Relationship between level curves and gradient

Recall: if θ is the angle between ∇f and $\mathbf{u}$, then

$$Df_{\mathbf{u}} = \|\nabla f\| \cos\theta$$

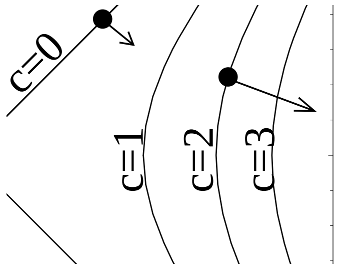
Assume $\|\nabla f\| \neq 0$.

$Df_{\mathbf{u}} = 0$ is zero if and only if $\cos\theta = 0$, i.e., when ∇f and $\mathbf{u}$ are perpendicular.

Along a level curve, $f(x,y)$ is constant.
(A level curve is defined by $f(x,y) = c$.)

Hence, the derivative in the direction tangent to a level curve is zero.

Therefore, the gradient is perpendicular to the level curve.



(Magnitude $\|\nabla f\|$ is large when level curves are close together.)

Tangent lines from gradient

We just showed the gradient is perpendicular to the tangent of the level curves.

We can use the gradient to find the equation for tangent lines.

Example: Find tangent line to $x^2 + 2y^2 = 22$ at the point (2,3).

$$\begin{aligned} f(x,y) &= x^2 + 2y^2 \\ \nabla f(x,y) &= (2x, 4y) \\ \nabla f(2,3) &= (4, 12) \end{aligned}$$

So line is through point $\mathbf{a} = (2,3)$ and perpendicular to vector $\mathbf{n} = (4,12)$.

Since we are in two dimensions ($\mathbb{R}^2$), this specifies the line. (Just the normal specifies a plane in $\mathbb{R}^3$.)

Points (x,y) on line must satisfy

$$((x,y) - \mathbf{a}) \cdot \mathbf{n} = 0$$

(again, just like plane in $\mathbb{R}^3$).

Can divide normal vector by 4: $\mathbf{n} = (1,3)$ Equation for the tangent line is

$$\begin{aligned} 0 &= ((x,y) - \mathbf{a}) \cdot \mathbf{n} \\ &= ((x,y) - (2,3)) \cdot (1,3) \\ &= (x-2, y-3) \cdot (1,3) \\ &= x - 2 + 3y - 9 \\ &= x + 3y - 11 \end{aligned}$$

I.e, equation is $x + 3y - 11 = 0$.

Tangent planes from gradient

We showed above that $Df_{\mathbf{u}} = 0$ is zero if and only if ∇f and $\mathbf{u}$ are perpendicular.

Since along a level surface (defined by $f(x,y,z) = c$) f is constant, $Df_{\mathbf{u}} = 0$ in any direction tangent to the level surface.

We conclude the gradient is perpendicular to level surfaces.

We can hence use the gradient to find tangent planes to level surfaces.

Example: Find tangent plane to

$$2x^2 + 3y^2 + z^2 = 20$$

at the point (2,1,3).

Let $f(x,y,z) = 2x^2 + 3y^2 + z^2$. The surface is the level surface $f(x,y,z) = 20$.

The gradient of f is perpendicular to the tangent plane.

$$\begin{aligned} \nabla f(x,y,z) &= (4x, 6y, 2z) \\ \nabla f(2,1,3) &= (8,6,6) \end{aligned}$$

The vector (8,6,6) or (4,3,3) is perpendicular to the tangent plane.

Let normal vector $\mathbf{n} = (4,3,3)$.

Equation for tangent plane with normal vector $\mathbf{n}$ through point $\mathbf{a} = (2,1,3)$ is

$$(\mathbf{x} - \mathbf{a}) \cdot \mathbf{n} = 0$$

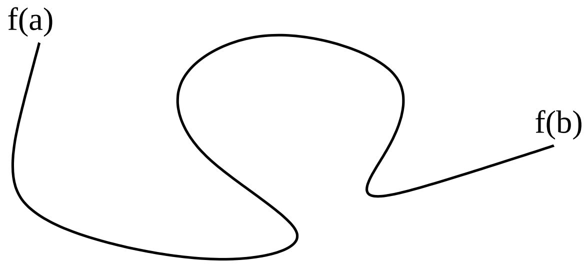
I.e., equation is

$$4(x-2) + 3(y-1) + 3(z-3) = 0$$

Paths (Section 5.1)

Let $\mathbf{f} : \mathbb{R} \to \mathbb{R}^2$

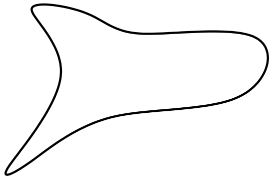
As let t go from a to b, $\mathbf{f}$ traces out a path:

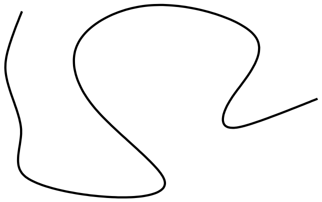


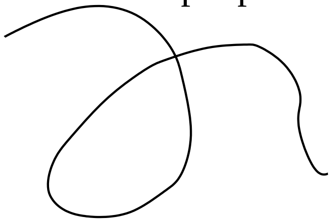
Write path as $\mathbf{f} : [a, b] \to \mathbb{R}^2$ to indicate t starts at a and ends at b.

The path $\mathbf{f} : [a, b] \to \mathbb{R}^2$ is closed if $f(a) = f(b)$.

The path is simple if it doesn't intersect itself ($f(a) = f(b)$ is OK).


Closed path


Simple path


Nonsimple path

The path is smooth is $\mathbf{f}'(t)$ is continuous and $\mathbf{f}'(t) \neq 0$ for all t.

Example: Is the path

$$\mathbf{f}(t) = (\sin t, \sin 2t) \qquad 0 \le t \le 2\pi$$

smooth? Is it simple?

$\mathbf{f}'(t) = (\cos t, 2\cos 2t)$

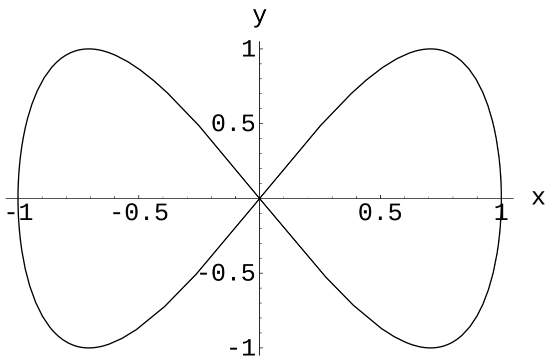
It is smooth (both $\cos t$ and $2\cos 2t$ are never zero).

$\mathbf{f}(0) = (0, 0)$.

$\mathbf{f}(\pi) = (0, 0)$

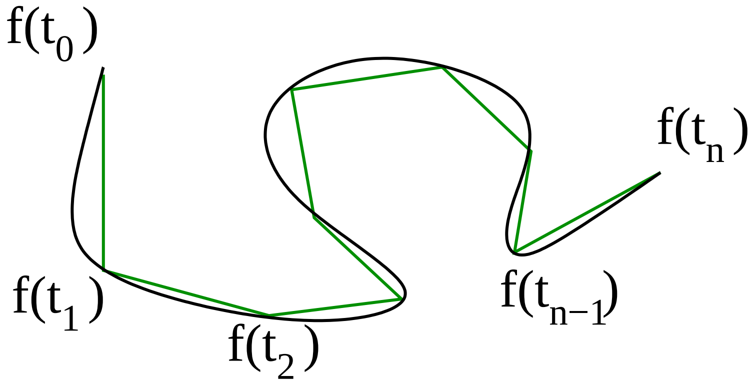
The path is not simple.

Plot it:

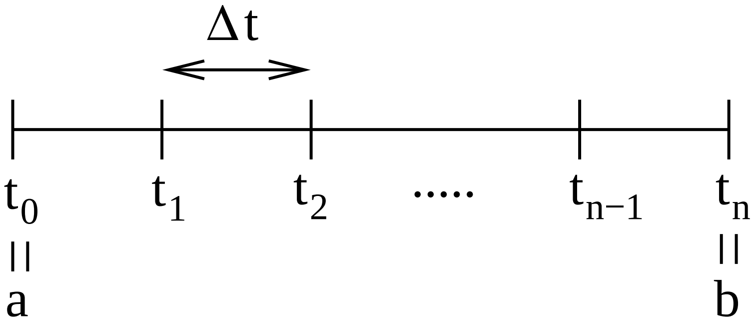


Path given by $\mathbf{f}(t)$, $a \leq t \leq b$.

What is the length of the path?



Divide interval $[a, b]$ into n segments of length Δt:



Connect each pair of points $\mathbf{f}(t_{i-1})$ and $\mathbf{f}(t_i)$ with line segments.

Length of path is approximately total length of line segments.

Length of path is approximately

$$\sum_{i=1}^{n} \|\mathbf{f}(t_i) - \mathbf{f}(t_{i-1})\|$$

$$= \sum_{i=1}^{n} \left\| \frac{\mathbf{f}(t_i) - \mathbf{f}(t_{i-1})}{t_i - t_{i-1}} \right\| \Delta t$$

since $\Delta t = t_i - t_{i-1}$.

Now divide the interval $[a, b]$ into smaller and smaller segments so that $n \to \infty$ and $\Delta t \to 0$.

Then

$$\frac{\mathbf{f}(t_i) - \mathbf{f}(t_{i-1})}{t_i - t_{i-1}} \to \mathbf{f}'(t_i)$$

so path length is approximately

$$\sum_{i=1}^{n} \|\mathbf{f}'(t_i)\| \Delta t$$

The last expression is an approximate sum of

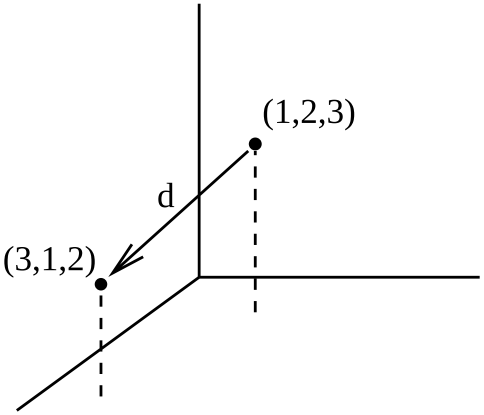
$$\int_a^b \|\mathbf{f}'(t)\| dt$$

Therefore, the length of the path given by $\mathbf{f}(t)$ for $a \le t \le b$ is

$$\int_a^b \|\mathbf{f}'(t)\| dt.$$

To be concrete, derivation was for paths in the plane. Everything works the same for paths in space.

Example: Write a parameterization for the straight-line path from the point (1,2,3) to the point (3,1,2). Find the path length.



$$\mathbf{d} = (3,1,2) - (1,2,3) = (2,-1,-1)$$

$$\mathbf{f}(t) = (1,2,3) + t(2,-1,-1) \qquad 0 \le t \le 1$$

To find path length calculate

$$\begin{aligned} \mathbf{f}'(t) &= (2,-1,-1) \\ \|\mathbf{f}'(t)\| &= \sqrt{2^2 + (-1)^2 + (-1)^2} \\ &= \sqrt{6} \end{aligned}$$

The length of the path is

$$\int_a^b \|\mathbf{f}'(t)\| dt = \int_0^1 \sqrt{6} dt = \sqrt{6}$$

Clearly, it was silly to calculate the length this way. Simply illustrates the method.

A path can be parameterized multiple ways.

Another parameterization for our path is

$$\mathbf{g}(t) = (1,2,3) + (e^t - 1)(2,-1,-1) \qquad 0 \le t \le \log 2$$

Find path length using this parameterization

$$\mathbf{g}'(t) = e^t(2,-1,1)$$
$$\|\mathbf{g}'(t)\| = e^t\|(2,-1,-1)\| = e^t\sqrt{6}$$

Path length is

$$\int_a^b \|\mathbf{g}'(t)\|dt = \int_0^{\log 2} e^t\sqrt{6}dt$$
$$= \sqrt{6}(e^{\log 2} - e^0)$$
$$= \sqrt{6}(2-1) = \sqrt{6}$$

We see that path length doesn't depend on parameterization. (Makes sense.)

Define an *arclength function* for a path $\mathbf{f}(t)$

$$L(t) = \int_a^t \|\mathbf{f}'(u)\|du.$$

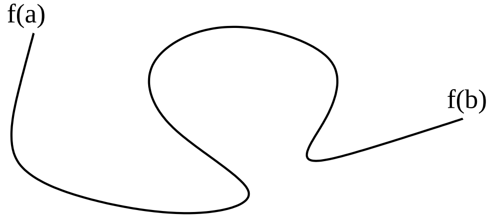
$L(t)$ is the length of the path up to the point $\mathbf{f}(t)$.

For the example $\mathbf{f} = (1,2,3) + t(2,-1-1)$

$$L(t) = \int_0^t \sqrt{6}du = t\sqrt{6}.$$

Path integrals (section 5.2)

Suppose path $\mathbf{f} : [a,b] \to \mathbb{R}^2$

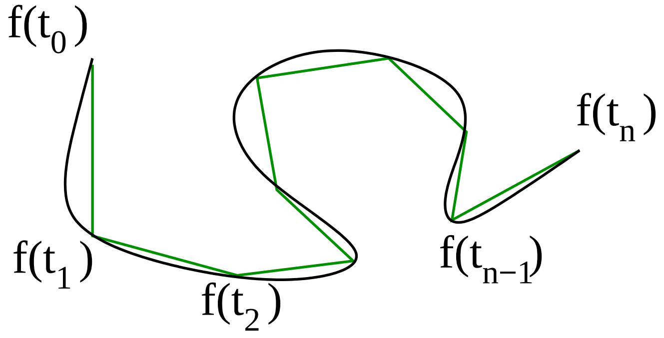


is a piece of wire.

Suppose have a function $g(x,y)$ that gives charge at point (x,y).

Goal: find the total charge on the wire.

As before, approximate curve by line segments



Look at piece from $\mathbf{f}(t_{i-1})$ to $\mathbf{f}(t_i)$.

Charge on this small piece of wire is approximately

$$g(\mathbf{f}(t_i))\|\mathbf{f}(t_i)-\mathbf{f}(t_{i-1})\|$$

Total charge on wire is approximately

$$\begin{aligned}&\sum_{i=1}^{n} g(\mathbf{f}(t_i))\|\mathbf{f}(t_i)-\mathbf{f}(t_{i-1})\|\\ &=\sum_{i=1}^{n} g(\mathbf{f}(t_i))\left\|\frac{\mathbf{f}(t_i)-\mathbf{f}(t_{i-1})}{t_i-t_{i-1}}\right\|\Delta t\end{aligned}$$

As with arclength derviation, we let $n \to \infty$, $\Delta t \to 0$.

The only thing different in this case is the $g(\mathbf{f}(t_i))$ factor.

Get that the total charge is

$$\int_a^b g(\mathbf{f}(t))\|\mathbf{f}'(t)\|dt.$$