AIMS Exercise Set # 2

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1. Explain why the equation $e^{-x} = x$ has a solution on the interval $[0,1]$. Use bisection to find the root to 4 decimal places. Can you prove that there are no other roots?

2. Find $\sqrt[6]{3}$ to 5 decimal places by setting up an appropriate equation and solving using bisection.

3. Find all real roots of the polynomial $x^5 - 3x^2 + 1$ to 4 decimal places using bisection.

4. Let $g(u)$ have a fixed point $u^\star$ in the interval $[0,1]$, with $g'(u^\star) \neq 1$. Define

$$G(u) = \frac{u\,g'(u) - g(u)}{g'(u) - 1}.$$

(*a*) Prove that, for an initial guess $u^{(0)}$ near $u^\star$, the fixed point iteration scheme $u^{(n+1)} = G(u^{(n)})$ converges to the fixed point u. (*b*) What is the order of convergence of this method? (*c*) Test this method on the non-convergent cubic scheme in Example 2.16.

5. Let $g(u) = 1 + u - \frac{1}{8}u^3$. (*a*) Find all fixed points of $g(u)$. (*b*) Does fixed point iteration converge? If so, to which fixed point(s)? What is the rate of convergence? (*c*) Predict how may iterates will be needed to get the fixed point accurate to 4 decimal places starting with the initial guess $u^{(0)} = 1$. (*d*) Check your prediction by performing the iteration.

6. Solve Exercise 1–3 by Newton's Method.

7. (*a*) Let $u^\star$ be a simple root of $f(u) = 0$. Discuss the rate of convergence of the iterative method (sometimes known as *Olver's Method*, in honor of the author's father) based on $g(u) = u + \dfrac{f(u)^2\, f''(u) - 2\, f(u)\, f'(u)^2}{2\, f'(u)^3}$ to $u^\star$. (*b*) Try this method on the equation in Exercise 3, and compare the speed of convergence with that of Newton's Method.