

## Opportunity for participation points

Earn a participation point at

- the beginning of each Monday lecture
- at the end of each Wednesday lecture.

(Each participation point is equivalent to an extra point on a midterm).

Green's theorem

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int \int_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dx dy$$

LHS: total circulation around  $C$

RHS: integral of tendency to circulate

## Stokes' theorem (Section 6.4)

Recall the one-variable Fundamental Theorem of Calculus

$$F(b) - F(a) = \int_a^b F'(x) dx$$

LHS: total change in  $F$

RHS:  $F'(x)$  is tendency for  $F$  to increase.

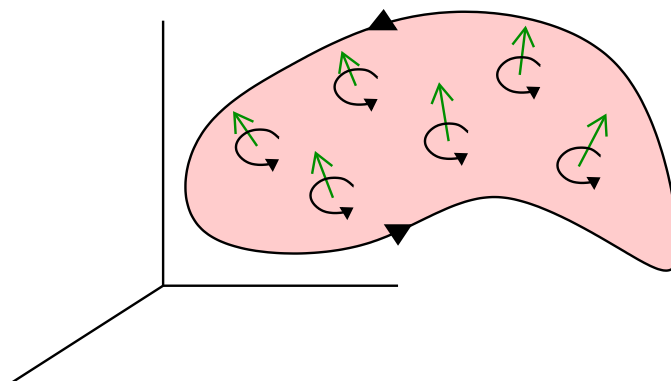
Interpret: total change in  $F$  is integral of its tendency to increase.

Stokes' theorem

$$\int_{\partial M} \mathbf{F} \cdot d\mathbf{x} = \int \int_M \text{curl}(\mathbf{F}) \cdot \mathbf{n} d\sigma,$$

LHS: total circulation around  $\partial M$  (boundary of  $M$ )

RHS: integral of tendency to circulate around normal of surface  $M$



Remember: need to get orientation right  
**positive side** of the surface: the side that the normal points out from

**positively oriented** boundary: goes counterclockwise when viewed from the positive side of the surface.

Stokes' theorem true for positively oriented boundary  $\partial M$ .

By Stokes' theorem, we need to compute

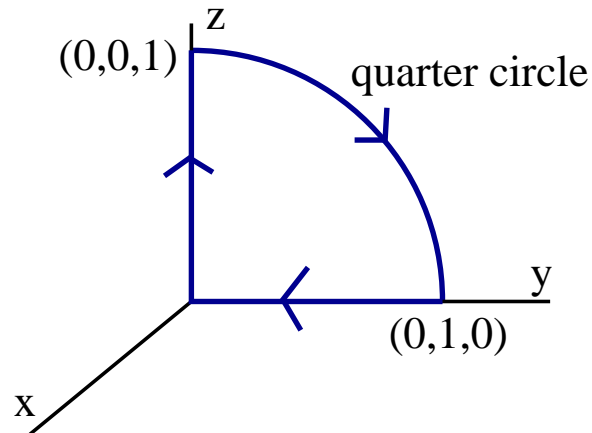
$$\int_M \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, d\sigma$$

where  $M$  is a surface with boundary  $C$ .

Let  $M$  be the quarter disk in the  $yz$ -plane.

Normal vector  $\mathbf{n}$  needs to point in which direction? Toward the positive or negative side of the  $x$ -axis?

Example. Let  $C$  be the closed curve



For  $\mathbf{F}(x, y, z) = (y, z, x)$ , compute

$$\int_C \mathbf{F} \cdot d\mathbf{x}$$

using Stokes' Theorem.

$$\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \nabla \times (y, z, x)$$

$$\begin{aligned}
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} \\
 &= \mathbf{i} \left( \frac{\partial}{\partial y} x - \frac{\partial}{\partial z} z \right) - \mathbf{j} \left( \frac{\partial}{\partial x} x - \frac{\partial}{\partial z} y \right) \\
 &\quad + \mathbf{k} \left( \frac{\partial}{\partial x} z - \frac{\partial}{\partial y} y \right) \\
 &= \mathbf{i}(-1) - \mathbf{j}(1) + \mathbf{k}(-1) \\
 &= (-1, -1, -1)
 \end{aligned}$$

Next, parameterize surface (quarter disk)

$$f(r, \theta) = (0, r \cos \theta, r \sin \theta)$$

for  $0 \leq r \leq 1$  and  $0 \leq \theta \leq \pi/2$ .

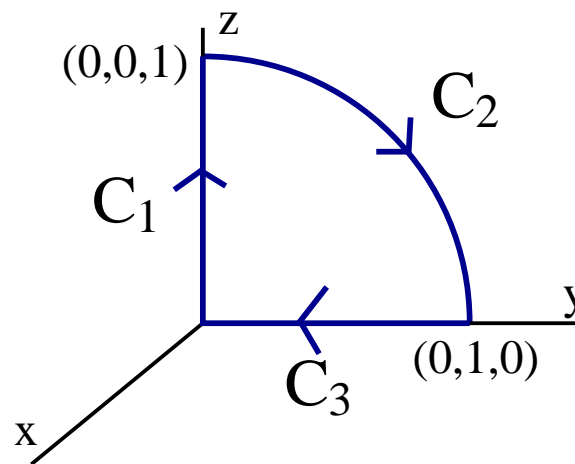
$$\frac{\partial f}{\partial r} = (0, \cos \theta, \sin \theta)$$

$$\frac{\partial f}{\partial \theta} = (0, -r \sin \theta, r \cos \theta)$$

$$\frac{\partial f}{\partial r} \times \frac{\partial f}{\partial \theta} = \mathbf{i}(r \cos^2 \theta + r \sin^2 \theta) = r \mathbf{i}$$

Now compute  $\int_C \mathbf{F} \cdot d\mathbf{x}$  directly.

Divide  $C$  into three parts.



Is the surface oriented properly?

If not, use  $\frac{\partial f}{\partial \theta} \times \frac{\partial f}{\partial r} = -r \mathbf{i}$ .

Should the integral  $\int_M \text{curl}(\mathbf{F}) \cdot \mathbf{n} d\sigma$  be positive or negative?

$$\begin{aligned} & \int_M \text{curl}(\mathbf{F}) \cdot \mathbf{n} d\sigma \\ &= \int_0^1 \int_0^{\pi/2} \text{curl} \mathbf{F}(f(r, \theta)) \cdot \left( \frac{\partial f}{\partial \theta} \times \frac{\partial f}{\partial r} \right) d\theta dr \\ &= \int_0^1 \int_0^{\pi/2} (-1, -1, -1) \cdot (-r, 0, 0) d\theta dr \\ &= \int_0^1 \int_0^{\pi/2} r d\theta dr = \frac{\pi}{4} \end{aligned}$$

Recall  $\mathbf{F}(x, y, z) = (y, z, x)$

First  $C_1$ . Parameterize

$$g(t) = (0, 0, t) \quad 0 \leq t \leq 1$$

$$g'(t) = (0, 0, 1)$$

$$\begin{aligned} \mathbf{F}(g(t)) \cdot g'(t) &= \mathbf{F}(0, 0, t) \cdot (0, 0, 1) \\ &= (0, t, 0) \cdot (0, 0, 1) \\ &= 0 \end{aligned}$$

Therefore

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{x} = \int_0^1 \mathbf{F}(g(t)) \cdot g'(t) dt = 0.$$

The integral for  $C_3$  is similar.

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{x} = 0$$

Last, do integral over  $C_2$ .

Parameterize as

$$g(\theta) = (0, \sin \theta, \cos \theta) \quad 0 \leq \theta \leq \pi/2$$

$$g'(\theta) = (0, \cos \theta, -\sin \theta)$$

$$\begin{aligned} \int_{C_2} \mathbf{F} \cdot d\mathbf{x} &= \int_0^{\pi/2} \mathbf{F}(g(\theta)) \cdot g'(\theta) d\theta \\ &= \int_0^{\pi/2} \mathbf{F}(0, \sin \theta, \cos \theta) \cdot (0, \cos \theta, -\sin \theta) d\theta \\ &= \int_0^{\pi/2} (\sin \theta, \cos \theta, 0) \cdot (0, \cos \theta, -\sin \theta) d\theta \\ &= \int_0^{\pi/2} \cos^2 \theta d\theta \\ &= \int_0^{\pi/2} \frac{1 + \cos 2\theta}{2} d\theta \\ &= \frac{\theta}{2} + \frac{\sin 2\theta}{4} \Big|_0^{\pi/2} = \frac{\pi}{4} \end{aligned}$$

Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \frac{\pi}{4}$$

in agreement with our Stokes' theorem answer.

One trick to use Stokes' theorem:

You are given a curve  $C$ .

How to select a surface?

(For Green's theorem, this wasn't an issue because everything was 2D).

A special case where this is relatively easy:  
When the curve  $C$  lies in a plane.

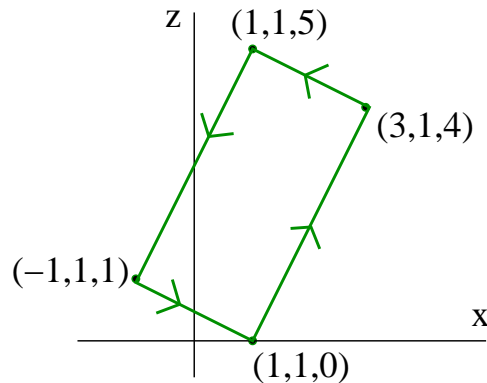
Especially easy:

that plane is parallel to a coordinate plane.

Example: Say want to use Stokes' theorem to compute  $\int_C \mathbf{F} \cdot d\mathbf{x}$  where  $C$  is polygon path connecting the following points:

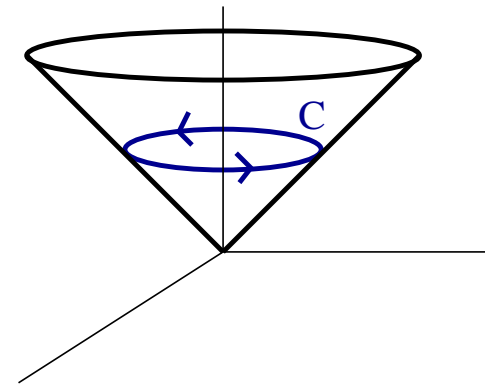
$(1,1,0)$ ,  $(3,1,4)$ ,  $(1,1,5)$ ,  $(-1,1,1)$

Does this curve lie in a plane?



In other cases, a surface is given explicitly in the problem.

Example:  $\int_C \mathbf{F} \cdot d\mathbf{x}$  where  $C$  is the curve in which the cone  $z^2 = x^2 + y^2$  intersects the plane  $z = 1$ . (Oriented counter clockwise viewed from positive  $z$ -axis).



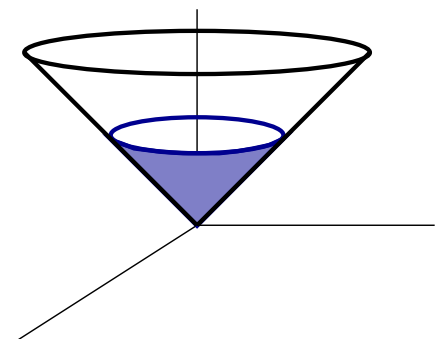
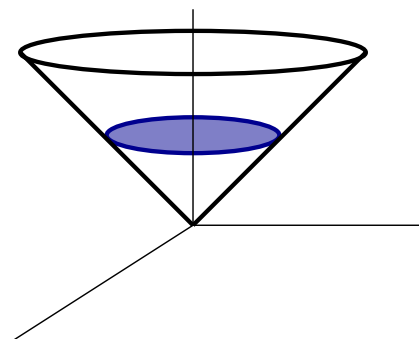
If one coordinate is constant, then curve is parallel to a coordinate plane.

( $xz$ -plane for above example).

For Stokes' theorem, use surface in that plane.

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_M \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, d\sigma$$

for what surface  $M$ ?



Portion of plane or portion of cone?

Use can use either surface.

Let  $P$  be the portion of the plane  $z = 1$  with  $x^2 + y^2 < 1$  with upward pointing normal.

Let  $Q$  be the portion of the cone  $z^2 = x^2 + y^2$  with  $0 < z < 1$  with upward angling normal.

How do  $\int_P \text{curl}(F) \cdot \mathbf{n} d\sigma$  and  $\int_Q \text{curl}(F) \cdot \mathbf{n} d\sigma$  compare?

$$\frac{\partial f}{\partial r} \times \frac{\partial f}{\partial \theta} = (0, 0, r)$$

$$\int_P \text{curl}(\mathbf{F}) \cdot \mathbf{n} d\sigma$$

$$= \int_0^1 \int_0^{2\pi} \text{curl}(\mathbf{F}(r \cos \theta, r \sin \theta, 1)) \cdot (0, 0, r) d\theta dr$$

$$= \int_0^1 \int_0^{2\pi} (r \cos \theta, -r \sin \theta, r^2) \cdot (0, 0, r) d\theta dr$$

$$= \int_0^1 \int_0^{2\pi} r^3 d\theta dr$$

$$= \int_0^1 2\pi r^3 dr = \frac{\pi}{2}$$

Continue example: Let

$$\mathbf{F}(x, y, z) = \left( \sin x - \frac{y^3}{3}, \cos y + \frac{x^3}{3}, xyz \right)$$

Compute  $\int_C \mathbf{F} \cdot d\mathbf{x}$ .

Can show that  $\text{curl}(\mathbf{F}) = (xz, -yz, x^2 + y^2)$ .

Use surface  $P$ , parameterized by

$$\mathbf{f}(r, \theta) = (r \cos \theta, r \sin \theta, 1)$$

for  $0 \leq r \leq 1, 0 \leq \theta \leq 2\pi$