Math 8385
September 22, 2021

Homework Assignment # 1

Exercises:

1. Suppose the Lagrangian $L(x, u, p) = L(p)$ depends only on the derivative variable, and so we are minimizing an integral of the form $J[u] = \int_a^b L(u')\,dx$ subject to fixed boundary conditions $u(a) = \alpha$, $u(b) = \beta$. Prove that either (a) the critical functions are straight lines, or (b) every function is a critical function. What can you say about the minimization problem in the latter case?

2. Find the minimizer for the variational problem $J[u] = \int_1^2 x\sqrt{1 + u'^2}\,dx$ subject to boundary conditions $u(1) = 0,\ \ u(2) = 1$. *Hint*: You will need to solve the boundary conditions numerically.

3. Find the function $u(x)$ that minimizes the integral

$$\mathcal{I}[u] = \int_0^\pi \left[\left(\frac{du}{dx}\right)^2 - 2\,x\,u\,\frac{du}{dx} + x^2 u \right] dx$$

subject to the boundary conditions $u(0) = 1,\ \ u'(\pi) = 0$.

4. Consider the variational problem of minimizing $J[u] = \int_0^5 \left(u'^2 - u^2 \right) dx$ subject to the boundary conditions $u(0) = 0,\ \ u(5) = 0$. (a) Find the critical function for this problem. (b) Find $J[u]$ when $u(x) = c\,x\,(5 - x)$, where c is a constant. (c) Explain why the critical function you found in part (a) is not a minimizer of this variational problem. Is it a maximizer? (d) Discuss the corresponding variational problem obtained by replacing the second boundary condition by $u(1) = 0$.

Due: Friday, October 1.