Math 5486
April 13, 2007

Homework #8

1. Let $u(x)$ be the function that minimizes the integral

$$I[u] = \int_0^2 \left[e^{-x} \left(\frac{du}{dx} \right)^2 - u \right] dx$$

among all C^2 functions that satisfy the boundary conditions $u(0) = 0, \; u(2) = 0$.

(a) Write out the boundary value problem satisfied by $u(x)$.

(b) Find the exact solution to your boundary value problem.

(c) Use the finite element method to approximate the solution. You should use an equally spaced mesh, and try at least three different mesh spacings. Compare your results with the exact solution.

(d) By inspecting the errors in your various approximations, can you predict how many mesh points would be required for 6 digit accuracy (assuming no round off error)?

2. Consider the boundary value problem

$$-u'' + \lambda u = x, \qquad \text{for} \qquad 0 < x < \pi, \qquad \text{with} \qquad u(0) = 0, \; u(\pi) = 0,$$

where λ is a real constant.

(a) For what values of λ does the system have a unique solution?

(b) For which values of λ can you find a minimization principle that characterizes the solution? Is the solution unique for all such values of λ?

(c) Using n equally spaced mesh points, write down the finite element equations for approximating the solution to the boundary value problem.

(d) Verify that the finite element equations agree with the equations obtained using the method of finite differences.

(e) For which values of λ does the finite element/difference system have a unique solution? *Hint*: Apply a lemma that we use for the numerical solutions to the heat equation.

(f) Select a value of λ for which the solution can be characterized by a minimization principle and verify that the finite element/difference approximation with $n = 10$ approximates the exact solution. What is the maximal error?

(g) Experiment with other values of λ. Does your numerical solution give a good approximation to the exact solution, when it exists? What happens at values of λ for which the solution does not exist or is not unique?

3. Find the three finite element functions $\omega_k^\nu(x, y)$ associated with the triangle with vertices $(0, 1), (1, -1)$ and $(-1, 0)$.

4. Write down the elemental stiffnesses for the following triangles: (a) the triangle with vertices $(0,1)$, $(-1,2)$, $(0,-1)$; (b) a $30-60-90$ degree right triangle.

5. A membrane is in the shape of an equilateral triangle T with unit length sides that are glued to the $x\,y$ plane. Each point on the membrane is subject to an external force equal to its distance from the center.

(a) Write down the boundary value problem and minimization principle satisfied by the equilibrium displacement $u(x,y)$. (Make sure you have the correct signs!)

(b) In order to approximate $u(x,y)$, the membrane is divided into smaller equilateral triangles, with n triangles on each side, and the reuslting finite element approximation is computed. How many triangles are in the triangulation? How many interior nodes? How many boundary nodes?

(c) For $n=6$, set up and solve the finite element linear system to find an approximation to the displacement of the center of the triangle.

6. A metal plate has the shape of a 3 cm square with a 1 cm square hole cut out of the middle. The plate is heated by making the inner edge have temperature 100° while keeping the outer edge at 0°. The equilibrium temperature is described by Laplace's equation $\Delta u = 0$.

(a) Find the (approximate) equilibrium temperature by using finite elements with a mesh width of $h = .5$ cm. Plot your approximate solution using MATLAB.

(b) Let C denote the square contour lying midway between the inner and outer square boundaries of the plate. At what point(s) on C is the temperature a (i) minimum, (ii) maximum, (iii) equal to the average of the two boundary temperatures?

(c) Repeat part (a) using a smaller mesh width of $h = .2$. How much does this affect your answers in part (b)?

Due: Monday, May 7

Text: B. Bradie, *A Friendly Introduction to Numerical Analysis.*

Second Midterm: Friday, May 4

Will cover sections 8.1–5, 10.1–2, 11.4 and Finite Elements.

You will be allowed to use one 8" $\times$ 11" sheet of notes.