7729 number 7 with details

First, express the following function using the unit step function $H(t)$ and then find its Laplace transform:

$$f(t) = \begin{cases} t+1 & 0 \leq t < 3 \\ (t-5)^2 & 3 \leq t < 7 \\ 4 \geq 7 & \end{cases}$$

The first thing we need to do to solve this problem is to rewrite it in terms of the Heavyside function. So:

$$\begin{aligned} &= (t+1)[1 - H(t-3)] + (t-5)^2[H(t-3) - H(t-7)] + 4[H(t-7)] \\ &= t + 1 + [(t-5)^2 - (t+1)][H(t-3)] + [4 - [(t-5)^2][H(t-3)] \\ &= t + 1 + (t^2 - 11t + 24)[H(t-3)] - (t^2 - 10t - 21)[H(t-7)] \end{aligned}$$

Notice, I must subtract every term preceding the function portion I'm looking for. There is some confusion about this. Many people took the final component 4 and subtracted $[(t-5)^2 - (t+1)]$. This is actually subtracting too much.

For example using the piece-wise definition $f(10) = 4$. Using my solution for the standard form of the Heavyside function is: $10 + 1 + 100 - 110 + 24 - 100 + 100 - 21 = 4$ This is good news. However if I subtracted $(t-5)^2 - (t+1)$ from 4 for my $H(t-7)$ component, I would have the equation

`INCORRECT!` $f(t) = t + 1 + (t^2 - 11t + 24)[H(t-3)] - (t^2 + 11t - 28)[H(t-7)]$ `INCORRECT!`

Which yields: $f(10) = 10 + 1 + 100 - 110 + 24 - 100 + 110 + 28 = 63$ Note: $4 \neq 63$! The general form looks something like this:

$$f(t) = \begin{cases} A & 0 \leq t < a \\ B & a \leq t < b \\ C & b \leq t < c \\ 0 & t \geq c \end{cases}$$

This yields the following Heavyside function:

$$\begin{aligned} f(t) &= A[1 - H(t-a)] + B[H(t-a) - H(t-b)] + C[H(t-b) - H(t-c)] \\ f(t) &= A + (B-A)H(t-a) + (C-B)H(t-b) + (0-C)H(t-c) \end{aligned}$$

OK! Back to the problem at hand. We know want to take the Laplace transform of $f(t)$. Recall from page 35 of your text: $\mathcal{L}\{f(t)H(t-c)\} = e^{-cs}\mathcal{L}\{f(t+c)\}$. So we must first pull out the e^{-cs} and apply the translation to the t variable. Then complete the Laplace transform. We always apply the transform to the t variable, even for inverse Laplace transforms. So in general, when you are doing translations only translate when you are working with the t variable.

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \mathcal{L}\{t\} + \mathcal{L}\{1\}\mathcal{L}\{(t^2-11t+24)[H(t-3)]\} - \mathcal{L}\{(t^2-10t-21)[H(t-7)]\} \\ &= \mathcal{L}\{t\} + \mathcal{L}\{1\} + e^{-3s}\mathcal{L}\{(t+3)^2 - 11(t+3) + 24\} + e^{-7s}\mathcal{L}\{-(t+7)^2 + 10(t+7) - 21\} \\ &= \tfrac{1}{s^2} + \tfrac{1}{s} + e^{-3s}\mathcal{L}\{t^2 - 5t\} + e^{-7s}\mathcal{L}\{t^2 + 4t\} \\ &= \tfrac{1}{s^2} + \tfrac{1}{s} + e^{-3s}[\tfrac{2}{s^3} - \tfrac{5}{s^2}] - e^{-7s}[\tfrac{2}{s^3} + \tfrac{4}{s^2}]\end{aligned}$$

This is our final solution.