$$\frac{\partial}{\partial x}(xyz) = yz,\ \frac{\partial}{\partial y}(xyz) = xz,\ \frac{\partial}{\partial z}(xyz) = xy$$

Evaluate

$$\int_C (yz, xz, xy) \cdot d\mathbf{x}$$

where C is the helix parameterized by

$$\mathbf{g}(t) = (\cos t, \sin t, t), \quad \pi/2 \le t \le \pi.$$

For participation point, hand in answer (or guess) on piece of paper. (**Not graded**.)

Line integral of $(yz, xz, xy) = \nabla f$, where $f(x, y, z) = xyz$.

Curve starts at $\mathbf{p} = \mathbf{g}(\pi/2) = (0, 1, \pi/2)$.
Curve ends at $\mathbf{q} = \mathbf{g}(\pi) = (-1, 0, \pi)$.

$$\int_C (yz, xz, xy) \cdot d\mathbf{x} = f(\mathbf{q}) - f(\mathbf{p}) = 0 - 0 = 0$$

Some tips you gave last Wednesday.

- Regularly review old material in class
- Hand out study sheets (hope to do that soon)
- Review sessions (I'll get back to you)

The divergence theorem

(Section 6.3)

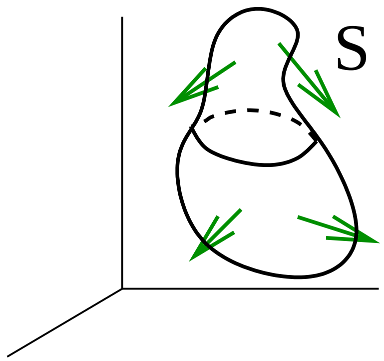
Let a vector field $\mathbf{F}(x, y, z)$ represent fluid flow.

Recall that the divergence

$$\operatorname{div} \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

measures the outflow per unit volume of a vector field at a point.

Given some solid S, we want to calculate the total flux of fluid out of S



This is the flux integral of $\mathbf{F}$ over the boundary of S, denoted ∂S:

$$\int\int_{\partial S} \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

Which choice of normal $\mathbf{n}$? Inward or outward pointing?

Divergence theorem: the total flux is the integral of the divergence over S.

$$\int\int_{\partial S} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int\int\int_S \text{div}\, \mathbf{F}\, dV$$

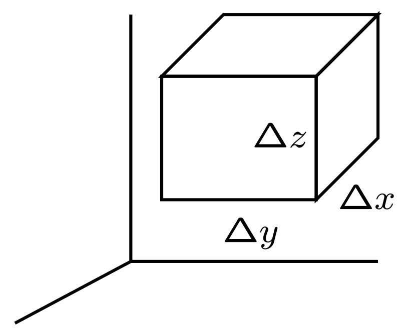
This makes sense intuitively.

Recall how we estimated the divergence of $\mathbf{F}$ at (x, y, z).

- Let M be a sphere (or a box) centered at (x, y, z), oriented outward.
- Calculate total flux out of the sphere: $\int\int_M \mathbf{F} \cdot \mathbf{n}\, d\sigma$.
- Let M shrink down to the point (x, y, z).
- Divergence is $\lim_{\text{Volume} \to 0} \dfrac{\int\int_M \mathbf{F} \cdot \mathbf{n}\, d\sigma}{\text{Volume of } M}$.

To sketch proof of divergence theorem, we'll show this estimate is the divergence.

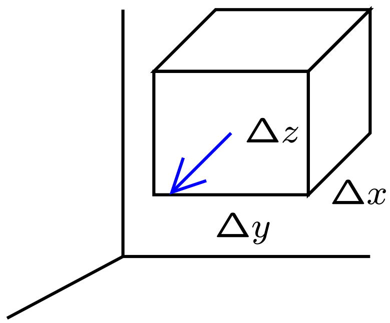
Calculate flux out of a little box



Take limit as $\Delta x, \Delta y, \Delta z \to 0$.

Calculate $\int_M \mathbf{F}\cdot\mathbf{n}\,d\sigma$ over box.

For front of box $\mathbf{n} = \mathbf{i}$,
so $\mathbf{F}\cdot\mathbf{n} = F_1$.

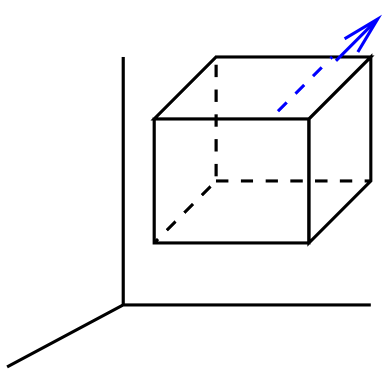


Say x is the position of back of box
Then $x + \Delta x$ is position of the front of box.

Total flux out of front of box is approximately

$$F_1(x + \Delta x, y, z)\Delta y \Delta z$$

What is normal out of back of box?



$\mathbf{n} = -\mathbf{j}$.

Therefore, $\mathbf{F}\cdot\mathbf{n} = -F_1$.

The total flux out the back of the box is approximately

$$-F_1(x, y, z)\Delta y \Delta z$$

Combine the front and back to get

$$[F_1(x + \Delta x, y, z) - F_1(x, y, z)]\Delta y \Delta z$$

Similarly, the total flux out the sides is

$$[F_2(x, y + \Delta y, z) - F_2(x, y, z)]\Delta x \Delta z.$$

The total flux out the top and bottom is

$$[F_3(x, y, z + \Delta z) - F_3(x, y, z)]\Delta x \Delta y$$

The total flux out of the box $\int\int_M \mathbf{F}\cdot\mathbf{n}\,d\sigma$ is the sum of these three terms.

Divide by the volume of the box ($\Delta x \Delta y \Delta z$)

$$\frac{\iint_M \mathbf{F}\cdot\mathbf{n}\,d\sigma}{\text{Volume}} = \frac{F_1(x+\Delta x,y,z)-F_1(x,y,z)}{\Delta x} + \frac{F_2(x,y+\Delta y,z)-F_2(x,y,z)}{\Delta y} + \frac{F_3(x,y,z+\Delta z)-F_3(x,y,z)}{\Delta z}$$

Total flux over boundary of surface ∂S is approximate sum over boxes B_i

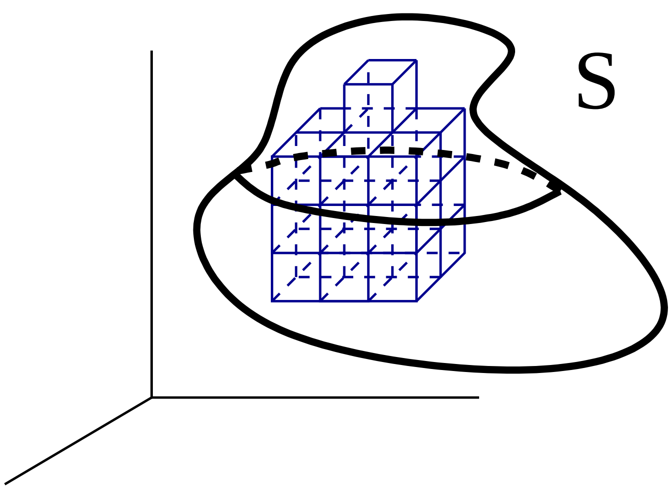
$$\int\int_{\partial S} \mathbf{F}\cdot\mathbf{n}\,d\sigma \approx \sum_i \int\int_{\partial B_i} \mathbf{F}\cdot\mathbf{n}\,d\sigma \approx \sum_i \operatorname{div}\mathbf{F}(x_i,y_i,z_i)\Delta x\Delta y\Delta z \approx \int\int\int_S \operatorname{div}\mathbf{F}\,dx\,dy\,dz$$

since last sum is a Riemann sum.

Take limit as $\Delta x, \Delta y, \Delta z \to 0$, get

$$\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z} = \operatorname{div}\mathbf{F}$$

To prove divergence theorem, chop up solid into many small boxes.



Example (Exercise 16, section 5.6)

Compute $\int\int_M \mathbf{F}\cdot\mathbf{n}\,d\sigma$ where

$$\mathbf{F} = (3x+z^2, y^2-x^2, xz+5)$$

and M is surface of box

$$0 \le x \le 1, \quad 0 \le y \le 3, \quad 0 \le z \le 2.$$

Use outward normal $\mathbf{n}$.

By divergence theorem,

$$\int\int_M \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int\int\int_B \text{div}\,\mathbf{F}\,dV$$

where B is the box

$$0 \le x \le 1, \quad 0 \le y \le 3, \quad 0 \le z \le 2.$$

$$\text{div}\,\mathbf{F} = 3 + 2y + x$$

$$\int\int_M \mathbf{F}\cdot\mathbf{n}\,d\sigma$$
$$= \int_0^1\int_0^3\int_0^2 (3 + 2y + x)dz\,dy\,dx$$
$$= \int_0^1\int_0^3 (6 + 4y + 2x)dy\,dx$$
$$= \int_0^1 (18 + 18 + 6x)dx$$
$$= 36 + 3 = 39$$

Let M be any closed surface. What is

$$\int\int_M \text{curl}(\mathbf{F})\cdot\mathbf{n}\,d\sigma$$

To ways to get answer:

Divergence theorem and fact that

$$\text{div}\,\text{curl}\,\mathbf{F} = 0.$$

Or Stokes' theorem and fact that M has no boundary.

Suppose a gas is expanding. What is $\text{div}\,\mathbf{F}$ if $\mathbf{F}$ is velocity of gas particles?

For a solid S, What do we know about the flux out its boundary ∂S? (positive, negative, zero, can't tell)

$$\int\int\int_S \text{div}\,\mathbf{F}\,dV > 0$$

so

$$\int\int_{\partial S} \mathbf{F}\cdot\mathbf{n}\,d\sigma > 0$$

Example: $\mathbf{F} = (xy^2, yz^2, x^2z)$.

Evaluate

$$\int\int_M \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

where M is the sphere of radius 3 centered at origin.

$\operatorname{div} \mathbf{F} = y^2 + z^2 + x^2$.

Integral is

$$\int\int\int_B (y^2 + z^2 + x^2) dV$$

where B is ball of radius 3.

To evaluate change to spherical coordinates. Then ball is

$$0 \le \rho \le 3, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi$$

$x^2 + y^2 + z^2 = \rho^2$,
$dV = \rho^2 \sin\phi\, d\phi\, d\theta\, d\rho$

Integral is

$$\int_0^3 \int_0^{2\pi} \int_0^{\pi} \rho^4 \sin\phi\, d\phi\, d\theta\, d\rho = \frac{972\pi}{5}$$

We have seen

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

show up in three places.

If $f(x,y,z)$ is a scalar function and $\mathbf{F}(x,y,z) = (F_1, F_2, F_3)$ is a vector-valued function, then

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \qquad \text{gradient of } f$$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \quad = \operatorname{div} \mathbf{F}$$

$$\nabla \times \mathbf{F} = \operatorname{curl} \mathbf{F}$$

$\nabla f(x,y,z)$ is a vector-valued function. Can take its divergence.

$$\nabla^2 f = \operatorname{div} \nabla f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

This is called the Laplacian of f.

Comes up a lot in equations describing physical laws (heat flow, wave propagation, etc.)

Could take curl of ∇f.

But, remember

$$\int_C \nabla f \cdot d\mathbf{x} =$$

for any closed curve C.

What does that say about curl(∇f)?

Other relationships:
see table in section 4.2.

Water is essentially an incompressible fluid.

If $\mathbf{F}$ represents water flow, what is div $\mathbf{F}$? Zero. What is $\int\int_{\partial S} \mathbf{F} \cdot \mathbf{n} \, d\sigma$? Zero.

In general, fluid flow is called incompressible (or divergence free) if div $\mathbf{F} = 0$.