First exam: next Wednesday, October 1

You may take exam either

- from 5:00 to 6:00 PM or
- from 6:10 to 7:10 PM.

Room: to be announced

Material covered: section 1.2 through section 3.4 (partial derivatives)

Sample exam is on website
(Help with Homework and Labs link)

Reminder: e-mail frank@math.umn.edu by tonight if you will attend review session Friday (see e-mail you received last Friday).

Homework clarification:
Problem 25, section 3.4, $G(x,y)$ should be

$$G(x,y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 1, & x^2+y^2 = 0. \end{cases}$$

Partial Derivatives (cont)

Let $g(\square, \bigcirc, \clubsuit) = e^{7\bigcirc} \sin \clubsuit$

Find:

$$\frac{\partial g}{\partial \clubsuit}(\square, \bigcirc, \clubsuit) = e^{7\bigcirc} \cos \clubsuit$$

$$\frac{\partial g}{\partial \bigcirc}(1,2,3) = 7e^{14} \sin 3$$

$$\frac{\partial g}{\partial \square}(a,b,c) = 0$$

Assume pressure P is given by a function of temperature T and volume V.

$$P = f(V,T)$$

Explain what the following partial derivatives mean physically.

$\dfrac{\partial f}{\partial V}$ = how much pressure changes as change volume (with temperature constant)

$\dfrac{\partial f}{\partial T}$ = how much pressure changes as change temperature (with volume constant)

Example

$$f(x,y) = \begin{cases} \dfrac{x^3 + x^4 - y^3}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Find $\dfrac{\partial f}{\partial x}(0,0)$.

Need to use definition of partial derivative.

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{h^3 + h^4 - 0^3}{h^2 + 0^2} - 0}{h} \\ &= \lim_{h\to 0} \frac{\frac{h^3 + h^4}{h^2}}{h} \\ &= \lim_{h\to 0} \frac{h^3 + h^4}{h^3} \\ &= \lim_{h\to 0} 1 + h \\ &= 1 \end{aligned}$$

Higher-order partial derivatives

Define higher-order partial derivatives just like higher-order ordinary derivatives.

For $f(x,y)$, have two first-order partial derivatives.

To form second-order derivatives, simply take partial derivative of these with respect to x and y (total of four possibilities).

Take partial derivatives of $\dfrac{\partial f}{\partial x}$:

$$\frac{\partial^2 f}{\partial x \partial x} = \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial x}\Big) \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial x}\Big)$$

Take partial derivatives of $\dfrac{\partial f}{\partial y}$:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial y}\Big) \qquad \frac{\partial^2 f}{\partial y \partial y} = \frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial y}\Big)$$

When take partial derivative with respect to same variable twice, typically write as

$$\frac{\partial^2 f}{\partial x^2} \qquad \text{or} \qquad \frac{\partial^2 f}{\partial y^2}.$$

Example: $f(x,y) = 2x^3 + 3xy - xy^2$

$$\frac{\partial f}{\partial x} = 6x^2 + 3y - y^2$$
$$\frac{\partial f}{\partial y} = 3x - 2xy$$

Find second-order partial derivatives by taking partial derivatives again.

Note that $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$.

Clairaut's Theorem: This is guaranteed to be true when $\dfrac{\partial^2 f}{\partial y \partial x}$ and $\dfrac{\partial^2 f}{\partial x \partial y}$ are continuous.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(6x^2 + 3y - y^2\right) = 12x$$

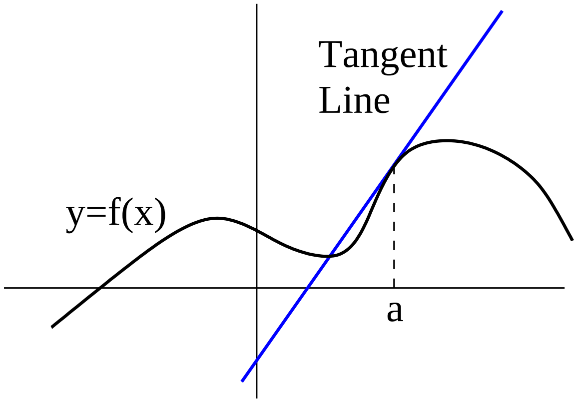
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(6x^2 + 3y - y^2\right) = 3 - 2y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(3x - 2xy\right) = 3 - 2y$$

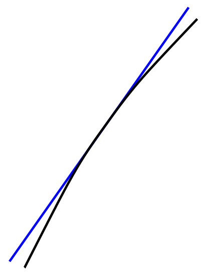
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(3x - 2xy\right) = 2x$$

The derivative of a function of several variables (Section 3.5)

In one-variable calculus, $y = f(x)$ is differentiable at a point $x = a$ if there is a line tangent to the graph at $x = a$.



One way to see that $f(x)$ is differentiable, is that its graph looks more and more like a line as you zoom in.



Close to the point a, $f(x)$ is closely approximated by the tangent line.

What about higher dimensions?

We'll look at $f : \mathbb{R}^2 \to \mathbb{R}$ so we can graph it.

A function $f(x, y)$ is differentiable at $(x, y) = (a, b)$ if there is a plane tangent to the graph at the point (a, b).

What is the equation for the tangent line?

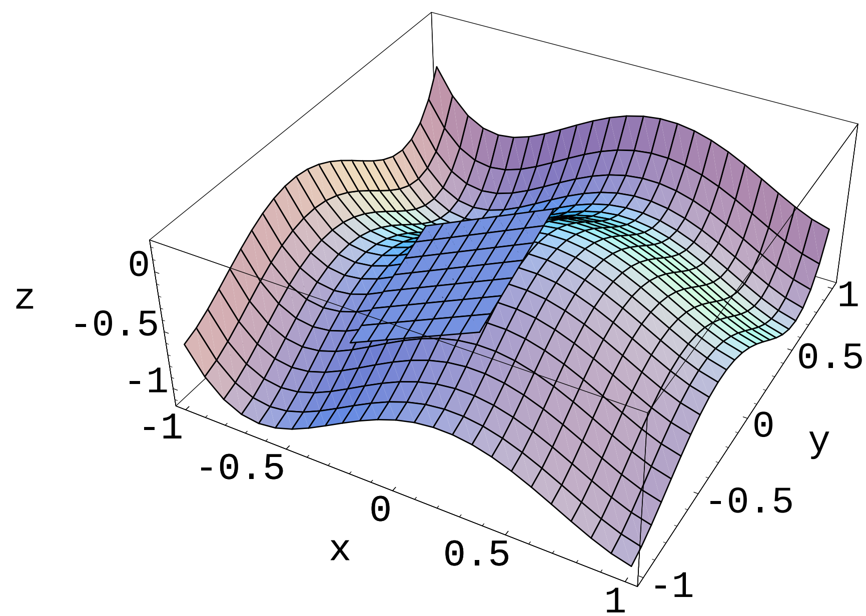
If slope is m, then $y = m(x - a) + f(a)$

If $f(x)$ is differentiable at a, then the difference between $y = m(x-a)+f(a)$ and $y = f(x)$ is small for x close to a.

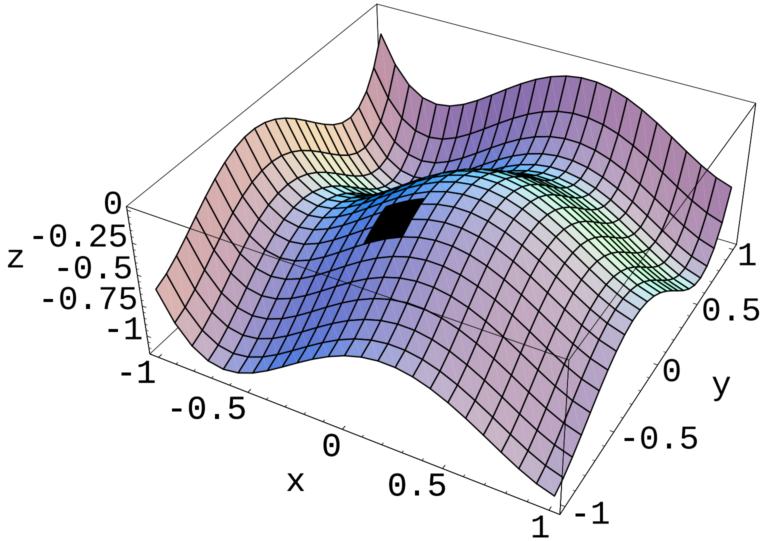
Subtracting the two expressions, we find that $f(x)$ is differentiable at $x = a$ if there exists an m so that $f(x) - f(a) - m(x - a)$ is small when x is near a.

What is the slope m? The derivative $f'(a)$.

Graph of $z = f(x, y)$ with tangent plane

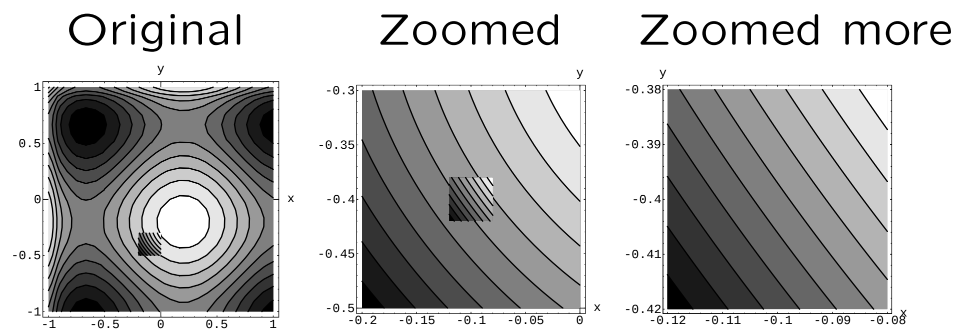


One way to see that $f(x,y)$ is differentiable is that its graph looks more and more like a plane as you zoom in.



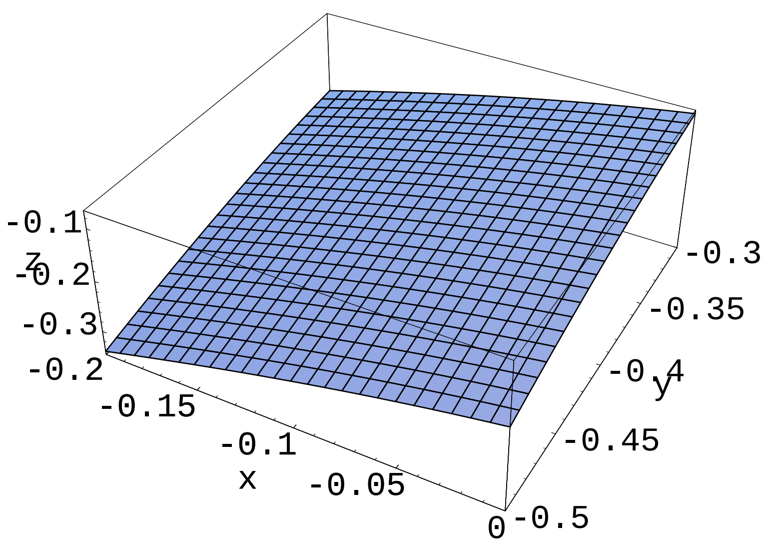
Can also look at it with level curves plots.

As zoom in, the level curves become parallel, evenly spaced, lines.

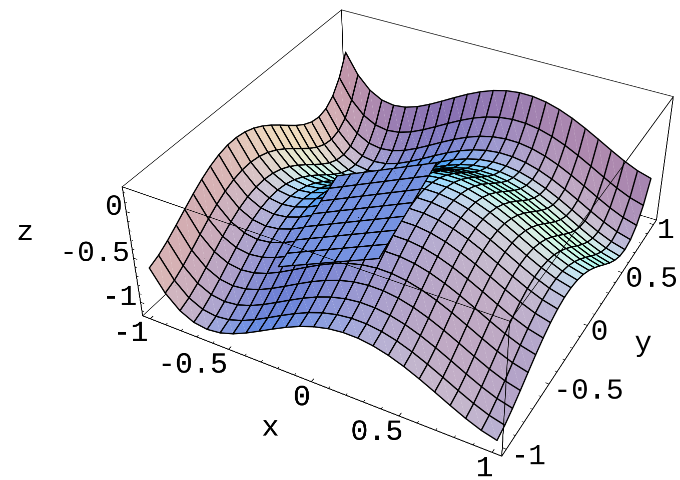


(Add more contour lines as zoom in.)

Close up of $f(x,y)$ looks like a plane.



Back to tangent plane



If $f(x,y)$ is differentiable at (a,b), this tangent plane exists, i.e., $f(x,y)$ is well approximated by the plane near (a,b).

What is the equation of the tangent plane?

It goes through $f(a, b)$ when $(x, y) = (a, b)$.
Let m be its slope in the x direction.
Let n be its slope in the y direction.

Then, equation for plane is
$z = f(a, b) + m(x - a) + n(y - b)$.

$f(x, y)$ is differentiable at (a, b) if can find m and n so that
$f(x, y) - f(a, b) - m(x - a) - n(y - b)$
is small for (x, y) near (a, b).

We can rewrite the equation as

$$z = mx + ny + c$$

where c incorporates all the constant terms.

The linear function

$$T(x, y) = mx + ny.$$

is called the derivative of f at (a, b).

What are the slopes?
The partial derivatives.

$$m = \frac{\partial f}{\partial x}(a, b), \qquad n = \frac{\partial f}{\partial y}(a, b).$$

The equation for the tangent plane is

$$z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$

This linear function can be represented by the matrix $[m \ n]$.

$$T(x, y) = \begin{bmatrix} m & n \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

This matrix is called the **Jacobian matrix** (or derivative matrix).

Denote the derivative by $Df(a, b)$.

The derivative $Df(a, b)$ can be represented by either the linear function $T(x, y)$ or the Jacobian matrix.

Putting back in the values for m and n, $Df(a,b)$ (the derivative of $f(x,y)$ at the point (a,b)) is the linear function

$$T(x,y) = \frac{\partial f}{\partial x}(a,b)x + + \frac{\partial f}{\partial y}(a,b)y$$

or the Jacobian matrix

$$\left[\begin{array}{cc} \frac{\partial f}{\partial x}(a,b) & \frac{\partial f}{\partial y}(a,b) \end{array} \right].$$

Note: $Df(a,b)$ exists only if f is **differentiable** at (a,b) (i.e., tangent plane exists).

Example. $f(x,y) = x^2 + y^2$. Find $Df(1,2)$.

$$\frac{\partial f}{\partial x}(x,y) = 2x$$
$$\frac{\partial f}{\partial x}(1,2) = 2$$
$$\frac{\partial f}{\partial y}(x,y) = 2y$$
$$\frac{\partial f}{\partial y}(1,2) = 4$$

So $Df(1,2)$ is the linear function

$$T(x,y) = 2x + 4y$$

or the Jacobian matrix $[\ 2 \quad 4\ ]$.

Since $f(1,2) = 1^2 + 2^2 = 5$, the equation for the tangent plane is

$$z = 5 + 2(x-1) + 4(y-2)$$

If looked at the point $(2,3)$, what changes?

$$\frac{\partial f}{\partial x}(3,2) = 4$$
$$\frac{\partial f}{\partial y}(3,2) = 6$$

$Df(2,3)$ is the linear function

$$T(x,y) = 4x + 6y$$

or the Jacobian matrix is $[\ 4 \quad 6\ ]$.