Derivatives from implicit equation
(End of section 3.6)

Let z be height. This time, we don't have an explicit function for height in terms of x and y. Instead, we have the implicit equation

$$f(x, y, z) = 0$$

We want to find how height changes as a function of x and y, i.e., view as $z = z(x, y)$ and calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Calculate

$$J_f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}$$

$$J_{\mathbf{g}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix}$$

Multiply to obtain

$$J_f J_{\mathbf{g}} = \begin{bmatrix} \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} \\ \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} \end{bmatrix}$$

Write implicit equation as

$$f(x, y, z(x, y)) = 0$$

This is a composition of functions $f(\mathbf{g}(x, y)) = 0$ where

$$\mathbf{g}(x, y) = (x, y, z(x, y)).$$

Differentiate and apply the chain rule.

$$0 = J_f(\mathbf{g}(x, y))J_{\mathbf{g}}(x, y).$$

Each component of $J_f J_{\mathbf{g}}$ must be zero.

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0$$
$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0$$

Solve for the partial derivatives of z:

$$\frac{\partial z}{\partial x} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{-\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}$$

Example: height z is given implicitly by

$$z^3 - xyz - 3 = 0$$

$$\frac{\partial f}{\partial x} = -yz \quad \frac{\partial f}{\partial y} = -xz \quad \frac{\partial f}{\partial z} = 3z^2 - xy$$

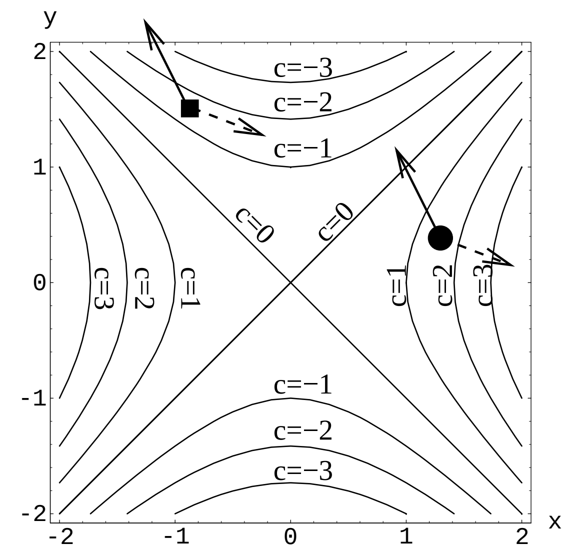
Rate of change in height as change x

$$\frac{\partial z}{\partial x} = \frac{yz}{3z^2 - xy}$$

Rate of change in height as change y

$$\frac{\partial z}{\partial y} = \frac{xz}{3z^2 - xy}$$

Estimate rate of change in different directions from level curve plot.



Rate of change depends both on point and direction. (Think mountain climbing.)

The directional derivative

Section 4.1

Partial derivatives: rate of change in the directions parallel to the coordinate axis.

Compute rate of change in an arbitrary direction?

Use a directional derivative.

Directional derivative is a one-dimensional derivative (like partial derivatives).

To compute derivative at point $\mathbf{x} = \mathbf{a}$ in direction given by $\mathbf{u}$ (where $\mathbf{u}$ is a unit vector $\|\mathbf{u}\| = 1$):

$$Df_{\mathbf{u}}(\mathbf{a}) = \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{u}) - f(\mathbf{a})}{h}$$

Change in function $f(\mathbf{x})$ as move in direction of $\mathbf{u}$ starting at point $\mathbf{x} = \mathbf{a}$.

What if take derivative in the positive x direction?

If $\mathbf{u} = \mathbf{i}$, then $Df_{\mathbf{u}}(\mathbf{a}) = \frac{\partial f}{\partial x}(\mathbf{a})$.

Positive y direction?

If $\mathbf{u} = \mathbf{j}$, then $Df_{\mathbf{u}}(\mathbf{a}) = \frac{\partial f}{\partial y}(\mathbf{a})$.

Relationship between directional derivative and partial derivatives?

The gradient

The gradient of a function $f(x, y)$ is the vector

$$\nabla f(x,y) = \frac{\partial f}{\partial x}(x,y)\mathbf{i} + \frac{\partial f}{\partial y}(x,y)\mathbf{j}$$

(analogous in higher dimensions).

The directional derivative is related to the gradient by

$$Df_{\mathbf{u}}(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{u}.$$

For $f(x, y)$ and $\mathbf{u} = (u_1, u_2)$

$$Df_{\mathbf{u}}(\mathbf{a}) = \frac{\partial f}{\partial x}(\mathbf{a})u_1 + \frac{\partial f}{\partial y}(\mathbf{a})u_2$$

Write this as a dot product between $\mathbf{u}$ and a new vector:

$$Df_{\mathbf{u}}(\mathbf{a}) = \left(\frac{\partial f}{\partial x}(\mathbf{a})\mathbf{i} + \frac{\partial f}{\partial y}(\mathbf{a})\mathbf{j}\right) \cdot (u_1\mathbf{i} + u_2\mathbf{j})$$

Example: $f(x, y) = x^2 y$.

Find $\nabla f(3, 2)$.
Find the derivative of f in the direction of (1,2) at the point (3,2).

$$\frac{\partial f}{\partial x}(x,y) = 2xy \qquad \frac{\partial f}{\partial y}(x,y) = x^2$$
$$\frac{\partial f}{\partial x}(3,2) = 12 \qquad \frac{\partial f}{\partial y}(3,2) = 9$$

Therefore,

$$\nabla f(3,2) = 12\mathbf{i} + 9\mathbf{j}$$

Directional derivative: Direction (1,2) is not a unit vector.
$\|(1,2)\| = \sqrt{1^2+2^2} = \sqrt{5}$.
Unit vector $\mathbf{u} = (1/\sqrt{5}, 2/\sqrt{5})$.

$$\begin{aligned} Df_{\mathbf{u}} &= \nabla f(3,2)\cdot \mathbf{u} \\ &= (12\mathbf{i}+9\mathbf{j})\cdot(u_1\mathbf{i}+u_2\mathbf{j}) \\ &= 12u_1 + 9u_2 \\ &= \frac{12}{\sqrt{5}} + \frac{18}{\sqrt{5}} = \frac{30}{\sqrt{5}} \end{aligned}$$

Note: gradient is closely related to the total derivative.

For our example, the total derivative $Df(3,2)$ is the linear function represented by the matrix

$$J_f(3,2) = [12 \quad 9].$$

Gradient is a vector, while the total derivative is a linear function or matrix.

Take derivative at same point in a different direction: $(2,1)$.
Unit vector is $\mathbf{u} = (2/\sqrt{5}, 1/\sqrt{5})$.

$$\begin{aligned} Df_{\mathbf{u}} &= \nabla f(3,2)\cdot \mathbf{u} \\ &= 12u_1 + 9u_2 \\ &= \frac{24}{\sqrt{5}} + \frac{9}{\sqrt{5}} = \frac{33}{\sqrt{5}} \end{aligned}$$

What does the gradient mean?

Remember that the direction derivative is $Df_{\mathbf{u}} = \nabla f \cdot \mathbf{u}$.

Let θ be the angle between ∇f and $\mathbf{u}$, then

$$\begin{aligned} Df_{\mathbf{u}} &= \|\nabla f\|\|\mathbf{u}\| \cos\theta \\ &= \|\nabla f\| \cos\theta \end{aligned}$$

When is $Df_{\mathbf{u}}$ the largest? When $\theta = 0$. Then

$$Df_{\mathbf{u}} = \|\nabla f\| \cos 0 = \|\nabla f\|$$

f increases most rapidly in the direction of ∇f.

The rate of increase is $\|\nabla f\|$.

Perpendicular to $\|\nabla f\|$, the directional derivative is zero.

$$Df_{\mathbf{u}} = \|\nabla f\| \cos(\pi/2) = \|\nabla f\| \cos(3\pi/2) = 0$$

Relationship between level curves and gradient

Recall: if θ is the angle between ∇f and $\mathbf{u}$, then

$$Df_{\mathbf{u}} = \|\nabla f\| \cos\theta$$

Assume $\|\nabla f\| \neq 0$.

$Df_{\mathbf{u}} = 0$ is zero if and only if $\cos\theta = 0$, i.e., when ∇f and $\mathbf{u}$ are perpendicular.

Our example: $f(x, y) = x^2 y$.
Recall $\nabla f(3, 2) = 12\mathbf{i} + 9\mathbf{j}$.

Directional derivative at (3,2) is largest in the direction (12,9).

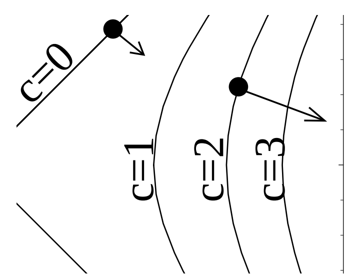
The rate of increase in that direction is

$$\begin{aligned} \|\nabla f\| &= \|(12, 9)\| \\ &= \sqrt{12^2 + 9^2} = \sqrt{225} = 15 \end{aligned}$$

Along a level curve, $f(x, y)$ is constant.
(A level curve is defined by $f(x, y) = c$.)

Hence, the derivative in the direction tangent to a level curve is zero.

Therefore, the gradient is perpendicular to the level curve.



(Magnitude $\|\nabla f\|$ is large when level curves are close together.)

Tangent lines from gradient

We just showed the gradient is perpendicular to the tangent of the level curves.

We can use the gradient to find the equation for tangent lines.

Example: Find tangent line to $x^2 + 2y^2 = 22$ at the point (2,3).

Can divide normal vector by 4: $\mathbf{n} = (1, 3)$
Equation for the tangent line is

$$\begin{aligned} 0 &= ((x, y) - \mathbf{a}) \cdot \mathbf{n} \\ &= ((x, y) - (2, 3)) \cdot (1, 3) \\ &= (x - 2, y - 3) \cdot (1, 3) \\ &= x - 2 + 3y - 9 \\ &= x + 3y - 11 \end{aligned}$$

I.e, equation is $x + 3y - 11 = 0$.

$$\begin{aligned} f(x, y) &= x^2 + 2y^2 \\ \nabla f(x, y) &= (2x, 4y) \\ \nabla f(2, 3) &= (4, 12) \end{aligned}$$

So line is through point $\mathbf{a} = (2, 3)$ and perpendicular to vector $\mathbf{n} = (4, 12)$.

Since we are in two dimensions ($\mathbb{R}^2$), this specifies the line. (Just the normal specifies a plane in $\mathbb{R}^3$.)

Points (x, y) on line must satisfy

$$((x, y) - \mathbf{a}) \cdot \mathbf{n} = 0$$

(again, just like plane in $\mathbb{R}^3$).

Tangent planes from gradient

We showed above that $Df_{\mathbf{u}} = 0$ is zero if and only if ∇f and $\mathbf{u}$ are perpendicular.

Since along a level surface (defined by $f(x, y, z) = c$) f is constant, $Df_{\mathbf{u}} = 0$ in any direction tangent to the level surface.

We conclude the gradient is perpendicular to level surfaces.

We can hence use the gradient to find tangent planes to level surfaces.

Example: Find tangent plane to

$$2x^2 + 3y^2 + z^2 = 20$$

at the point (2,1,3).

Let $f(x, y, z) = 2x^2 + 3y^2 + z^2$. The surface is the level surface $f(x, y, z) = 20$.

The gradient of f is perpendicular to the tangent plane.

$$\nabla f(x, y, z) = (4x, 6y, 2z)$$
$$\nabla f(2, 1, 3) = (8, 6, 6)$$

The vector (8,6,6) or (4,3,3) is perpendicular to the tangent plane.

Let normal vector $\mathbf{n} = (4, 3, 3)$.

Equation for tangent plane with normal vector $\mathbf{n}$ through point $\mathbf{a} = (2, 1, 3)$ is

$$(\mathbf{x} - \mathbf{a}) \cdot \mathbf{n} = 0$$

I.e., equation is

$$4(x - 2) + 3(y - 1) + 3(z - 3) = 0$$