Worksheet number 2
"7825 Linear Equations"

1

Solve the initial value problem $t\frac{dy}{dt} + 3y = 5t^2$.

First we need standard form:
$\frac{dy}{dt} + \frac{3}{t}y = 5t$ done.

Next find integration factor. So take the coefficient of the y variable and integrate it. $\int \frac{3}{t}dt = 3ln(t)$. Now raise it to the e. $e^{ln(t^3)} = t^3$. done.

Next we multiply the standard form by the integration factor:

$$\begin{aligned} t^3\frac{dy}{dt} + 3t^2y &= 5t^4 \\ \frac{d}{dt}(t^3y) &= 5t^4 \\ \int \frac{d}{dt}(t^3y) &= \int 5t^4 \\ t^3y &= t^5 + c \\ y &= t^2 + ct^{-3} \text{note :} t \neq 0 \end{aligned}$$

Now we have our standard form. Next we use the IV (initial value) to solve for c. When $t = 1$ and $y = 5$, $y = t^2 + 4t^{-3}$ with $y < 0$. And when $t = -1$ and $y = -5$, $y = t^2 + 6t^{-3}$ for $t < 0$.

2

Solve the initial value problem $\frac{dy}{dt} + \frac{3}{100+t}y = 12$ when $t > -100$ and $y(0) = 500$. Why do we say $t > -100$?
Notice: $(100 + t)^{-1}$ is not defined for $t = -100$.
The formula is already in standard form. The integration factor is $(100 + t)^3$. So the equation we want to integrate is: $(100 + t)^3\frac{dy}{dt} + 3(100 + t)^2y = 12(100 + t)^3$.

$$\begin{aligned} \int(100 + t)^3\frac{dy}{dt}dt + \int 3(100 + t)^2ydt &= \int 12(100 + t)^3dt \\ (100 + t)^3y &= 3(100 + t)^4 + C \\ y &= 3(100 + t) + c(100 + t)^{-3} \end{aligned}$$

So the initial value of $Y(0) = 500$ implies $y = 3(100 + t) + 200(100)^3(100 + t)^{-3}$ and we are done.

3

Solve the initial value problem $\frac{dy}{dt} = \frac{2y+5}{3t+2}$ and $y(2) = \frac{-13}{2}$. Solve for the y in the solution. For what values of t is this a solution of the initial value problem? Also find solutions satisfying $y(-1) = -7$.

This is a seperable equation. So:

$$\begin{aligned} \frac{dy}{sy+5} &= \frac{dt}{3t+2} \text{ for } t \neq \frac{-2}{3} \\ \frac{1}{2} ln|2y+5| &= \frac{1}{3} ln|3t+2| + ln|c| \text{ for } t \neq \frac{-2}{3}, y \neq \frac{-5}{2}, c > 0 \end{aligned}$$

So the IVP of $y(2) = \frac{-13}{2}$ implies $c = -2$. Then $y = \frac{-5}{2} - (3t+2)^{2/3}$. Notice this is a solution for $t > \frac{-2}{3}$. Notice, y' does not exist when $t = \frac{-2}{3}$.

Also: IVP with $y(-1) = 7$ gives $c = -9$ so we get $y = \frac{-5}{2} - \frac{9}{2}(3t+2)^{2/3}$ with $t < \frac{-2}{3}$. Notice here $y'(\frac{-2}{3})$ DNE. And we are done.