Basics of harmonic polynomials and spherical functions

I present a beautiful, elegant, and rather simple theory of spherical harmonics, which I heard on one of Dynkin's seminars I was attending in my third year at the university.

By $\mathbb{R}^d$ we denote the Euclidean space of points $x = (x^1, ..., x^d)$. When it makes sense, for real-valued $u(x)$ on $\mathbb{R}^d$ we denote

$$u_{x^i} = \frac{\partial u}{\partial x^i}, \quad u_{x^i x^j} = \frac{\partial}{\partial x^j} u_{x^i}, \quad \Delta u = u_{x^1 x^1} + ... + u_{x^d x^d}.$$

1. Spherical harmonics and the Laplace-Beltrami operator

Denote by $\mathcal{P}_n$ the set of polynomials of degree (at most) n defined in $\mathbb{R}^d$, $d \geq 2$. Denote

$$B_R = \{x \in \mathbb{R}^d : |x| < R\}, \quad S_R = \{x \in \mathbb{R}^d : |x| = R\}.$$

Lemma 1.1 (maximum principle). *If $u \in C^2_{\text{loc}}(B_R) \cap C(\bar{B}_R)$ and $\Delta u \geq 0$ in B, then in B*

$$u \leq \max_{S_R} u.$$

In particular, if $\Delta u = 0$ in B and $u = 0$ on S_R, then $u = 0$ in B.

Proof. Let u at some points in B_R be strictly bigger than $\max_{S_R} u$. Then, by continuity, for $\varepsilon > 0$ small enough $u - \varepsilon(R^2 - |x|^2)$ will also be at some points in B_R strictly bigger than its maximum over S_R. It follows that the maximum point, say x_ε of $u - \varepsilon(R^2 - |x|^2)$ over $\bar{B}_R$ lies in B_R. At x_ε the second order derivatives and, hence, the Laplacian is nonpositive:

$$0 \geq \Delta(u - \varepsilon(R^2 - |x|^2))|_{x=x_\varepsilon} = \Delta u(x_\varepsilon) + 2\varepsilon d.$$

But this is impossible, since $\Delta u(x_\varepsilon) \geq 0$, which proves the lemma. $\square$

Lemma 1.2. *Let f and g be polynomials. Then there exists a unique polynomial h, such that $\Delta h = f$ in B_1 and $h = g$ on S_1.*

Proof. By considering $h - g$ we reduce the problem to the one with $g \equiv 0$. Let $f \in \mathcal{P}_n$.

Observe that the operator $T : v \to Tu = \Delta\big[\big(1 - |x|^2\big)v\big]$ maps $\mathcal{P}_n$ into $\mathcal{P}_n$. Furthermore, by the maximum principle, if $v \in \mathcal{P}_n$ and $Tv = 0$, then $\big(1 - |x|^2\big)v \equiv 0$, meaning that T is a one-to-one mapping. Since $\mathcal{P}_n$ is a finite-dimensional linear space, the equation $Tv = f$ has a unique solution in $\mathcal{P}_n$, which proves the lemma in light of the fact that $1 - |x|^2$ is a polynomial. $\square$

It is interesting to discuss the following without referring to the above lemma.

Remark 1.3. If $n \geq 2$, $u \in \mathcal{P}_n$ and $u = 0$ on S_1, then $u = (1 - |x|^2)v$, where $v \in \mathcal{P}_{n-2}$.

Indeed, then $f := \Delta u \in \mathcal{P}_{n-2}$, there is a unique polynomial v such that $Tv = f$, and this $v \in \mathcal{P}_{n-2}$.

Remark 1.4. If $f \equiv 0$ and g is a polynomial of degree n, then h is also a polynomial of degree (at most) n.

Indeed, if $n = 0, 1$, then $h = g$, and, if $n \geq 2$, then $\Delta(g - h) = p$, where p is a polynomial of degree $n - 2$ and it follows from the above proof that $g - h = \big(1 - |x|^2\big)v$, where v is a polynomial of degree $n - 2$, so that h is a polynomial of degree (at most) n. In addition, if $n \geq 2$,

$$g = h + \big(1 - |x|^2\big)v = h + \big(1 - |x|^2\big)h_1 + \big(1 - |x|^2\big)^2 v_1 = ...$$

Example 1.5. If $g = |x|^2$, $h \equiv 1$.

Remark 1.6. Every harmonic polynomial $h \in \mathcal{P}_n$ can be uniquely represented as the sum $h_n + ... + h_0$, where h_k are *homogeneous* of degree k harmonic polynomials.

Definition 1.7. The set of harmonic polynomials is denoted by $\mathcal{H}$. By $\mathcal{H}_n$ we denote the set of homogeneous polynomials of order n which are harmonic. Any element of $\mathcal{H}_n$ restricted to S_1 is called a *spherical harmonic* of degree n. The set of those is denoted by $\mathcal{H}_n(S)$

Corollary 1.8. *The set*

$$\bigcup_{n=0}^{\infty} \{h_0 + ... + h_n : h_i \in \mathcal{H}_i(S), i = 0, ..., n\}$$

is dense in $L_2(S_1)$ since the set of polynomials is dense there.

Definition 1.9. The Laplace-Beltrami operator Δ_S on S_1 is introduced on smooth functions ϕ given on S_1 by the formula

$$\Delta_S \phi(x) = \Delta\Big(\phi\Big(\frac{x}{|x|}\Big)\Big)\Big|_{|x|=1}.$$

Example 1.10. Let us compute $\Delta_S h$ for $h \in \mathcal{H}_n$. Observe that by Euler $nh(x) = x^i h_{x^i}(x)$ and on functions $\phi(x) = \phi(r)$, $r = |x|$, we have

$$\Delta \phi = \phi'' + \frac{d-1}{r}\phi'.$$

Also

$$\Delta(\psi\eta) = \psi\Delta\eta + \eta\Delta\psi + 2\psi_{x^i}\eta_{x^i}, \qquad \Big(\frac{1}{|x|^n}\Big)_{x^i} = -\frac{nx^i}{|x|^{n+2}}.$$

Then for $|x| = 1$ we have

$$\Delta\Big(h\Big(\frac{x}{|x|}\Big)\Big) = \Delta\Big(\frac{1}{|x|^n}h(x)\Big) = \big[n(n+1) - (d-1)n\big]h - 2nx^i h_{x^i},$$

$$\Delta_S h = -n(n+d-2)h,$$

so that h is an eigenfunction of Δ_S with eigenvalue

$$\lambda_n = -n(n+d-2).$$

Theorem 1.11. *The operator Δ_S is formally self-adjoint. Moreover, for any smooth ϕ, ψ given on $\mathbb{R}^d$*

$$I := \int_{S_1} \phi\Delta_S\psi\, dS = -\int_{S_1} (D^t\phi, D^t\psi)\, dS, \tag{1.1}$$

where $D^t\phi(x)$, $x \in S_1$, is the projection of the gradient ϕ_x of ϕ at point x on the tangent plane to S_1 at x, that is

$$(D^t\phi(x))^i = \Big[\phi\Big(\frac{x}{|x|}\Big)\Big]_{x^i} = \phi_{x^j}(x)(\delta^{ij} - x^i x^j).$$

Proof. By using polar coordinates we write that for $r > 1/2$

$$\int_{B_r\setminus B_{1/2}} \phi\Big(\frac{x}{|x|}\Big)\Delta\Big[\psi\Big(\frac{x}{|x|}\Big)\Big]\, dx = \int_{1/2}^r \int_{S_\rho} \phi\Big(\frac{x}{|x|}\Big)\Delta\Big[\psi\Big(\frac{x}{|x|}\Big)\Big]\, dS_\rho\, d\rho.$$

It follows that for $r = 1$

$$I = \frac{d}{dr}\int_{B_r\setminus B_{1/2}} \phi\Big(\frac{x}{|x|}\Big)\Delta\Big[\psi\Big(\frac{x}{|x|}\Big)\Big]\, dx.$$

We use Green's formula and observe that the boundary terms disappear because $x/|x|$ does not change along the normals to the boundary of $B_r \setminus B_{1/2}$. Hence, I equals

$$-\frac{d}{dr}\int_{B_r\setminus B_{1/2}} \Big(\Big(\phi\Big(\frac{x}{|x|}\Big)\Big)_x, \Big(\phi\Big(\frac{x}{|x|}\Big)\Big)_x\Big)\, dx$$

$$= -\int_{S_1} \Big(\Big(\phi\Big(\frac{x}{|x|}\Big)\Big)_x, \Big(\phi\Big(\frac{x}{|x|}\Big)\Big)_x\Big)\, dS,$$

which yields (1.1). The theorem is proved. □

Corollary 1.12. *If $h \in \mathcal{H}_n$, $g \in \mathcal{H}_m$ and $n \neq m$, then $h \perp g$, that is*

$$\int_{S_1} hg\, dS = 0.$$

Indeed,

$$\lambda_n \int_{S_1} hg\, dS = \int_{S_1} \Delta_S hg\, dS = \lambda_m \int_{S_1} hg\, dS,$$

which implies the result since $\lambda_n \neq \lambda_m$.

Theorem 1.13. *Any function $g \in L_2(S_1)$ has a unique representation*

$$g = \sum_{n=0}^{\infty} h_n, \tag{1.2}$$

where $h_n \in \mathcal{H}_n$ and the series converges in $L_2(S_1)$-sense.

To prove this theorem it suffices to refer to Corollaries 1.12 and 1.8. □

Corollary 1.14. *If ϕ is a smooth function on S_1 such that $\Delta_S \phi = \lambda\phi$, where λ is a constant, then either $\phi = \text{const}$ and $\lambda\phi = 0$, or $\phi \neq \text{const}$ and there exists an $m = 1, 2, ...$ such that $\phi \in \mathcal{H}_m(S_1)$ and $\lambda = \lambda_m$.*

Indeed, for $m \geq 1$ and the projection ϕ_m of ϕ on $\mathcal{H}_m(S_1)$ we have

$$\|\phi_m\|^2_{L_2(S_1)} = (\phi, \phi_m)_{L_2(S_1)} = \lambda_m^{-1}(\phi, \Delta_S \phi_m)_{L_2(S_1)} = \lambda_m^{-1}(\Delta_S \phi, \phi_m)_{L_2(S_1)}$$
$$= \lambda\lambda_m^{-1}\|\phi_m\|^2_{L_2(S_1)}.$$

Hence, $\phi_m = 0$ if $\lambda \neq \lambda_m$. Therefore, if $\lambda \notin \{\lambda_m : m \geq 1\}$, then $\phi \in \mathcal{H}_0(S_1)$, $\phi = \text{const}$, $\Delta_S \phi = 0 = \lambda\phi$. However, if $\lambda = \lambda_{m_0}$ for some $m_0 \geq 1$, then $\phi \perp \mathcal{H}_m(S_1)$ for $m \neq m_0$, since, for $h \in \mathcal{H}_m$,

$$\lambda_{m_0} \int_{S_1} h\phi\, dS = \int_{S_1} \Delta_S h\phi\, dS = \lambda_m \int_{S_1} h\phi\, dS.$$

Hence, $\phi \in \mathcal{H}_{m_0}(S_1)$. □

For $\rho > 0$ denote by

$$⨍_{S_\rho} u(x)\, dS_\rho$$

the average value of u on S_ρ, that is its integral over S_ρ divided by the surface of S_ρ. Similarly introduce

$$⨍_{B_\rho} u(x)\, dx = \frac{1}{\mathrm{Vol}\,(B_\rho)} \int_{B_\rho} u(x)\, dx.$$

Theorem 1.15 (mean value theorem)**.** *If $h \in \mathcal{H}$, $a \in \mathbb{R}^d$, and $\rho > 0$, then*

$$⨍_{S_\rho} h(x+a)\, dS_\rho = ⨍_{B_\rho} h(x+a)\, dx = h(a). \tag{1.3}$$

Indeed, it suffices to consider the case where $a = 0$ and $h(0) = 0$. In that case, the equality

$$\fint_{S_\rho} h(x)\, dS_\rho = 0 \tag{1.4}$$

for all $\rho > 0$ implies

$$\int_{S_\rho} h(x)\, dS_\rho = 0, \quad \int_{B_\rho} h(x)\, dx = 0, \quad \fint_{B_\rho} h(x)\, dx = 0.$$

Therefore, we only need to prove (1.4). Scalings imply that it suffices to concentrate on $\rho = 1$. For $h \in \mathcal{H}$ there exists $n \geq 0$ such that $h = h_n + ... + h_0$, where $h_i \in \mathcal{H}_i$. Since $h(0) = 0$ and, obviously, $h_i(0) = 0$, $i \geq 1$, it holds that $h_0 = 0$ and it suffices to prove that (1.4) holds for $\rho = 1$, $n \geq 1$, and $h \in \mathcal{H}_n$. In that case (1.4) follows from Corollary 1.12 since $1 \in \mathcal{H}_0$. □

2. Dirichlet problem

Any d-tuple $\beta = (\beta_1, ..., \beta_d)$ consisting of $\beta_i \in \{0, 1, ...\}$ is called a multi-index. For any multi-index β we set

$$|\beta| = \sum_{i=1}^{d} \beta_i, \quad D^\beta = D_1^{\beta_1} \cdot ... \cdot D_d^{\beta_d}, \quad D_i = \frac{\partial}{\partial x^i} \quad \beta! = \beta_1! \cdot ... \cdot \beta_d!.$$

Lemma 2.1. *Let $R \in (0, \infty)$, g, u be polynomials such that $\Delta u = 0$ and $u = g$ on ∂B_R. Then there exists a constant $N = N(d)$ such that in B_R for any multi-index β*

$$\left| D^\beta u(x) \right| \leq \left(\frac{N|\beta|}{R - |x|} \right)^{|\beta|} \sup_{\partial B_R} |g|. \tag{2.1}$$

Proof. We employ Bernstein's method which uses only the maximum principle. First we note that scalings show that it suffices to concentrate on $R = 1$. In that case take any $\zeta \in C_0^\infty(\mathbb{R}^d)$ with support in B_1, assume that $\zeta(0) = 1$, and consider the function

$$w := \zeta^2 |u_x|^2 + \lambda |u|^2,$$

where $\lambda > 0$ is a constant to be chosen later. We have $\Delta u = 0$, $\Delta u_{x^i} = 0$, and

$$\Delta w = |u_x|^2 \Delta(\zeta^2) + \zeta^2 \big[2u_{x^i}\Delta u_{x^i} + 2\sum_{i,k} |u_{x^i x^k}|^2\big]$$

$$+ 8\zeta\zeta_{x^i} u_{x^k}\, u_{x^i x^k} + 2\lambda |u_x|^2 + 2\lambda u \Delta u$$

$$= \big[2\lambda |u_x|^2 + |u_x|^2 \Delta(\zeta^2)\big] + 2\zeta^2 \sum_{i,k} |u_{x^i x^k}|^2$$

$$+ 8[\zeta_{x^i} u_{x^k}]\,[\zeta u_{x^i x^k}]$$

$$\geq |u_x|^2 \big[2\lambda + \Delta(\zeta^2) - 8|\zeta_x|^2\big]$$

(we have used $2a^2 + 8ab \geq -8b^2$). We see how to take $\lambda = \lambda(d)$ so that $\Delta w \geq 0$. Fix such a λ. Then by the maximum principle

$$\big|u_x(0)\big|^2 \leq \max_{\bar{B}_1} w \leq \max_{\partial B_1} w = \lambda \max_{\partial B_1} |g|^2.$$

This yields (2.1) for $|\beta| = 1$, $R = 1$, and $x = 0$. Then (2.1) holds for $|\beta| = 1$ and any $R \in (0, \infty)$ if $x = 0$. Moving the origin and observing that $\Delta u = 0$ in $B_{R-|x_0|}(x_0)$ for any $x_0 \in B_R$ we get that (2.1) holds for $|\beta| = 1$ and any $x \in B_R$.

For higher values of $|\beta|$, one obtains (2.1) by splitting $B_R \backslash B_{|x|}$ into $|\beta|$ rings of width $\big(R - |x|\big)/|\beta|$ and estimating the derivatives $D_{j_1} \cdot \ldots \cdot D_{j_k} u$ inside the k-th ring by using the above result obtained for $|\beta| = 1$ and the fact that $\Delta D_{j_1} \cdot \ldots \cdot D_{j_{k-1}} u = 0$ in B_R. The lemma is proved. □

Corollary 2.2. *For any $g \in C(\bar{B}_R)$ there exists a unique $u \in C^2_{\text{loc}}(B_R) \cap C(\bar{B}_R)$, that satisfies $\Delta u = 0$ in B_R and is equal to g on ∂B_R. Furthermore, any such function is in $C^\infty_{\text{loc}}(B_R)$ and the assertions of Lemma 2.1 hold true. Also, for any $a \in B_R$ and $\rho > 0$ such that $B_\rho(a) \subset B_R$ we have (mean value theorem)*

$$⨍_{S_\rho} u(x + a)\, dS_\rho = ⨍_{B_\rho} u(x + a)\, dx = u(a)$$

Uniqueness is a consequence of the maximum principle, which also assures that, if the polynomials g_n converge uniformly to g on ∂B_R and u_n are the corresponding polynomials from Lemma 2.1, then

$$\sup_{\bar{B}_R} |u_n - u_m| \leq \sup_{\partial B_R} |g_n - g_m| \to 0$$

as $n, m \to \infty$. Hence, u_n converge in $\bar{B}_R$ uniformly to, say u, which is continuous and equal g on ∂B_R. Applying estimates (2.1) to $u_n - u_m$, we see that also all derivatives of u_n converge uniformly on any compact subset of B_R. Of course, this implies that $u \in C^\infty_{\text{loc}}(B_R)$ and u is the desired solution. After that estimates (2.1) for u and the last assertions are immediate. □

Corollary 2.3. *If $u \in C^2_{\text{loc}}(\mathbb{R}^d)$, $\Delta u = 0$ in $\mathbb{R}^d$, $\alpha \in [0, 1)$, $n \in \{0, 1, 2, ...\}$ and*

$$\overline{\lim_{|x|\to\infty}} \frac{|u(x)|}{|x|^{n+\alpha}} < \infty,$$

then $u \in \mathcal{H}_n$.

Indeed, as $R \to \infty$, (2.1) implies that all derivatives of order $n + 1$ vanish.

Corollary 2.4. *If $u \in C^2_{\text{loc}}(B_R)$ is harmonic in B_R, then it is real analytic.*

This follows from (2.1) (applied in a smaller ball) and the fact that, for any constant N and $|x| \le (eN)^{-1}$,

$$\frac{N^k |x|^k k^k}{k!} \to 0$$

as $k \to \infty$.

Corollary 2.5. *Let $u_n(x)$, $n = 1, 2, ...$, be a sequence of harmonic functions in B_1 such that they are uniformly bounded in B_R for any $R < 1$ and at any point of B_1 they converge as $n \to \infty$ to a function $u(x)$. Then $u(x)$ is infinitely differentiable, harmonic in B_1, and any derivative of u_n converges to the corresponding derivative of u locally uniformly in B_1.*

Theorem 2.6 (Harnack's inequality)**.** *Let $u \in C^2_{\text{loc}}(B_R)$ be a nonnegative harmonic in B_R. Then $u(x) \le 5^d u(y)$ if $|x|, |y| \le R/5$.*

Proof. We have $B_{R/2}(y) \supset B_{R/10}(x)$ and

$$u(y) = \fint_{B_{R/2}(y)} u(z)\, dz \ge M \fint_{B_{R/10}(x)} u(z)\, dz = M u(x),$$

where

$$M = \frac{\operatorname{Vol} B_{R/10}}{\operatorname{Vol} B_{R/2}} = 5^{-d}.$$

This proves the theorem. □

Corollary 2.7 (one-sided Liouville's theorem). *If $u \in C^2_{\mathrm{loc}}(\mathbb{R}^d)$ is harmonic in $\mathbb{R}^d$ which is bounded from below, then $u = \mathrm{const}$.*

Indeed, in that case $v := u - \inf_{\mathbb{R}^d} u$ is also a harmonic in $\mathbb{R}^d$ and there is a sequence y_n such that $v(y_n) \to 0$ as $n \to \infty$. By Theorem 2.6 we have $v(x) \leq N v(y_n)$, which implies that $(0 \leq) v \leq 0$, $v \equiv 0$, and $u \equiv \inf_{\mathbb{R}^d} u$.

A different argument. One can assume that $u \geq 0$. Then Harnack's inequality implies that u is bounded, and, by Corollary 2.3, $u \in \mathcal{H}_0$, that is $u = \mathrm{const}$. □

Next, we are interested in solving the Dirichlet problem with more general boundary data.

Theorem 2.8. *For any function $g \in L_2(S_1)$ there exists a unique harmonic function u in B_1 such that $u(tx) \to g(x)$ in $L_2(S_1)$-sense as $t \uparrow 1$.*

Proof. Existence. We take the right-hand side of (1.2) and consider it in B_1. Obviously, $h_n \perp h_m$ in $L_2(B_1)$ if $n \neq m$. Furthermore,

$$\int_{B_1} |h_n|^2 \, dx = c \int_0^1 r^{d-1} \Big(\int_{S_1} |h_n(rx)|^2 \, dS \Big) dr$$

$$= c \int_0^1 r^{d+2n-1} \, dr \int_{S_1} |h_n(x)|^2 \, dS = c(d+2n)^{-1} \|h_n\|^2_{L_2(S_1)}.$$

It follows that the series in (1.2) converges in $L_2(B_1)$.

Finite sums Σ_n of this series are well defined harmonic functions satisfying

$$\Sigma_n(x) = \fint_{B_r(x)} \Sigma_n(y) \, dy.$$

By Corollary 2.5, Σ_n converge uniformly in any B_r, $r < 1$, to a harmonic function. Call it u. Then, for $x \in S_1$ and $t \in [0, 1)$ we have

$$u(tx) = \sum_{n=0}^{\infty} t^n h_n(x).$$

Hence

$$\|u(t\cdot) - g\|^2_{L_2(S_1)} = \sum_{n=0}^{\infty} (1 - t^n) \|h_n\|^2_{L_2(S_1)},$$

which goes to zero as $t \uparrow 1$ indeed.

Uniqueness. Observe that by Green's formula for $t < 1$

$$\frac{d}{dt} \int_{S_1} |u(tx)|^2 \, dS = 2 \int_{S_1} u(tx)(x, u_x(tx)) \, dS = 2t \int_{B_1} |u_x(tx)|^2 \, dx \geq 0$$

and this yields the result. The theorem is proved. □

3. Case $d = 2$

Let s denote the point on S_1 which is at the distance s along the circle in the counterclockwise direction from the point $(1, 0)$. As easily follows from (1.1), the Laplace-Beltrami operator on S_1 is just the second-order derivative with respect to s. Therefore, $h \in \mathcal{H}_n(S_1)$ are (the only, see Corollary 1.14) solutions of

$$h'' = \lambda_n h = -n^2 h, \quad h(s) = c_1 \sin ns + c_2 \cos ns,$$

and (1.2) is a usual representation of a function $f \in L_2(0, 2\pi)$ by its Fourier series. Amazing, at the first sight, the theory of Fourier series has nothing to do with the Laplacian in $d = 2$.

Also observe that since, generally, $\mathcal{H}_n$ are obviously invariant under orthogonal transformation in this case for $d = 2$ and $n = 1$ for any t we have

$$\sin(t + s) = b(t) \cos s + c(t) \sin s$$

and one can find $b(t)$ and $c(t)$ by using substitutions. Say, for $s = 0$ we get $\sin t = b(t)$. Knowing that and interchanging s and t we get $c(t) = \cos t$.

One more remarkable thing is that $\sin ns$, which is in $\mathcal{H}_n(S)$ by Corollary 1.14, as a solution of the appropriate equation, and is therefore the trace on S_1 of a homogeneous nth-order harmonic *polynomial*. For $n = 2$ this means that $\sin 2s$ and $\cos 2s$ are linear combinations of the traces of $x^2 - y^2$ and xy, that is of $\cos^2 s - \sin^2 s$ and $\cos s \sin s$.

4. Basis in $\mathcal{H}_n$ and its dimension

Let $\mathcal{P}_n^{\text{hom}}$ be the set of *homogeneous* polynomials of degree n.

Lemma 4.1. *For $n \geq 2$ every $p \in \mathcal{P}_n^{\text{hom}}$ has a unique representation*

$$p = h + |x|^2 r,$$

where $h \in \mathcal{H}_n$ and $r \in \mathcal{P}_{n-2}^{\text{hom}}$.

Proof. Existence. By Lemma 1.2 there is a harmonic polynomial u of degree n such that $p = u$ on S_1. By Remark 1.3 we have $p - u = (1 - |x|^2)q$, where $q \in \mathcal{P}_{n-2}$. Let h be the homogeneous part of u of order n and r be the homogeneous part of q of order $n - 2$. Then

$$p = h - |x|^2 r + [(u - h) + r + (1 - |x|^2)(q - r)]$$

$$= h - |x|^2 r + p_{n-1},$$

where $p_{n-1} \in \mathcal{P}_{n-1}$. By homogeneity $p_{n-1} = 0$ and we get the desired representation.

Uniqueness. We need to prove that if $h \in \mathcal{H}_n$, $r \in \mathcal{P}_{n-2}^{\text{hom}}$ and $h + |x|^2 r \equiv 0$, then $h = r \equiv 0$.

More generally, let $k \geq 1$ be an integer such that h admits the representation $h = -|x|^{2k} r$, where r is a polynomial. We want to show that $h = 0$. Obviously $n - 2k \geq 0$ and

$$0 = \Delta(|x|^{2k} r) = |x|^{2k} \Delta r + 2k(2n - 2k + d - 2)|x|^{2k-2} r,$$

Here $2k(2n-2k+d-2) > 0$, so that $r_1 := c\Delta r$ ($c^{-1} = -2k(2n-2k+d-2)$) is a polynomial, and $h = -|x|^{2(k+1)} r_1$. By induction we see that for any $k \geq 1$ there exists polynomials r_k such that $h = |x|^{2k} r_k$, which is only possible if $h \equiv 0$ and this proves the lemma. □

As a simple corollary of Lemma 4.1 obtained by iteration we come to the following.

Theorem 4.2. *Let $p \in \mathcal{P}_n$. Then there exist unique $h_{n-2i} \in \mathcal{H}_{n-2i}$, $i = 0, 1, ..., k$, where $k = \lfloor n/2 \rfloor$ such that*

$$p = h_n + |x|^2 h_{n-2} + ... + |x|^{2k} h_{n-2k}.$$

Example 4.3. Take $p = x^1 x^2 x^3$. It has a representation $p = h + |x|^2 h_1$, where $h \in \mathcal{H}_3$ and h_1 is an affine homogeneous function of $x = (x^1, x^2, x^3)$. By symmetry it follows that $h_1 = c(x^1 + x^2 + x^3)$ and $x^1 x^2 x^3 - c|x|^2(x^1 + x^2 + x^3)$ is harmonic.

Theorem 4.4. *If $n \geq 2$, then*

$$\dim \mathcal{H}_n = \binom{d+n-1}{d-1} - \binom{d+n-3}{d-1}.$$

Proof. First we find $\dim \mathcal{P}_n^{\text{hom}}$, which is the number of different monomials

$$x^\alpha = (x^1)^{\alpha_1} \cdot ... \cdot (x^d)^{\alpha_d}$$

such that $\alpha_1 + ... + \alpha_d = n$. We consider a row of $d+n-1$ seats, numbered from 1 to $d+n-1$, choose arbitrarily $d-1$ of them $i_1 < i_2 < ... < i_{d-1}$ and then define α_k as the number of seats strictly between i_k and i_{k+1}: $\alpha_1 = i_1 - 1$, $\alpha_2 = i_2 - i_1 - 1$, ..., $\alpha_d = d + n - 1 - i_{d-1}$. The number of such arrangement will be exactly the number of different monomials x^α such that $\alpha_1 + ... + \alpha_d = n$. This number is

$$\binom{d+n-1}{d-1}.$$

By Lemma 4.1 to obtain $\dim \mathcal{H}_n$ it suffices to subtract from this number $\dim \mathcal{P}_{n-2}^{\text{hom}}$. This yields the result. □

Remark 4.5. If $d = 2$ and $n \geq 1$, $\dim \mathcal{H}_n = 2$.

Now we want to find a basis in $\mathcal{H}_n$.

Lemma 4.6. *For $d > 2$ and any multi-index α the function*

$$h := |x|^{2n+d-2} D^\alpha \frac{1}{|x|^{d-2}}$$

belongs to $\mathcal{H}_n$, where $n = |\alpha|$.

Proof. That $h \in \mathcal{P}_n^{\text{hom}}$ is checked by induction. Then, since $D^\alpha \frac{1}{|x|^{d-2}}$ is harmonic in $\mathbb{R}^d \setminus \{0\}$ and homogeneous of degree $-(n + d - 2)$, we have

$$\begin{aligned}
|x|^{-n-d+4} \Delta h &= [(2n+d-2)(2n+d-3) + (d-1)(2n+d-2)] D^\alpha \frac{1}{|x|^{d-2}} \\
&\quad + 2(2n + d - 2) x^j D_j D^\alpha \frac{1}{|x|^{d-2}} \\
&= [(2n + d - 2)(2n + d - 3) + (d - 1)(2n + d - 2) \\
&\quad - 2(2n + d - 2)(n + d - 2)] D^\alpha \frac{1}{|x|^{d-2}} = 0.
\end{aligned}$$

□

Lemma 4.7. *Let $d \geq 3$ and let $u \in C^2(\mathbb{R}^d)$ have compact support. Then for any $x \in \mathbb{R}^d$*

$$u(x) = -c_d \int_{\mathbb{R}^d} \frac{1}{|y - x|^{d-2}} \Delta u(y)\, dy, \tag{4.1}$$

(Newton's potential of $-\Delta u$) where c_d is a constant depending only on d.

Proof. First note that it suffices to prove (4.1) for $x = 0$. In that case take $\zeta \in C_0^\infty(\mathbb{R}^2)$ such that $\zeta = 1$ in B_1 and for $\varepsilon > 0$ set $\zeta_\varepsilon(y) = \zeta(y/\varepsilon)$. Observe that if in (4.1) we take $(1 - \zeta_\varepsilon)u$ in place of u, then we can integrate by parts ($x = 0$) and, using the fact that $\Delta |y|^{2-d} = 0$, conclude that the integral in (4.1) is zero. Hence, the integral in (4.1) in its original form equals

$$I := \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d} \frac{1}{|y|^{d-2}} \Delta(\zeta_\varepsilon u)(y)\, dy.$$

We use that

$$\varepsilon^{-1} \int_{\mathbb{R}^d} \frac{1}{|y|^{d-2}} \zeta_{x^i}(y/\varepsilon) u_{x^i}(y)\, dy = \varepsilon \int_{\mathbb{R}^d} \frac{1}{|y|^{d-2}} \zeta_{x^i}(y) u_{x^i}(\varepsilon y)\, dy \to 0,$$

$$\int_{\mathbb{R}^d} \frac{1}{|y|^{d-2}} \zeta(y/\varepsilon) \Delta u(y)\, dy \to 0,$$

and conclude that

$$I := \lim_{\varepsilon \downarrow 0} \varepsilon^{-2} \int_{\mathbb{R}^d} \frac{1}{|y|^{d-2}} u(y) (\Delta \zeta)(y/\varepsilon)\, dy.$$

Observe that for N being the Lipschitz constant of u

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-2}\int_{\mathbb{R}^d}\frac{1}{|y|^{d-2}}|u(y)-u(0)|\,|\Delta\zeta|(y/\varepsilon)\,dy$$
$$\leq N\lim_{\varepsilon\downarrow 0}\varepsilon\int_{\mathbb{R}^d}\frac{1}{|y|^{d-2}}|y|\,|\Delta\zeta|(y)\,dy=0.$$

Hence,

$$I=u(0)\lim_{\varepsilon\downarrow 0}\varepsilon^{-2}\int_{\mathbb{R}^d}\frac{1}{|y|^{d-2}}(\Delta\zeta)(y/\varepsilon)\,dy=u(0)\int_{\mathbb{R}^d}\frac{1}{|y|^{d-2}}\Delta\zeta(y)\,dy.$$

This proves the lemma. □

Recall that we denote by $\mathcal{H}_n(S_1)$ the set of restrictions of the functions in $\mathcal{H}_n$ to S_1.

Lemma 4.8. *If $d>2$, the linear span of*

$$\{D^\alpha|x|^{2-d}:|\alpha|=n\}$$

is $\mathcal{H}_n(S_1)$.

Proof. If $n=0$, the assertion is obvious. For $n\geq 1$ assume the contrary. Then there exists $h\in\mathcal{H}_n(S_1)$ such that

$$\int_{S_1}hD^\alpha|x|^{2-d}\,dS=0 \tag{4.2}$$

for all α such that $|\alpha|=n$. By Lemma 4.6 and Corollary 1.12 equation (4.2) also holds for $|\alpha|\neq n$.

Next, the function

$$F(x):=\int_{S_1}h(y)|y-x|^{2-d}\,dS$$

is harmonic outside S_1 and as such is real analytic there. Equation (4.2) says that all derivatives of F vanish at 0. It follows that $F=0$ in B_1. Furthermore, F is a bounded continuous function on $\mathbb{R}^d$ since bounded continuous

$$\int_{S_1}h(y)|y-x|^{2-d}I_{|y-x|\geq\varepsilon}\,dS$$

converge uniformly on $\mathbb{R}^d$ to F as $\varepsilon\downarrow 0$. In addition, $F\to 0$ as $|x|\to\infty$, hence, by the maximum principle $F=0$ in B_1^c, since it is harmonic in $\bar{B}_1^c$. Thus, $F\equiv 0$.

Now take $u\in C_0^\infty(\mathbb{R}^d)$ and integrate the equality

$$0=\int_{S_1}h(y)|y-x|^{2-d}\,dS\Delta u(x)$$

over $\mathbb{R}^d$. By using Lemma 4.7 we get

$$\int_{S_1} h(y)u(y)\, dS = 0$$

and the arbitrariness of u implies that $h \equiv 0$. $\square$

Theorem 4.9. *For $d \geq 3$ the set*

$$\{D^\alpha |x|^{2-d} : |\alpha| = n, \alpha_1 \leq 1\} \tag{4.3}$$

is a basis in $\mathcal{H}_n(S_1)$.

Proof. The number of elements in (4.3) is not greater than the number of multi-indices α such that $|\alpha| = n$ and $\alpha_1 \leq 1$. The latter number is easily shown to be equal to $\dim \mathcal{H}_n(S_1)$ (use the same interpretation as in the proof of Theorem 4.4). Therefore, due to Lemma 4.8 we only need to show that $D^\alpha |x|^{2-d}$ belongs to the span of (4.3) for any α_1 if $|\alpha| = n$. This is trivial because $|x|^{2-d}$ is harmonic and for any even α_1 we have

$$D_1^{\alpha_1} |x|^{2-d} = (-D_2^2 - ... - D_d^2)^{\alpha_1/2} |x|^{2-d},$$

whereas if $\alpha_1 = 2k + 1$ $(\leq n)$,

$$D_1^{\alpha_1} |x|^{2-d} = D_1(-D_2^2 - ... - D_d^2)^k |x|^{2-d}.$$

$\square$

5. An unexpected formula for the scalar product of spherical harmonics

Recall that for any multi-index $\alpha = (\alpha_1, ..., \alpha_d)$ and $x \in \mathbb{R}^d$ we set

$$x^\alpha = (x^1)^{\alpha_1} \cdot ... \cdot (x^d)^{\alpha_d}, \quad |\alpha| = \alpha_1 + ... + \alpha_d, \quad \alpha! = \alpha_1! \cdot ... \cdot \alpha_d!\,.$$

Suppose

$$p = \sum_{|\alpha|=n} b_\alpha x^\alpha, \quad q = \sum_{|\alpha|=n} c_\alpha x^\alpha$$

are in $\mathcal{H}_n$. We want to find their scalar product in $L_2(S_1)$. It looks like the best we can do is to write

$$(p, q)_{L_2(S_1)} = \sum_{\alpha,\beta} b_\alpha c_\beta \int_{S_1} x^{\alpha+\beta}\, dS.$$

The integral over S_1 of the monomial $x^{\alpha+\beta}$ was explicitly calculated by Hermann Weyl in Section 3 of [2] (1939). Using that result would complete the formula for $(p, q)_{L_2(S_1)}$. There is however a shorter way, which I take from [1].

Lemma 5.1. *If $n > 0$ and $p, q \in \mathcal{H}_n$, then*

$$I := \int_{S_1} pq\, dS = \frac{1}{n(d+2n-2)} \sum_{j=1}^{d} \int_{S_1} D_j p D_j q\, dS. \tag{5.1}$$

Proof. Let Du denote the gradient of u. By Theorem 1.11 and Example 1.10

$$-\lambda_n I = \int_{S_1} \Big(D\Big(\frac{1}{|x|^n} p(x)\Big), D\Big(\frac{1}{|x|^n} q(x)\Big)\Big)\, dS$$

$$= \int_{S_1} \Big(-nxp(x) + Dp(x), -nxq(x) + Dq(x)\Big)\, dS.$$

By using that $(x, Dp(x)) = np(x)$ we see that

$$-\lambda_n I = -n^2 I + \int_{S_1} (Dp, Dq)\, dS,$$

which is equivalent to (5.1). □

Theorem 5.2. *Let $p = \sum_\alpha b_\alpha x^\alpha$ and $q = \sum_\alpha c_\alpha x^\alpha$ be harmonic polynomials. Then*

$$(p, q)_{L_2(S_1)} = \sum_\alpha b_\alpha c_\alpha w_\alpha,$$

where

$$w_\alpha = \frac{\alpha!}{d(d+2)\cdot \ldots \cdot (d+2|\alpha|-2)} \qquad \alpha \neq 0, \quad w_0 = 1.$$

Proof. Owing to Corollary 1.12 it suffices to prove the theorem under the assumption that $p, q \in \mathcal{H}_n$.

If $n = 0$, then p, q are constant and the desired result obviously holds.

If $n > 0$ observe that $D_j p$ and $D_j q$ are harmonic polynomials in $\mathcal{H}_{n-1}$, so that Lemma 5.1 is applicable. By induction we get for e_j being the jth basis vector in $\mathbb{R}^d$ that

$$\sum_{j=1}^{d} (D_j p, D_j q)_{L_2(S_1)} = \sum_{j=1}^{d} \int_{S_1} \Big(\sum_\alpha b_\alpha \alpha_j x^{\alpha - e_j}\Big)\Big(\sum_\alpha c_\alpha \alpha_j x^{\alpha - e_j}\Big)$$

$$= \sum_{j=1}^{d} \sum_\alpha b_\alpha c_\alpha \alpha_j^2 \frac{(\alpha - e_j)!}{d(d+2)\cdot \ldots \cdot (2+2n-4)}$$

$$= \sum_\alpha b_\alpha c_\alpha \sum_{j=1}^{d} \alpha_j \frac{\alpha!}{d(d+2)\cdot \ldots \cdot (2+2n-4)}$$

$$= \sum_{\alpha} b_\alpha c_\alpha \frac{\alpha! n}{d(d+2) \cdot \ldots \cdot (2+2n-4)}.$$

By combining this with Lemma 5.1, we get the result. □

References

[1] Sheldon Axler, Paul Bourdon, and Wade Ramey, "Harmonic function theory", Springer, 2001.

[2] Hermann Weyl, *On the volume of tubes*, American Journal of Mathematics, 61 (1939), 461-472.