F_t = tF(x) + x$ and $G_t = tG(x) + x$ are one-to-one on $\bar{\Omega}$ and, by the implicit function theorem, have C^1-inverse mappings on Ω. Because of that they map $\partial\Omega$ onto the boundary of $F_t(\Omega)$ which is $F_t(\partial\Omega)$ and is, of course, the same as the

1991 *Mathematics Subject Classification.* 47H10.

Key words and phrases. Brouwer's theorem, fixed pont theorem.

boundary of $G_t(\Omega)$. Furthermore, for small t the intersection of $F_t(\Omega)$ and $G_t(\Omega)$ is obviously nonempty and since they are connected and have the same boundary, $F_t(\Omega) = G_t(\Omega)$ and

$$\operatorname{Vol} F_t(\Omega) = \operatorname{Vol} G_t(\Omega)$$

for small t. We express this equality in terms of DF_t and DG_t as

$$\int_\Omega \det(tDF + I)\,dx = \int_\Omega \det(tDG + I)\,dx, \tag{1}$$

where I is the unit $d \times d$-matrix, then use the fact both parts of (1) are polynomials in t. Since they coincide for small t, they are identical and by comparing the coefficients of t^d we get the result. The lemma is proved. □

What follows is borrowed from pages 467–470 of N. Dunford and J.T. Schwartz [1].

Corollary 3. *For the domain Ω from Lemma 2 there is no $C^1(\bar\Omega)$ function $G : \bar\Omega \to \partial\Omega$ such that $G(x) = x$ on $\partial\Omega$.*

Indeed, if we assume the contrary, then for $F(x) = x$ we find

$$\operatorname{Vol}\Omega = \int_\Omega \det DG\,dx.$$

However, the condition $G : \bar\Omega \to \partial\Omega$ implies that all partial derivatives of G are tangent to $\partial\Omega$, in particular, all d-columns of $DG(x)$ are tangent to $\partial\Omega$ at the point $G(x)$. But the tangent plane to $\partial\Omega$ at any point is only $(d-1)$-dimensional, so that the columns of DG are linearly dependent and, hence, $\det DG = 0$. This yields a contradiction, $\operatorname{Vol}\Omega = 0$, and proves our claim. □

Now comes a particular case of Theorem 1, which implies it, as is explained above.

Theorem 4. *Let $f : \bar B \to \bar B$ be a continuous mapping. Then f has fixed points in $\bar B$.*

Proof. First assume that f is smooth. Assume that there are no fixed points and for each $x \in \bar B$ define $G(x) \in \bar B$ as

$$G(x) = x - t(x)(f(x) - x),$$

where $t(x) \geq 0$ is the root of the equation

$$|x - t(f(x) - x)| = 1.$$

From the geometric picture it is clear that this equation has always two distinct roots ($f(x) \neq x$) one is strictly negative and the other is nonnegative (zero if $x \in \partial B$). This means that the discriminant of the quadratic equation

$$|x|^2 - 2t(x, f(x) - x) + t^2|f(x) - x|^2 = 1$$

is strictly positive, smooth, and its square root is smooth, so that

$$t(x) = \frac{(x, f(x) - x) + \sqrt{(x, f(x) - x)^2 + (1 - |x|^2)|f(x) - x|^2}}{|f(x) - x|^2}$$

is a smooth function along with $f(x)$ and $G(x)$. Then we can use Corollary 3 and finish the proof in the case of smooth f.

In the general case, let f_n be a sequence of polynomials such that

$$|f_n - f| \leq 1/n$$

on $\bar{B}$. By replacing f_n with $(n/(n+1))f_n$, if necessary, we may assume that the f_n's map $\bar{B}$ into itself. Then there exist $x_n \in \bar{B}$ such that $f_n(x_n) = x_n$. Obviously any converging subsequence of x_n converges to a fixed point of f. The theorem is proved. □

Now we comment on some steps some people may regard as missing in the proof of Lemma 2.

Remark 1. First question: Why (for small t) is F_t one-to-one? Assume that there are two points $x', x'' \in \bar{\Omega}$ such that $F_t(x') = F_t(x'')$. Then by denoting by N_F the Lipschitz constant of F and taking t such that $tN_F \leq 1/2$, we get

$$|x' - x''| = t|F_t(x') - F_t(x'')| \leq N_F t|x' - x''| \leq (1/2)|x' - x''|$$

and $|x' - x''| = 0$.

Second question: Why (for small t) does F_t map $\partial\Omega$ onto $\partial F_t(\Omega)$? Assume that there is a point $x_0 \in \partial\Omega$ such that $F_t(x_0) \notin \partial F_t(\Omega)$. Then $F_t(x_0) =: y_0 \in F_t(\Omega)$ (which is an open set in $\mathbb{R}^d$ by the implicit function theorem) and consequently there is $x_1 \in \Omega$ such that $F_t(x_1) =: y_0$. We have $x_1 \in \Omega$, $x_0 \in \partial\Omega$, so that $x_1 \neq x_0$, contradicting the one-to-one property. Thus, $F_t(\partial\Omega) \subset \partial F_t(\Omega)$.

That, conversely, any $y_0 \in \partial F_t(\Omega)$ is in $F_t(\partial\Omega)$ follows from the fact that there is a sequence $x_n \in \Omega$ such that $y_n := F_t(x_n) \to y_0$ as $n \to \infty$ and for any subsequence of x_n converging, say to $x_0 \in \bar{\Omega}$ we have $F_t(x_0) = y_0$, which leaves only one possibility for x_0: $x_0 \in \partial\Omega$, since $y_0 \notin F_t(\Omega)$.

Third question: Why (for small t) does the equality $\partial F_t(\Omega) = \partial G_t(\Omega)$ imply that $F_t(\Omega) = G_t(\Omega)$? Here we use that Ω is connected and first prove that

$$F_t(\Omega) \cap G_t(\Omega) \neq \emptyset.$$

For that we fix any $x_0 \in \Omega$ and show that if t is sufficiently small, then

$$F_t(x_0) \in G_t(\Omega). \tag{2}$$

Let t be so small that, for any $x \in \Omega$, the distance of $x_0 + tF(x_0) - tG(x)$ to $\partial\Omega$ is at least half the distance of x_0 to $\partial\Omega$. Then define

$$x_{k+1} = x_0 + tF(x_0) - tG(x_k), \quad k = 0, 1, ...$$

Decreasing t if necessary we may assume that $t|G(x') - G(x'')| \leq (1/2)|x' - x''|$ and then it follows that the sequence x_k converges and the limit point, say x' is in Ω and satisfies $F_t(x_0) = G_t(x')$. This proves (2).

Now assume that $F_t(\Omega) \not\subset G_t(\Omega)$. Then there is $y_1 = F(x_1) \in F_t(\Omega)$ such that $y_1 \notin G_t(\Omega)$. Take a broken line x_s, $s \in [0,1]$, inside Ω connecting x_0 and x_1. On the one end of this broken line $F(x_1) \notin G_t(\Omega)$ and on the other

$F_t(x_0) \in G_t(\Omega)$. Hence there is $s \in [0,1]$ such that $x_s \in \Omega$, $F_t(x_s) \in F_t(\Omega)$ and $F_t(x_s) \in \partial G_t(\Omega) = \partial F_t(\Omega)$, the latter contradicting $F_t(x_s) \in F_t(\Omega)$. It follows that $F_t(\Omega) \subset G_t(\Omega)$. By symmetry, $G_t(\Omega) \subset F_t(\Omega)$ (if t is small enough) and $G_t(\Omega) = F_t(\Omega)$.

Remark 2. An analytic proof of Lemma 2 can be obtained as in [1] if one proves that for smoother $\mathbb{R}^d$-valued H on $\mathbb{R}^d$

$$\det DH = (1/d)\operatorname{div}\hat{H}, \quad \hat{H}_j = H^i A_{ij}, \quad \sum_{j=1}^{d} D_j A_{ij} = 0,$$

where A_{ij} are the cofactors of $D_j H^i$ in the matrix DH.

Indeed, in that case

$$\frac{d}{dt}\det D[tF + (1-t)G] = \sum_{i,j=1}^{d} A_{ij} D_j[F^i - G^i] = \sum_{i,j=1}^{d} D_j\big(A_{ij}[F^i - G^i]\big),$$

the integral over Ω of the last divergence is reduced to the integral over its boundary and and $A_{ij}[F^i - G^i] = 0$ on $\partial\Omega$.

References

[1] N. Dunford and J.T. Schwartz, "Linear operators, Part 1", New York, Interscience, 1958.

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