Math 5588
May 5, 2005

Homework Assignment # 6 — Updated

Use a computer for the numerical problems.

Exercises: Strauss pp. 367: 3, 8.

1. A cylindrical Coke can must contain exactly 12 fluid ounces = 354.778 cubic centimeters. What should be the height and radius of the can in order that the Coke inside stays coolest the longest?

2. Consider the following initial-boundary value problem

$$\frac{\partial u}{\partial t} = \frac{1}{5}\,\frac{\partial^2 u}{\partial x^2}\,, \qquad u(t,-1) = u(t,1) = 0, \qquad u(0,x) = \begin{cases} 1, & 0 < x < 1, \\ -1, & -1 < x < 0. \end{cases}$$

(*a*) Write down the analytic formula for the solution and discuss its behavior.

(*b*) Construct a numerical approximation to the solution based on (*i*) the explicit scheme; (*ii*) the implicit scheme; (*iii*) the Crank–Nicholson scheme. Use $\Delta x = .1, .05$ and $.025$ and an appropriate time step Δt.

(*c*) Compare your answers with each other and the exact solution. Are the errors in accordance with your expectations?

3. Consider the following initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{9}\,\frac{\partial^2 u}{\partial x^2}\,, \qquad u(t,-1) = u(t,1) = 0,$$

$$u(0,x) = \begin{cases} 1, & 0 < x < 1, \\ -1, & -1 < x < 0, \end{cases} \qquad \frac{\partial u}{\partial t}\,(0,x) = 0.$$

(*a*) Write down the analytic formula for the solution and discuss its behavior.

(*b*) Construct a numerical approximation to the solution based on the finite difference method discussed in class. Use $\Delta x = .1, .05$ and $.025$ and an appropriate time step Δt.

(*c*) Compare your answers with each other and the exact solution. Are the errors in accordance with your expectations?

4. A metal plate has the shape of a 0.5-inch square with a 0.1-inch square hole cut out of the middle. The plate is placed at location $0 \le x, y \le .5$ and is subjected to a constant heat source $f(x, y) = x\,y$ while keeping its inner and outer edges at 0°. Find the (approximate) equilibrium temperature by using a finite element method where the elements are right isoceles triangles of size $h = 0.1$ inch. How much does your answer change if the mesh size is decreased to 0.05 inch?

5. Prove by implicit differentiation that if $u(t, x)$ satisfies the implicit equation

$$u = f(x - t\,u),$$

then it solves the initial value problem

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = 0, \qquad u(0, x) = f(x).$$

Due: Thursday, May 12

Text: Walter A. Strauss, *Partial Differential Equations: an Introduction*, John Wiley & Sons, New York, 1992.