Introduction to Partial Differential Equations

by Peter J. Olver

Corrections to First Printing (2014)

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⋆⋆⋆ Page ix ⋆⋆⋆

Line 7: change "In order the make progress ..." to "In order to make progress ..."

Line -3: change "... no knowledge of measure theory or Lebesgue integration ..." to "... knowledge of measure theory and Lebesgue integration ..."

⋆⋆⋆ Page xv ⋆⋆⋆

Line 29: correct spelling of "especially"

⋆⋆⋆ Page xvii ⋆⋆⋆

Line 16: change "... discover(s) ..." to "... discoverer(s) ..."

⋆⋆⋆ Page 6 ⋆⋆⋆

line 2: change $D = \{ t > 0 \} \subset \mathbb{R}^2$ to $D = \{ (t,x) \mid t > 0 \} \subset \mathbb{R}^2$

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Exercise 1.10(a): change $4t^2 - x^2$ to $4t^2 + x^2$.

Exercise 1.16: change the final term in the displayed formula

$$\frac{\partial^2 u}{\partial x \, \partial y}(0,0) = 1 \neq -1 = \frac{\partial^2 u}{\partial y \, \partial x}(0,0).$$

⋆⋆⋆ Page 18 ⋆⋆⋆

Exercise 2.1.9: correct the formula for $D_a = D \cap \{ (t,a) \mid t \in \mathbb{R} \}$

⋆⋆⋆ Page 23 ⋆⋆⋆

Exercise 2.2.11: reword part (c) as follows:

On the other hand, if the initial data is negative somewhere, so $f(x) < 0$ at some $x \in \mathbb{R}$, then the solution *blows up* in finite time: $\lim_{t \to \tau^-} u(t,y) = -\infty$ for some $\tau > 0$ and some $y \in \mathbb{R}$.

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Exercise 2.2.31(b): insert $= 0$ in equation: $u_t + y\,u_x - x\,u_y = 0$.

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Line before Example 2.6: change “ther” to “the”

⋆⋆⋆ Page 33 ⋆⋆⋆

Second row of Figure 2.11: change $t = 5$ to $t = 4.9$

⋆⋆⋆ Page 57 ⋆⋆⋆

In the last two displayed formulas, the first term on the right hand side of the equals sign is missing a minus sign:

$$\frac{\partial v}{\partial \xi}(\xi,\eta) = -\,\frac{1}{2\,c}\,\frac{\partial u}{\partial t}\left(\frac{\eta-\xi}{2\,c},\frac{\eta+\xi}{2}\right) + \frac{1}{2}\,\frac{\partial u}{\partial x}\left(\frac{\eta-\xi}{2\,c},\frac{\eta+\xi}{2}\right),$$

and so, in particular,

$$\frac{\partial v}{\partial \xi}(\xi,\xi) = -\,\frac{1}{2\,c}\,\frac{\partial u}{\partial t}(0,\xi) + \frac{1}{2}\,\frac{\partial u}{\partial x}(0,\xi) = 0,$$

⋆⋆⋆ Page 59 ⋆⋆⋆

Last displayed equation: add in movie symbol

⋆⋆⋆ Page 82 ⋆⋆⋆

Exercise 3.2.14: insert after first sentence:

(See (3.81) for the definition of the function $\operatorname{sign} x$.)

⋆⋆⋆ Page 87 ⋆⋆⋆

In Example 3.16: the Fourier coefficient $a_0 = 4/\pi$, so

$$a_k = \frac{2}{\pi}\int_0^{\pi} \sin x \cos k\,x\,dx = \begin{cases} \dfrac{4}{(1-k^2)\,\pi}\,, & k \text{ even},\\ 0, & k \text{ odd}. \end{cases}$$

⋆⋆⋆ Page 88 ⋆⋆⋆

Exercise 3.2.47: add final sentence “If not, explain why.”

⋆⋆⋆ Page 97 ⋆⋆⋆

In caption to Figure 3.10: delete “3” in “extension”

⋆⋆⋆ Page 98 ⋆⋆⋆

Lines -6 and -4: change $v^\star$ to $\mathbf{v}^\star$

⋆⋆⋆ Page 101 ⋆⋆⋆

Reword Proposition 3.28 for clarity:

Suppose the series $\sum_{k=1}^{\infty} u_k(x) = f(x)$ converges pointwise. If the differentiated series $\sum_{k=1}^{\infty} u_k'(x) = g(x)$ is uniformly convergent, then the original series is also uniformly convergent, and, moreover, $f'(x) = g(x)$.

⋆⋆⋆ Page 103 ⋆⋆⋆

Exercise 3.5.1: change $v^\star$ to $\mathbf{v}^\star$ (twice)

⋆⋆⋆ Page 104 ⋆⋆⋆

Exercise 3.5.18: change $v^\star$ to $\mathbf{v}^\star$

⋆⋆⋆ Page 105 ⋆⋆⋆

Line -4: change "second derivative" to "first derivative"

⋆⋆⋆ Page 108 ⋆⋆⋆

Line 2 before (3.106): remove square root from "… equal to $\dfrac{1}{2\pi}\displaystyle\int_a^b |\,\varphi(x)\,|^2\,dx$."

⋆⋆⋆ Page 116 ⋆⋆⋆

Change the displayed equation after (3.129) and the following line to:

$$\begin{aligned} s_n(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,\frac{\sin\left(n+\frac12\right)(x-y)}{\sin\frac12(x-y)}\,dy \\ &= \frac{1}{2\pi}\int_{-\pi-x}^{\pi-x} f(x+\widehat{y})\,\frac{\sin\left(n+\frac12\right)\widehat{y}}{\sin\frac12\widehat{y}}\,d\widehat{y} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+y)\,\frac{\sin\left(n+\frac12\right)y}{\sin\frac12 y}\,dy. \end{aligned}$$

The second equality is the result of changing the integration variable to $\widehat{y} = y - x$ and canceling the minus signs in the resulting trigonometric fraction;

⋆⋆⋆ Page 119 ⋆⋆⋆

Exercises 3.5.41 and 3.5.42: change $v^\star$ to $\mathbf{v}^\star$

⋆⋆⋆ Page 127 ⋆⋆⋆

Correct the formula for b_1 in equation (4.28):

$$u(t,x) \approx b_1 \exp\left(-\,\frac{\gamma\pi^2}{\ell^2}\,t\right)\,\sin\frac{\pi x}{\ell}\,, \quad \text{where} \quad b_1 = \frac{2}{\ell}\int_0^\ell f(x)\,\sin\frac{\pi x}{\ell}\,dx. \tag{4.28}$$

⋆⋆⋆ Page 128 ⋆⋆⋆

Line 10: change "According to the discussion at the end of Section 3.3, …" to "According to Corollary 3.32, …"

⋆⋆⋆ Page 130 ⋆⋆⋆

Equation (4.34): add in movie symbol $\uplus$

⋆⋆⋆ Page 131 ⋆⋆⋆

In (4.37), change final + to −:

$$u(t,x) \approx \tfrac{1}{2}\,a_0 + e^{-t}\,(a_1 \cos x + b_1 \sin x) = \tfrac{1}{2}\,a_0 + r_1\,e^{-t}\cos(x - \delta_1), \tag{4.37}$$

⋆⋆⋆ Page 152 ⋆⋆⋆

Line -6: change "… appear the context of boundary value problems." to "… appear in the context of boundary value problems."

⋆⋆⋆ Page 153 ⋆⋆⋆

Line 5 after (4.86): change "heat flux out of a plate" to "heat flux into a plate"

⋆⋆⋆ Page 163 ⋆⋆⋆

Last line of table: change $x^4 - 4\,x^2 y^2 + y^4$ to $x^4 - 6\,x^2 y^2 + y^4$

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There is a subtle logical flaw in the argument leading from Theorem 4.8 to the Uniqueness Theorem 4.10 via the Maximum Principle 4.9. The proof of Theorem 4.8 relies on the Poisson integral formula (4.126) leading to formula (4.130). However, it is not a priori guaranteed that *every* solution to the Poisson boundary value problem on the unit disk is given by the Poisson formula; indeed, this follows from uniqueness, but this leads to a circular argument. A fully rigorous approach is to rely instead on the following direct proof of Theorem 4.8. Once this is established without recourse to the Poisson solution formula, the Maximum Principle and the Uniqueness Theorem follow as in the text.

Thanks to Christopher Grant for pointing this out.

Direct Proof of Theorem 4.8: Given the harmonic function $u(x,y)$, consider the scalar function

$$g(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x_0 + a\cos\theta, y_0 + a\sin\theta)\,d\theta,$$

which is well defined for $a > 0$ sufficiently small. Since $u \in \mathrm{C}^2$, we can calculate the derivative of g as follows:

$$\begin{aligned} g'(a) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\cos\theta\,\frac{\partial u}{\partial x}\,(x_0{+}a\cos\theta, y_0{+}a\sin\theta) + \sin\theta\,\frac{\partial u}{\partial y}\,(x_0{+}a\cos\theta, y_0{+}a\sin\theta)\right] d\theta \\ &= \frac{1}{2\pi a}\oint_C \frac{\partial u}{\partial \mathbf{n}}\,ds, \end{aligned}$$

where $\mathbf{n} = (\cos\theta, \sin\theta)$ defines the unit normal to C at the point $(x_0 + a\cos\theta, y_0 + a\sin\theta)$ and $ds = a\,d\theta$ is the arc length element. Letting $D = \{\,(x - x_0)^2 + (y - y_0)^2 \leq a^2\,\}$ denote the disk of radius a, so $C = \partial D$ is its boundary, the divergence identity (6.89) (an easy consequence of Green's Theorem) implies that the latter integral equals

$$\oint_C \frac{\partial u}{\partial \mathbf{n}}\, ds = \iint_D \Delta u \, dx\, dy = 0$$

because u is harmonic. Thus, $g'(a) = 0$ for all $a > 0$ sufficiently small, which implies $g(a) = c$ is constant. But $g(a)$ represents the average of $u(x,y)$ on the circle C of radius a centered at (x_0, y_0) and hence $g(a) \to u(x_0, y_0)$ as $a \to 0$. We conclude that $g(a) = u(x_0, y_0)$ for all $a > 0$ such that $u(x,y)$ is harmonic in the disk of radius a, which establishes (4.131) for all such harmonic functions. *Q.E.D.*

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Exercise 4.3.25(b): change $x^2 + y^2 = 1;$ to $x^2 + y^2 = 2;$

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Exercise 4.4.12(a): switch t and x in the function: $u_n(t,x) = \dfrac{\cosh n\pi t\ \sin n\pi x}{n}$.

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Line 2 before (5.14): change
"… heat equation (5.14) …" to "… heat equation (5.7) …".

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Example 5.4: change rest of sentence after displayed formula to

"…used earlier in Example 4.1, along with homogeneous Dirichlet boundary conditions, so $u(t,0) = u(t,1) = 0$."

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Equation (5.28): change $\mathrm{O}\big((\Delta t)^2\big)$ to $\mathrm{O}(\Delta t)$:

$$\frac{\partial u}{\partial t}(t_j, x_m) \approx \frac{u(t_j, x_m) - u(t_{j-1}, x_m)}{\Delta t} + \mathrm{O}(\Delta t). \tag{5.28}$$

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Example 5.5, line 4: change $t = .2, .4, .6$ to $t = .02, .04, .06$

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Exercise 5.2.1(b): change "… one value of Δx in the allowed …" to
"… one value of Δt in the allowed …"

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Exercise 5.2.8(a): correct spelling of "approximation"
Exercise 5.2.9(a): correct spelling of "approximations"

⋆⋆⋆ Page 195 ⋆⋆⋆

Change sentence after equation (5.40): "We use step sizes $\Delta t = \Delta x = .005$, set $\ell = 1$, and try four different values of the wave speed."

⋆⋆⋆ Page 196 ⋆⋆⋆

Figure 5.5 is missing the last column of plots. The corrected page can be found at

`http://www.math.umn.edu/~olver/book_/p196.pdf`

Line -5: change "The are two ways ..." to "There are two ways ..."

⋆⋆⋆ Page 198 ⋆⋆⋆

Equation (5.45): change denominator to $2\,\Delta x$:

$$\frac{\partial u}{\partial x}(t_j, x_m) \approx \frac{u_{j,m+1} - u_{j,m-1}}{2\,\Delta x} + \mathrm{O}\big((\Delta x)^2\big). \tag{5.45}$$

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Line 4 after (5.50): reverse the inequality: $\Delta x / \Delta t \geq |\, c_{j,m} \,|$

⋆⋆⋆ Page 210 ⋆⋆⋆

Correct last displayed equation by switching indices on the $u_{i,j}$:

$$\begin{array}{lll} u_{1,1} = .1831, & u_{2,1} = .2589, & u_{3,1} = .1831, \\ u_{1,2} = .3643, & u_{2,2} = .5152, & u_{3,2} = .3643, \\ u_{1,3} = .5409, & u_{2,3} = .7649, & u_{3,3} = .5409, \end{array}$$

⋆⋆⋆ Page 211 ⋆⋆⋆

Equation (5.78): the sub- and super-diagonal matrix elements should be -1, not $-\rho^2$:

$$B_\rho = \begin{pmatrix} 2(1+\rho^2) & -1 & & & & & \\ -1 & 2(1+\rho^2) & -1 & & & & \\ & -1 & 2(1+\rho^2) & -1 & & & \\ & & -1 & 2(1+\rho^2) & -1 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -1 & 2(1+\rho^2) & -1 \\ & & & & & -1 & 2(1+\rho^2) \end{pmatrix} \tag{5.78}$$

⋆⋆⋆ Page 212 ⋆⋆⋆

Equation (5.82): replace $\mathbf{w}^{(n-1)}$ by $U_{n-1}\mathbf{w}^{(n-1)}$ and U_j by U_k:

$$\begin{array}{lll} \mathbf{z}^{(1)} = \widehat{\mathbf{f}}^{(1)}, & \mathbf{z}^{(j+1)} = \widehat{\mathbf{f}}^{(j+1)} - L_j \mathbf{z}^{(j)}, & j = 1, 2, \ldots, n-2, \\ U_{n-1}\mathbf{w}^{(n-1)} = \mathbf{z}^{(n-1)}, & U_k \mathbf{w}^{(k)} = \mathbf{z}^{(k)} - \rho^2\, \mathbf{w}^{(k+1)}, & k = n-2, n-3, \ldots, 1, \end{array} \tag{5.82}$$

Equation (5.83): replace L_j by L_k:

$$\mathbf{w}^{(k)} = L_k\big(\mathbf{w}^{(k+1)} - \rho^{-2}\,\mathbf{z}^{(k)}\big), \qquad k = n-2, n-3, \ldots, 1. \tag{5.83}$$

⋆⋆⋆ Page 229 ⋆⋆⋆

Equation (6.40): delete initial fraction:

$$\int_{-\pi}^{\pi} s_n(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{1}{2}x}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-n}^{n} e^{\,\mathrm{i}\,k\,x}\,dx = 1, \tag{6.40}$$

⋆⋆⋆ Page 231 ⋆⋆⋆

Add the following footnote:

Definition 6.5 only requires continuity of the test functions, whereas in (6.44) they need to be C^1, and so the notion of weak convergence here is slightly more refined. One often restricts further to allow only C^∞ test functions.

Also, delete the *Remark* at bottom of the page, which is *not* correct. (Indeed, if valid, it would imply that 0 is always a point of convergence of a Fourier series of any continuous function, which, by translation, would imply that such Fourier series converge everywhere, in contradiction to known examples of continuous functions whose Fourier series does not converge everywhere.)

⋆⋆ Thanks to Joost Hulshof for pointing this out.

⋆⋆⋆ Page 238 ⋆⋆⋆

In the first integral in the displayed equation after (6.65) change $\sinh\omega y$ to $\sinh\omega\xi$:

$$u(x) = \int_0^x \frac{\sinh\omega\,(1-x)\sinh\omega\,\xi}{\omega\sinh\omega}\,d\xi \;+\; \int_x^1 \frac{\sinh\omega\,x\,\sinh\omega\,(1-\xi)}{\omega\sinh\omega}\,d\xi$$

⋆⋆⋆ Page 250 ⋆⋆⋆

Displayed formula after Theorem 6.17: change $\mathbb{R}^2$ to Ω:

$$u(x,y) = -\iint_\Omega G_0(x,y;\xi,\eta)\,\Delta u(\xi,\eta)\,d\xi\,d\eta.$$

Equation (6.108): change $\mathbb{R}^2$ to Ω twice:

$$\iint_\Omega \delta(x-\xi)\,\delta(y-\eta)\,u(\xi,\eta)\,d\xi\,d\eta = \iint_\Omega -\Delta G_0(x,y;\xi,\eta)\,u(\xi,\eta)\,d\xi\,d\eta. \tag{6.108}$$

⋆⋆⋆ Page 255 ⋆⋆⋆

Exercise 6.3.5: replace $\beta > 0$ by $\beta \neq 0$

⋆⋆⋆ Page 258 ⋆⋆⋆

Equation (6.135): correct left hand side:

$$\frac{\partial G}{\partial \rho}\,(r,\theta;1,\phi) = -\,\frac{1}{2\pi}\;\frac{1-r^2}{1+r^2-2\,r\cos(\theta-\phi)}\,, \tag{6.135}$$

⋆⋆⋆ Page 260 ⋆⋆⋆

Exercise 6.3.21: change $u_{xx} - u_{yy} = f(x,y)$ to $-u_{xx} - u_{yy} = f(x,y)$
Exercise 6.3.28: change "... of the point $(a,0)$." to "... of the point $(0,a)$."

⋆⋆⋆ Page 270 ⋆⋆⋆

Equation (7.33): delete 2 in numerator of fraction:

$$\lim_{a\to 0}\sqrt{\frac{2}{\pi}}\;\frac{a}{k^2+a^2} = \begin{cases} 0, & k\neq 0, \\ \infty, & k = 0. \end{cases} \tag{7.33}$$

⋆⋆⋆ Page 272 ⋆⋆⋆

The Table entry for the Fourier transform of $\tan^{-1} x$ is *not* correct. The correct Fourier transform

$$-\,\mathrm{i}\sqrt{\frac{\pi}{2}}\;\frac{e^{-\,|\,k\,|}}{k}$$

can be found in Example 7.11 on page 277. Thanks to Ulrich Gerlach for noticing this.

⋆⋆⋆ Page 273 ⋆⋆⋆

First displayed equation and line immediately after: change $x - \frac{1}{2}\,\mathrm{i}\,k$ to $x + \frac{1}{2}\,\mathrm{i}\,k$:

$$\widehat{g}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\,x^2-\,\mathrm{i}\,k\,x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\,(x+\,\mathrm{i}\,k/2)^2 - k^2/4}\,dx$$
$$= \frac{e^{-\,k^2/4}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\,y^2}\,dy = \frac{e^{-\,k^2/4}}{\sqrt{2}}\,.$$

The next-to-last equality employed the change of variables† $y = x + \frac{1}{2}\,\mathrm{i}\,k$, while the final step used formula (2.100).

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The proof of Proposition 7.10 is flawed. Since $k\,\delta(k) \equiv 0$, dividing equation (7.47) by $\mathrm{i}\,k$ could also introduce a multiple of the delta function, and it seems difficult to dismiss this term. A better proof uses the Convolution Theorem 7.13, as follows.

The first step is to note that we can write the integral of $f(x)$ as a convolution with the step function:

$$g(x) = \int_{-\infty}^{x} f(\xi)\,d\xi = \int_{-\infty}^{\infty} \sigma(x-\xi)\,f(\xi)\,d\xi,$$

Thus, according to the convolution formula (7.55),

$$\widehat{g}(k) = \sqrt{2\pi}\,\widehat{\sigma}(k)\,\widehat{f}(k).$$

Consulting our Table of Fourier transforms, we find

$$\widehat{g}(k) = \sqrt{2\pi}\left(\sqrt{\frac{\pi}{2}}\,\delta(k) - \frac{\mathrm{i}}{\sqrt{2\pi}\,k}\right)\widehat{f}(k) = -\frac{\mathrm{i}}{k}\,\widehat{f}(k) + \pi\widehat{f}(k)\,\delta(k) = -\frac{\mathrm{i}}{k}\,\widehat{f}(k) + \pi\widehat{f}(0)\,\delta(k),$$

which establishes the desired formula.

⋆⋆ Thanks to William Young for alerting me to this issue and for sharing the above convolution-based proof.

⋆⋆⋆ Page 277 ⋆⋆⋆

In the displayed equation immediately above the Exercises, delete one factor of $1/k$ in the first term after the equals sign:

$$\widehat{f}(k) = \left(-\frac{\mathrm{i}}{k}\sqrt{\frac{\pi}{2}}\,e^{-|\,k\,|} + \frac{\pi^{3/2}}{\sqrt{2}}\,\delta(k)\right) - \frac{\pi^{3/2}}{\sqrt{2}}\,\delta(k) = -\,\mathrm{i}\sqrt{\frac{\pi}{2}}\,\frac{e^{-|\,k\,|}}{k}\,.$$

⋆⋆⋆ Page 278 ⋆⋆⋆

Exercise 7.2.12: insert factor of $\sqrt{2\pi}$ in formula $\widehat{f}(k) = \sqrt{2\pi}\displaystyle\sum_{n=-\infty}^{\infty} c_n\,\delta(k-n)$.

⋆⋆⋆ Page 279 ⋆⋆⋆

Line before (7.51): change "… table of Fourier transform," to "… table of Fourier transforms,"

⋆⋆⋆ Page 287 ⋆⋆⋆

Line after (7.67): change "… is the *variance*, that is, the statistical deviation …" to "… which is the probability density's *variance*, is the statistical deviation …"

⋆⋆⋆ Page 308 ⋆⋆⋆

Line 3: delete "symmetryscaling"

⋆⋆⋆ Page 310 ⋆⋆⋆

Second displayed formula after equation (8.63): insert missing factor of $\frac{1}{2}$:

$$u(t,x) = c_1 + c_2\,\mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right).$$

⋆⋆⋆ Page 314 ⋆⋆⋆

Exercise 8.3.2: correct spelling of "interpretation"

⋆⋆⋆ Page 337 ⋆⋆⋆

Exercise 8.5.16: correct spelling of "speeds"

$\star\star\star$ Page 351 $\star\star\star$

Fifth displayed formula: change second equation $v_2 - 2v_3 = b_2$ to $v_2 - 2v_3 = 0$

$\star\star\star$ Page 353 $\star\star\star$

Exercise 9.1.21: change boundary conditions to $u'(1) = u'(2) = 0$.
Line -5: change "equal their own transpose," to "equal to their own transpose,"

$\star\star\star$ Page 363 $\star\star\star$

Insert parenthetical comment at end of page:

(The case $q(x) \equiv 0$ can also be positive definite, when subject to suitable boundary conditions, but is treated differently, in accordance with the weighted inner product construction appearing in Example 9.23.)

$\star\star\star$ Page 371 $\star\star\star$

Line 4: change "... heat and s." to "... heat and wave equations."
Line 8: change "... the self-adjointn of ..." to "... the self-adjointness of ..."

$\star\star\star$ Page 389 $\star\star\star$

Change (9.131–132) to the following:

$$\begin{aligned} u(t,x) &= \sum_{k=1}^{\infty} \left[\, c_k u_k(t,x) + d_k \widetilde{u}_k(t,x) \,\right] \\ &= \sum_{k=1}^{\infty} \left[\, c_k \cos(\omega_k t) + d_k \sin(\omega_k t) \,\right] v_k(x) = \sum_{k=1}^{\infty} r_k \cos(\omega_k t - \delta_k)\, v_k, \end{aligned} \tag{9.131}$$

where (r_k, δ_k) are the polar coordinates of (c_k, d_k):

$$c_k = r_k \cos\delta_k, \qquad d_k = r_k \sin\delta_k. \tag{9.132}$$

$\star\star\star$ Page 391 $\star\star\star$

Change (9.145) to the following:

$$0 = \langle\, h - 2a\omega_k v_k \,, v_k \,\rangle = \langle\, h \,, v_k \,\rangle - 2a\omega_k \,\|\, v_k \,\|^2, \qquad \text{and hence} \qquad a = \frac{\langle\, h \,, v_k \,\rangle}{2\omega_k \,\|\, v_k \,\|^2}\,, \tag{9.145}$$

$\star\star\star$ Page 392 $\star\star\star$

Correct sign errors in (9.149):

$$v_\star(x) = \frac{\sin k\pi x}{k^2\pi^2c^2 - \omega^2}\,, \qquad \text{so that} \qquad u_\star(t,x) = \frac{\cos\omega t\ \sin k\pi x}{k^2\pi^2c^2 - \omega^2}\,, \tag{9.149}$$

and in the last displayed equation:

$$z(0,x) = f(x) \;-\; \frac{\sin k\,\pi\,x}{k^2\,\pi^2\,c^2 - \omega^2}\,, \qquad\qquad \frac{\partial z}{\partial t}\,(0,x) = g(x),$$

⋆⋆⋆ Page 397 ⋆⋆⋆

Exercise 9.5.30: change "... you may wish return here ..." to
"... you may wish to return here ..."

⋆⋆⋆ Page 402 ⋆⋆⋆

Line 7: change "ndTwo ..." to "Two ..."

⋆⋆⋆ Page 411 ⋆⋆⋆

Line 9 in paragraph beginning "The first ...": change "vertexvertices" to "vertices".

⋆⋆⋆ Page 413 ⋆⋆⋆

Last equation in (10.32): change y_k to y_l:

$$\begin{aligned} \omega_l^\nu(x_i,y_i) &= \alpha_l^\nu + \beta_l^\nu\,x_i + \gamma_l^\nu\,y_i = 0,\\ \omega_l^\nu(x_j,y_j) &= \alpha_l^\nu + \beta_l^\nu\,x_j + \gamma_l^\nu\,y_j = 0,\\ \omega_l^\nu(x_l,y_l) &= \alpha_l^\nu + \beta_l^\nu\,x_l + \gamma_l^\nu\,y_l = 1. \end{aligned} \tag{10.32}$$

⋆⋆⋆ Page 416 ⋆⋆⋆

The first term in the integral in (10.38) is the Euclidean norm of a vector:

$$\begin{aligned} Q[w] = Q\left[\sum_{i=1}^n c_i\,\varphi_i\right] &= \iint_\Omega \left[\left\|\sum_{i=1}^n c_i\,\nabla\varphi_i\right\|^2 - f(x,y)\left(\sum_{i=1}^n c_i\,\varphi_i\right)\right] dx\,dy \\ &= \frac{1}{2}\sum_{i,j=1}^n k_{ij}\,c_i\,c_j - \sum_{i=1}^n b_i\,c_i = \tfrac{1}{2}\,\mathbf{c}^T K\,\mathbf{c} - \mathbf{b}^T\mathbf{c}. \end{aligned} \tag{10.38}$$

⋆⋆⋆ Page 417 ⋆⋆⋆

Correct last line in (10.45):

$$\begin{aligned} k_{ij}^\nu &= \frac{1}{2}\,\frac{(y_j-y_l)(y_l-y_i)+(x_l-x_j)(x_i-x_l)}{(\Delta_\nu)^2}\;|\,\Delta_\nu\,| = -\,\frac{(\mathbf{x}_i-\mathbf{x}_l)\cdot(\mathbf{x}_j-\mathbf{x}_l)}{2\,|\,\Delta_\nu\,|}\,, \quad i\neq j,\\ k_{ii}^\nu &= \frac{1}{2}\,\frac{(y_j-y_l)^2+(x_l-x_j)^2}{(\Delta_\nu)^2}\;|\,\Delta_\nu\,| = \frac{\|\,\mathbf{x}_j-\mathbf{x}_l\,\|^2}{2\,|\,\Delta_\nu\,|}\\ &= \frac{(\mathbf{x}_i-\mathbf{x}_l)\cdot(\mathbf{x}_j-\mathbf{x}_l)+(\mathbf{x}_l-\mathbf{x}_j)\cdot(\mathbf{x}_i-\mathbf{x}_j)}{2\,\Delta_\nu} = -\,k_{ij}^\nu - k_{il}^\nu\,. \end{aligned} \tag{10.45}$$

⋆⋆⋆ Pages 418–9 ⋆⋆⋆

Change first sentence in Example 10.7 to

A metal plate has the shape of an oval running track, consisting of a square, with side lengths 2 m, and two semi-circular disks glued onto opposite sides, as sketched in Figure 10.9.

⋆⋆⋆ Page 426 ⋆⋆⋆

Exercise 10.3.16: change $n = 2$ in part (b) to $n = 3$ and change $n = 3$ in part (c) to $n = 4$.

⋆⋆⋆ Page 430 ⋆⋆⋆

Line before (10.68): change "… atrictly positive …" to "… strictly positive …"

⋆⋆⋆ Page 434 ⋆⋆⋆

Exercise 10.4.3(c): change "… wave equation." to "… transport equation."

⋆⋆⋆ Page 448 ⋆⋆⋆

Figure 11.2: add in movie symbol
Line -3: correct spelling of "discontinuities"

⋆⋆⋆ Page 450 ⋆⋆⋆

Exercise 11.2.11: change "… circular ends at held …" to
"… circular ends are held …"

⋆⋆⋆ Page 450 ⋆⋆⋆

Exercise 11.2.12: correct boundary conditions:

$$u(t,0,y) = u(t,\pi,y) = 0 = u(t,x,0), \quad u(t,x,\pi) = f(x), \quad 0 < x, y < \pi, \quad t > 0.$$

⋆⋆⋆ Pages 464–5 ⋆⋆⋆

In equation (11.91) and the subsequent displayed formula, change all s, t, r to a, b, c:

$$a_0\, r\,(r-1) + b_0\, r + c_0 = 0, \tag{11.91}$$

where, referring back to (11.71),

$$a_0 = a(x_0), \qquad b_0 = b(x_0), \qquad c_0 = c(x_0),$$

⋆⋆⋆ Page 465 ⋆⋆⋆

Case (*iii*): change $r_2 = r_1 + k$ to $r_1 = r_2 + k$; change "smaller" to "larger", and change x^{r_2} to $(x - x_0)^{r_2}$ in equation (11.93):

(*iii*) Finally, if $r_1 = r_2 + k$, where $k > 0$ is a positive integer, then there is a nonzero solution $\widehat{u}(x)$ with a convergent Frobenius expansion corresponding to the larger index r_1. One can construct a second independent solution of the form

$$\widetilde{u}(x) = c \log(x - x_0)\, \widehat{u}(x) + v(x), \quad \text{where} \quad v(x) = (x - x_0)^{r_2} + \sum_{n=1}^{\infty} v_n (x - x_0)^{n + r_2} \tag{11.93}$$

is a convergent Frobenius series, and c is a constant, which may be 0, in which case the second solution $\widetilde{u}(x)$ is also of Frobenius form.

⋆⋆⋆ Page 466 ⋆⋆⋆

Correct formulas after equation (11.96) as follows:

$$u'' + \left(\frac{1}{x} + \frac{x}{2}\right) u' + u = v \left[\widehat{u}'' + \left(\frac{1}{x} + \frac{x}{2}\right) \widehat{u}' + \widehat{u} \right] + v' \left[2\,\widehat{u}' + \left(\frac{1}{x} + \frac{x}{2}\right) \widehat{u} \right] + v''\, \widehat{u}$$
$$= e^{-x^2/4} \left[v'' + \left(\frac{1}{x} - \frac{x}{2}\right) v' \right].$$

If u is to be a solution, v' must satisfy a linear first-order ordinary differential equation:

$$v'' + \left(\frac{1}{x} - \frac{x}{2}\right) v' = 0, \qquad \text{and hence} \qquad v' = \frac{c}{x}\, e^{x^2/4}, \qquad v = c \int \frac{e^{x^2/4}}{x} + d,$$

where c, d are arbitrary constants. We conclude that the general solution to the original differential equation is

$$\widetilde{u}(x) = v(x)\, \widehat{u}(x) = \left(c \int \frac{e^{x^2/4}}{x} + d \right) e^{-x^2/4}. \tag{11.97}$$

⋆⋆ Thanks to Manuel Mañas for alerting me to this error.

⋆⋆⋆ Page 481 ⋆⋆⋆

First line of Section 11.5: change "As we learned in Section 4.1, ..." to
"As we learned in Section 8.1, ..."

⋆⋆⋆ Page 482 ⋆⋆⋆

Line after (11.133): Change "... both function (11.132) ..." to
"... both functions (11.132) ..."

⋆⋆⋆ Page 483 ⋆⋆⋆

In Theorem 11.13, first displayed formula: change $u(t, x, y)$ to $u(0, x, y)$:

$$u_t = \gamma\, \Delta u, \qquad u(0, x, y) = f(x, y), \qquad (x, y) \in \mathbb{R}^2,$$

⋆⋆⋆ Page 485 ⋆⋆⋆

Exercise 11.5.9: replace $\beta > 0$ by $\beta \neq 0$

⋆⋆⋆ Page 489 ⋆⋆⋆

Add movie symbol ⊎ to Figure 11.10.

⋆⋆⋆ Page 499 ⋆⋆⋆

Example 11.15: correct equation 2 lines from the end: $\zeta_{0,1}/\zeta_{0,2} \approx .43565$

⋆⋆⋆ Page 500 ⋆⋆⋆

Exercise 11.6.41: switch indices on $\omega_{i,j}$:

(a) $\omega_{0,4}$, (b) $\omega_{2,4}$, (c) $\omega_{4,2}$, (d) $\omega_{3,3}$, (e) $\omega_{5,1}$.

⋆⋆⋆ Page 507 ⋆⋆⋆

Exercise 12.1.11(b): change $\displaystyle\iiint_\Omega ||| \nabla u |||^2 \,dx\,dy\,dz$ to $\displaystyle\iiint_\Omega \| \nabla u \|^2 \,dx\,dy\,dz$

⋆⋆⋆ Page 511 ⋆⋆⋆

After equation (12.29): add "with corresponding eigenvalue parameter $\mu = n\,(n+1)$."

⋆⋆⋆ Page 513 ⋆⋆⋆

After equation (12.31): add

"The eigenvalue parameter for $P_n^m(t)$ is also $\mu = n\,(n+1)$."

⋆⋆⋆ Page 521 ⋆⋆⋆

A direct proof of Theorem 12.4 that does not rely on the solution formula (12.51) can be constructed along the same lines of the proof of Theorem 4.8 given above. As noted there, this is preferable for logical consistency in establishing the Maximum Principle of Theorem 12.5 and the consequential Uniqueness Theorem.

Proof of Theorem 12.4: Let us denote the average of u over the sphere of radius a by

$$\begin{aligned} g(a) &= \frac{1}{4\pi a^2}\iint_{S_a} u \; dS \\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi}\int_0^{\pi} u(x_0 + a\sin\varphi\,\cos\theta, y_0 + a\sin\varphi\,\sin\theta, z_0 + a\cos\varphi)\,\sin\varphi\,d\varphi\,d\theta. \end{aligned}$$

By continuity, as the radius $a \to 0$, the average of u on the sphere S_a tends to its value at the center: $g(a) \to u(\mathbf{x}_0)$. On the other hand, since $u \in \mathrm{C}^2$ and harmonic in $B_a \subset \Omega$, the derivative

$$\begin{aligned} g'(a) &= \frac{1}{4\pi}\int_{-\pi}^{\pi}\int_0^{\pi}\left(\sin\varphi\,\cos\theta\,\frac{\partial u}{\partial x} + \sin\varphi\,\sin\theta\,\frac{\partial u}{\partial y} + \cos\varphi\,\frac{\partial u}{\partial z}\right)\sin\varphi\,d\varphi\,d\theta \\ &= \frac{1}{4\pi a^2}\iint_{S_a}\frac{\partial u}{\partial \mathbf{n}}\,dS = \frac{1}{4\pi a^2}\iiint_{B_a}\Delta u\,dx\,dy\,dz = 0, \end{aligned}$$

where $\mathbf{n}$ denotes the unit outwards normal to $S_a = \partial B_a$, and we used the divergence identity in Exercise 12.1.11(a). We conclude that $g(a)$ is constant, and hence $g(a) = u(\mathbf{x}_0)$ for any $a > 0$ provided $B_a \subset \Omega$. *Q.E.D.*

⋆⋆⋆ Page 532 ⋆⋆⋆

Line -3: change comma to semicolon in $v(\mathbf{x};\boldsymbol{\xi})$

⋆⋆⋆ Page 533 ⋆⋆⋆

Theorem 12.11: change "... the homogeneous Dirichlet boundary value problem" to "... the Dirichlet boundary value problem" to avoid possible confusion. (The boundary conditions are homogeneous, but the Poisson equation is not.)

⋆⋆⋆ Page 541 ⋆⋆⋆

The entries in the fourth column, corresponding to $m = 3$ of the table of spherical Bessel roots is missing. The corrected table is below. Thanks to Ted Kroon for pointing this out.

Spherical Bessel Roots $\sigma_{m,n}$

$n \backslash m$	0	1	2	3	4	5	6	7	8	
1	3.1416	4.4934	5.7635	6.9879	8.1826	9.3558	10.5128	11.6570	12.7908	...
2	6.2832	7.7253	9.0950	10.4171	11.7049	12.9665	⋮	⋮	⋮	
3	9.4248	10.9041	12.3229	⋮	⋮	⋮				
4	12.5664	⋮	⋮							
⋮	⋮									

⋆⋆⋆ Page 548 ⋆⋆⋆

Equation (12.131): add t dependence to $u_{0,0,n}$ and $\widehat{u}_{0,0,n}$, and correct denominators in final expressions:

$$\begin{aligned} u_{0,0,n}(t,r,\varphi,\theta) &= \cos(c\,n\,\pi\,t)\;S_0(n\,\pi\,r) = \frac{\cos c\,n\,\pi\,t\;\sin n\,\pi\,r}{n\,\pi\,r}, \\ \widehat{u}_{0,0,n}(t,r,\varphi,\theta) &= \sin(c\,n\,\pi\,t)\;S_0(n\,\pi\,r) = \frac{\sin c\,n\,\pi\,t\;\sin n\,\pi\,r}{n\,\pi\,r}, \end{aligned} \qquad n = 1,2,3,\ldots. \qquad (12.131)$$

⋆⋆⋆ Page 553 ⋆⋆⋆

In equation (12.145) and the displayed equation immediately after, the limit should be as $t \to 0$:

$$\lim_{t \to 0} \mathrm{M}_{ct}[\,f\,] = \mathrm{M}_0[\,f\,] = f(\mathbf{0}). \tag{12.145}$$

$$\lim_{t \to 0} \langle\, u(t,\cdot)\,, f\,\rangle = \langle\, u(0,\cdot)\,, f\,\rangle = 0 \qquad \text{for } \textit{all} \text{ functions} \quad f,$$

⋆⋆⋆ Page 555 ⋆⋆⋆

Replace the period in equation (12.151) by a comma, and replace the following sentence by

where $\mathrm{M}_{ct}^{\mathbf{x}}[\,g\,]$ denotes the average of the initial velocity function g over the sphere $S_{ct}^{\mathbf{x}} = \{\,\|\,\boldsymbol{\xi} - \mathbf{x}\,\| = c\,t\}$ of radius $c\,t$ centered at the point $\mathbf{x}$. Thus, the value of our solution at position $\mathbf{x}$ and time $t > 0$ only depends upon the initial data a distance ct away from the point $\mathbf{x}$.

⋆⋆⋆ Page 564 ⋆⋆⋆

Section 12.7, second line: change "... owing to it relatively tiny size," to "... owing to its relatively tiny size,"

⋆⋆⋆ Page 579 ⋆⋆⋆

At the end of the statement of Theorem B.15, add "; for the triangle equality, the scalar multiples must be nonnegative."

⋆⋆⋆ Page 584 ⋆⋆⋆

In equation (B.38) and the line before: change $v^\star$ to $\mathbf{v}^\star$ (4 times)

⋆⋆⋆ Page 585 ⋆⋆⋆

lines 2, 4, 5: change $v^\star$ to $\mathbf{v}^\star$ (6 times)

⋆⋆⋆ Page 599 ⋆⋆⋆

Line 12: change $\mathbf{N}$ to $\mathbb{N}$ (the symbol for natural numbers)

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