Math 8385
November 17, 2021

Homework Assignment # 3

1. Find and classify (maximum, minimum, neither; weak or strong) the extremals (solutions to the Euler–Lagrange equation) for the following variational problem:

$$J[u] = \int_0^2 \frac{u}{u'^2}\, dx, \qquad u(0) = 1, \qquad u(2) = 4.$$

Hint: Use envelopes to find conjugate points.

2. Find a minimizer with one corner point for the functional

$$J[u] = \int_0^4 (1 - u'^2)^2\, dx, \qquad u(0) = 0, \qquad u(4) = 2.$$

Are there local minimizers with more than one corner? If so, which one is the global minimizer?

3. Suppose that $u(x)$ satisfies the Euler–Lagrange equation for the variational problem

$$J[u] = \int_a^b L(x, u, u')\, dx, \qquad u(a) = \alpha, \qquad u(b) = \beta,$$

with the following properties

(a) $\dfrac{\partial^2 L}{\partial p^2}(x, u(x), p) > 0$ for all $p \in \mathbb{R}$.

(b) There are no conjugate values in $(a, b]$.

Prove that this implies $u(x)$ is a strong minimizer of the variational problem.

4. Compute the Euler–Lagrange equation for the weighted norm of the gradient $\iint \| w(x,y)\, \nabla u \|^2\, dx\, dy$, where $w(x,y) > 0$ is a fixed function. What are the associated natural boundary conditions?

5. Compute the Euler–Lagrange equation for the variational problem $\iint (u_{xx} u_{yy} - u_{xy}^2)\, dx\, dy$. What can you deduce from your result?

Due: Monday, December 6.