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Recursion Operators and Hamiltonian Systems

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Abstract

This paper reviews the basic concepts in the theory of recursion operators and their applications to infinite dimensional Hamiltonian systems. Connections with the Poisson complex are explained in detail. The theory is illustrated by new results for first order hyperbolic systems, including the equations of gas dynamics.

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1. Introduction.

Generalized symmetries first make their appearance the original paper of E. Noether on the correspondence between symmetries of variational problems and conservation laws of the associated Euler-Lagrange equations. The terminology *generalized* refers to the fact that the infinitesimal generators are allowed to depend on derivatives of the dependent variables, which makes the corresponding group transformations nonlocal. Recurson operators were first introduced in [9] to provide a mechanism for generating infinite families of generalized symmetries. A fundamental advance in the subject was the work of Magri, [7], who showed how recursion operators could be constructed for systems with two compatible Hamiltonian structures. In this paper, we develop the general theory of recursion operators and biHamiltonian systems, based on the important constuct of the Poisson complex. New Hamiltonian structures and recursion operators for systems of hyperbolic conservation laws, including the equations of gas dynamics and one-dimensional elasticity are found. Many of the topics in this article are more extensively developed in [11], [12], [14] to which we refer the interested reader for a more complete exposition of the theory, applications and history.

2. Generalized Symmetries, Recursion Operators and Conservation Laws.

Consider a system of partial differential equations

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \ldots, m, \tag{2.1}$$

defined on some open subset $M^{(n)}$ of the jet space, whose coordinates $(x,u^{(n)})$ consist of the independent variables $x = (x^1,\ldots,x^p)$, the dependent variables $u = (u^1,\ldots,u^q)$, and their partial derivatives $u_J^\alpha = \partial^J u^\alpha / \partial x^J$ up to order n. A *generalized vector field* is a partial differential operator of the form

$$v_Q = \sum_{\alpha=1}^{q} Q_\alpha(x,u^{(k)}) \frac{\partial}{\partial u^\alpha}, \tag{2.2}$$

in which the *characteristic* $Q[u] = (Q_1,\ldots,Q_q)$ is a q-tuple of *differential functions*, meaning smooth functions of x, u, and derivatives of u. The vector field v_Q generates a oneparameter group of transformations on a suitable space of functions. Specifically, if $u = f(x)$ is a prescribed function, then the transformed function $\tilde{f}_\varepsilon(x) = g_\varepsilon \cdot f(x) = f(x,\varepsilon)$ is found by evaluating the solution $u = f(x,\varepsilon)$ to the Cauchy problem

$$\frac{\partial u^\alpha}{\partial \varepsilon} = Q_\alpha(x, u^{(k)}), \qquad \alpha = 1, \dots, q, \qquad u(x, \varepsilon = 0) = f(x). \tag{2.3}$$

at time ε. (We ignore complications involving the existence and uniqueness of solutions of the Cauchy problem (2.3).) The vector field $\mathbf{v}_Q$ is called an (infinitesimal) *generalized symmetry* of the system (2.1) if it takes solutions to solutions (at least formally), i.e. if $u = f(x)$ is a solution, so is $u = g_\varepsilon \cdot f(x)$. In particular, if the characteristic takes the special form

$$Q_\alpha(x, u^{(1)}) = \varphi_\alpha(x,u) - \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial u^\alpha}{\partial x^i}, \qquad \alpha = 1, \dots, q,$$

then the group corresponds to the geometrical group of transformations generated by the vector field

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi_\alpha(x,u) \frac{\partial}{\partial u^\alpha}.$$

Other generalized symmetries act non-locally on functions.

To obtain the infinitesimal invariance criterion for the system of differential equation (2.1), we prolong $\mathbf{v}_Q$ to the infinite jet space $M^{(\infty)}$, which we realize as the direct limit of the finite jet spaces $M^{(n)}$ as $n \to \infty$, leading to the partial differential operator

$$\operatorname{pr} \mathbf{v}_Q = \sum_{\alpha=1}^{q} \sum_{J} D_J Q_\alpha \frac{\partial}{\partial u_J^\alpha}.$$

Here $D_J = D_{j_1} \cdot \ldots \cdot D_{j_k}$ denotes the k^{th} order total derivative corresponding to the multi-index $J = (j_1, \dots, j_k)$.

Theorem 1. Suppose the system of partial differential equations (2.1) is totally nondegenerate in the sense of [11; Definition 2.83]. Then $\mathbf{v}_Q$ is a generalized symmetry of the system if and only if

$$\mathrm{pr}\,\mathbf{v}_Q(\Delta_\nu) = 0, \qquad \nu = 1, \ldots, m, \tag{2.4}$$

for all solutions $u = f(x)$ to the system (2.1).

The nondegeneracy condition required for the validity of the theorem is very mild, and is satisfied by well-nigh every system of partial differential equations which arises in physical applications. The symmetry conditions (2.4) constitute a large, over-determined system of elementary partial differential equations, called the *determining equations*, for the characteristic Q of $\mathbf{v}_Q$. In practice, these can be systematically solved to determine all the generalized symmetries of the system (2.1).

Define the Fréchet derivative of $\Delta[u] = (\Delta_1, \ldots, \Delta_m)$ to be the differential operator

$$\mathbf{D}_\Delta(v) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \Delta[u + \varepsilon v].$$

Note the elementary identity

$$\mathrm{pr}\,\mathbf{v}_Q(\Delta) = \mathbf{D}_\Delta(Q),$$

so that the symmetry condition (2.4) can be rewritten as

$$\mathbf{D}_\Delta(Q) = 0 \qquad \text{whenever } u \text{ is a solution to } \Delta = 0. \tag{2.5}$$

There is a natural Lie bracket operation between generalized vector fields. Specifically, if $\mathbf{v}_Q$ and $\mathbf{v}_R$ are generalized vector fields, their Lie bracket $\mathbf{v}_S = [\mathbf{v}_Q, \mathbf{v}_R]$ is the generalized vector field with characteristic

$$S = \mathrm{pr}\,\mathbf{v}_Q(R) - \mathrm{pr}\,\mathbf{v}_R(Q). \tag{2.6}$$

The reader can verify the usual binlinearity, skew-symmetry and Jacobi identity for this bracket. In particular, if $\mathbf{v}_Q$ and $\mathbf{v}_R$ are generalized symmetries of the system (2.1), so is the vector field $\mathbf{v}_S = [\mathbf{v}_Q, \mathbf{v}_R]$.

By definition, a *recursion operator* $\mathfrak{R}$ for a system of differential equations is a linear operator which maps symmetries to symmetries; in other words if $\mathbf{v}_Q$ is a generalized symmetry, and $\tilde{Q} = \mathfrak{R} \cdot Q$, then $\mathbf{v}_{\tilde{Q}}$ is also a generalized symmetry. There is a simple criterion for detemining when a given operator is a recursion operator. The proof is an elementary application of formula (2.5).

Theorem 2. A linear operator $\mathfrak{R}$ is a recursion operator for the system of differential equations $\Delta[u] = 0$ if there is a second linear operator $\tilde{\mathfrak{R}}$ such that the identity

$$\mathbf{D}_\Delta \cdot \mathfrak{R} = \tilde{\mathfrak{R}} \cdot \mathbf{D}_\Delta \tag{2.7}$$

holds on solutions to Δ.

Example 3. We shall illustrate our concepts using the elementary example of the *Riemann equation*

$$u_t = u\, u_x, \tag{2.8}$$

which is the simplest possible scalar nonlinear conservation law. Later we will see how many of the results for this very simple equation have direct counterparts in the case of two-dimensional hyperbolic systems, including the equations for polytropic gas dynamics and one-dimensional elasticity .

We begin by determining all the generalized symmetries of the Riemann equation. Since (2.5) is required to hold only on solutions to (2.8), we can always replace t-derivatives of u by equivalent expressions involving only x-derivatives. If $\Delta = u_t - uu_x$, then the Fréchet derivative operator is $\mathbf{D}_\Delta = D_t - u \cdot D_x - u_x$. Therefore, a differential function $Q[u] = Q(x,t,u,u_x,\ldots,u_k)$, where $u_k \equiv d^k u/dx^k$, is the characteristic of a generalized symmetry if and only if Q is a solution to the first order partial differential equation

$$D_t Q = u \cdot D_x Q + u_x \cdot Q$$

whenever u solves (2.8). Expanding the total derivatives, and replacing t-derivatives of u by x-derivatives, we see that Q must be a solution to the first order partial differential equation

$$\mathbf{w}(Q) \equiv Q_t + u \cdot Q_x + u_x^2 \cdot Q_{u_x} + 3u_x \cdot u_{xx} \cdot Q_{u_{xx}} + \ldots + \{D_x^k(u \cdot u_x) - u \cdot u_{k+1}\} \cdot Q_{u_k} = u_x \cdot Q, \tag{2.9}$$

subscripts on Q denoting partial derivatives. By the method of characteristics for a first order linear partial differential equation, it is easy to determine the general solution:

Theorem 4. Define the rational differential functions

$$I_0 = u, \qquad I_1 = x - t\cdot u, \qquad I_2 = \frac{u}{u_x} - x, \qquad I_3 = \frac{u_{xx}}{u_x^3}, \qquad I_{j+1} = \frac{1}{u_x} D_x I_j, \qquad j \geq 3.$$

A differential function Q is the characteristic of a generalized symmetry $\mathbf{v}_Q = Q\cdot\partial_u$ of (2.8) if and only if

$$Q = u_x \cdot G(I_0, I_1, I_2, \ldots, I_{k+1}), \tag{2.10}$$

where G is an arbitrary smooth function of its arguments.

It turns out that there are several recursion operators for the Riemann equation. Two zeroth order ones are given by

$$\mathcal{R}_1 = 2u + u_x \cdot D_x^{-1}, \qquad \mathcal{R}_2 = u^2 + u\, u_x \cdot D_x^{-1}. \tag{2.11}$$

For instance, to prove (2.7) for $\mathcal{R}_1$, we note that since $u\cdot D_x + u_x = D_x \cdot u$, the commutator

$$[D_\Delta, \mathcal{R}_1] = [D_t - u\cdot D_x - u_x, 2u + u_x \cdot D_x^{-1}]$$
$$= 2u_t + u_{xt} \cdot D_x^{-1} - 2u\cdot u_x - (u\cdot u_{xx} + u_x^2)\cdot D_x^{-1}$$

vanishes on solutions, which proves (2.7), with $\tilde{\mathcal{R}} = \mathcal{R}_1$. The verification for $\mathcal{R}_2$ is similar. Therefore, starting with the translational symmetry, with characteristic $Q_0 = u_x$, we generate a hierarchy of higher order symmetries with characteristics $Q_n = u^n \cdot u_x$; explicitly

$$\mathcal{R}_1(Q_n) = \frac{2n+3}{n+1} Q_{n+1}, \qquad \mathcal{R}_2(Q_n) = \frac{n+2}{n+1} Q_{n+2}.$$

(Interestingly, even though there are two independent recursion operators, the two hierarchies happen to coincide. However, this is special to the polynomial symmetries; on other symmetries, these recursion operators will act differently.)

The Riemann equation admits an additional first order recursion operator

$$\mathfrak{R} = D_x \cdot \frac{1}{u_x} . \tag{2.12}$$

This latter operator acts on the hierarchy Q_n according to

$$\mathfrak{R}(Q_n) = n\, Q_{n-1},$$

and so, up to multiple, "inverts" the first order recursion operator $\mathfrak{R}_1$. Again, this is special to the polynomial hierarchy. For instance, starting with the rational second order generalized symmetry with characteristic $\hat{Q}_2 = u_x \cdot I_3 = u_x^{-2} \cdot u_{xx}$, the recursion operator $\mathfrak{R}$ generates the additional hierarchy of higher order symmetries $\hat{Q}_k = u_x \cdot I_{k+1}$, $k = 2, 3, \dots$, whereas $\mathfrak{R}_1$ and $\mathfrak{R}_2$ lead to yet other second order symmetries.

Given a system of partial differential equations (2.1), a *conservation law* is a p-tuple of differential functions $P[u] = (P_1, \dots, P_p)$ whose divergence

$$\text{Div}\, P = \sum_{i=1}^{p} D_i P_i = 0,$$

vanishes on all solutions to (2.1). For dynamic problems, the conservation law takes the form

$$D_t T + \text{Div}\, X = 0.$$

(Div here refers to the spatial variables.) The t-component of such a conservation law is referred to as the conserved density, and, for suitable solutions, (in particular those for which the flux X vanishes on the boundary) the integral $\int T[u]\, dx$ provides a constant of the motion. A conserved density is called *trivial* if it is a (spatial) divergence $T = \text{Div}\, Y$ on solutions. In the Lagrangian framework, Noether's Theorem, cf. [11; Theorem 5.42], provides a complete correspondence between generalized symmetries of a variational problem and conservation laws of the associated Euler-Lagrange equations. In the next section, we shall see how this extends to the Hamiltonian framework.

Example 5. For the Riemann equation (2.8), any conserved density T, which, without loss of generality, we can take to depend only on $x, t, u, \dots, u_n$, must satisfy

$$D_t T + D_x X = 0$$

on solutions to the equation for some flux X. Writing this out, and replacing t-derivatives by the corresponding expressions in terms of x-derivatives, we find

$$u \cdot D_x T + \mathbf{w}(T) + D_x X = 0,$$

where $\mathbf{w}$ is the vector field given in (2.9). Let $X = Y - u \cdot T$, so this becomes

$$\mathbf{w}(T) - u_x \cdot T + D_x Y = 0.$$

If we rewrite $T = u_x \cdot F(t, I_0, I_1, I_2, \ldots, I_{k+1})$, in terms of the invariants I_j of $\mathbf{w}$, and the single parametric variable t, then $\mathbf{w}(T) - u_x \cdot T = u_x \cdot G_t$, hence the functions

$$u_x \cdot F(t, I_0, I_1, I_2, \ldots, I_{k+1}) - u_x \cdot F(0, I_0, I_1, I_2, \ldots, I_{k+1}) = D_x \{ \int_0^t Y \, ds \}$$

differ by a trivial conserved density. Setting $t = 0$ in the formula for T, we conclude that it is equivalent to a conserved density of the form

$$T = u_x \cdot G(I_0, I_1, I_2, \ldots, I_k).$$

Thus, surprisingly, for the Riemann equation the expressions for symmetries and conserved densities are the same! In particular, we note the infinite sequence of zero$^{\text{th}}$ order conserved densities

$$H_n(u) = u^n, \qquad n = 1,2,3, \ldots . \tag{2.13}$$

3. The Poisson Complex and Hamiltonian Systems.

We now present an approach to the theory of Hamiltonian systems based on the important Poisson complex, which plays as fundamental a role here as the deRham complex does in the theory of differential forms. The Poisson complex, though, involves the dual objects to differential forms, which are known as multi–vectors, or, in the infinite–dimensional case, functional multi–vectors. This complex, in the finite–dimensional case, is due to Lichnerowicz, [6], and was generalized to infinite dimensions in Olver, [10]. We begin by recalling the basic definitions; see [11] for many of the details.

On the infinite jet space, $M^{(\infty)}$, the space $\bigwedge_*^0$ of *functionals* is defined as the cokernel of the total divergence operator, so that two differential functions $L[u]$ and $\tilde{L}[u]$ define the same functional $\mathcal{L}[u] = \int L[u]\, dx$ if and only if they differ by a total divergence: $L = \tilde{L} + \operatorname{Div} P$. More generally, define a *vertical k-form* to be a finite sum

$$\hat{\omega} = \sum P_J^{\alpha}[u]\, du_{J_1}^{\alpha_1} \wedge \ldots \wedge du_{J_k}^{\alpha_k},$$

where the coefficients P_J^{α} are arbitrary differential functions. The total derivatives D_i act as Lie derivatives on the vertical forms, and the space Λ_*^k of *functional k-forms* is analogously defined as the cokernel of the total divergence. In particular, an easy integration by parts argument shows that any functional one-form is uniquely equivalent to one in the form

$$\omega = \int \Big\{ \sum_{\alpha=1}^{q} P_{\alpha}[u]\, du^{\alpha} \Big\}\, dx = \int \{P \cdot du\}\, dx. \tag{3.1}$$

Similarly, it can be shown that any functional 2-form can be placed into canonical form

$$\Omega = \frac{1}{2} \int \Big\{ \sum_{\alpha,\beta} du^{\alpha} \wedge \mathcal{D}_{\alpha\beta} \cdot du^{\beta} \Big\}\, dx = \frac{1}{2} \int \{du \wedge \mathcal{D} \cdot du\}\, dx, \tag{3.2}$$

uniquely determined by the skew-adjoint matrix differential operator $\mathcal{D} = (\mathcal{D}_{\alpha\beta})$.

The *vertical differential* $\hat{d}$ takes a vertical k-form to a vertical (k+1)-form, and is induced by its action

$$\hat{d} P = \sum_{\alpha, J} \frac{\partial P}{\partial u_J^{\alpha}}\, du_J^{\alpha}$$

on differential functions. It can be shown that $\hat{d}$ commutes with each total derivative D_i, and hence induces a well-defined map

$$\delta : \Lambda_*^k \longrightarrow \Lambda_*^{k+1}$$

on the spaces of functional forms, called the *variational differential.* An easy argument based on the finite-dimensional Poincaré lemma shows that the *variational complex*

$$0 \longrightarrow \Lambda_*^0 \xrightarrow{\ \delta\ } \Lambda_*^1 \xrightarrow{\ \delta\ } \Lambda_*^2 \xrightarrow{\ \delta\ } \Lambda_*^3 \xrightarrow{\ \delta\ } \ldots .$$

is locally exact, meaning that, over suitable (star-shaped) subdomains $\delta\omega = 0$ if and only if $\omega = \delta\zeta$ for some functional form ζ. In particular, if $\mathcal{L}[u] = \int L[u]\, dx$ is a functional, its variational differential is the one-form

$$\delta\mathcal{L} = \int \{E(L) \cdot du\}\, dx \qquad (3.3)$$

determined by the Euler-Lagrange expression $E(L)$ or variational derivative of $\mathcal{L}$. The exactness of the varational complex at the Λ_*^1 - stage leads to the well-known Helmholtz conditions for a differential equation to be the Euler-Lagrange equation for some variational problem, [11; Theorem 5.68], namely $\Delta = E(L)$ for some Lagrangian L if and only if its Fréchet derivative is self-adjoint: $\mathbf{D}_\Delta = \mathbf{D}_\Delta^*$.

Our main interest here is not in the functional forms, but rather in the dual objects - the functional multi-vectors. By definition, a *functional k-vector* is an alternating, k-linear map from the space Λ_*^1 of functional one-forms to the space Λ_*^0 of functionals. It can be shown that each functional k-vector can be written in the form

$$\Theta = \int \Big\{ \sum R_J^\alpha[u]\, \theta_{J_1}^{\alpha_1} \wedge \dots \wedge \theta_{J_k}^{\alpha_k} \Big\}\, dx,$$

where the θ_J^α form a basis for the *vertical vectors*, dual to the basis du_J^α of vertical forms. We find

$$\langle \Theta; \omega_1 \wedge \dots \wedge \omega_k \rangle = \int \Big\{ \sum R_J^\alpha[u] \cdot \det(D_{J_i} P_{\alpha_i}^j) \Big\}\, dx,$$

where $\omega_j = \int \{ \sum P_\alpha^j[u]\, du^\alpha \}\, dx$ are functional one-forms written in canonical form (3.1). The total derivatives act as Lie derivatives on vertical multi-vectors, and so the space Λ_k^* of functional k-vectors is again the cokernel of the total divergence operator. In particular, integation by parts can be used to place any functional uni-vector (i.e. $k = 1$) in the canonical form

$$v_Q = \int \Big\{ \sum_{\alpha=1}^{q} Q_\alpha[u]\, \theta^\alpha \Big\}\, dx = \int \{Q \cdot \theta\}\, dx, \qquad (3.4)$$

and the space Λ_1^* can be identified with the space of generalized vector fields. Similarly, any functional bi-vector, i.e. element of Λ_2^*, can be placed in canonical form

$$\Theta = \Theta_{\mathcal{D}} = \frac{1}{2}\int \{\theta \wedge \mathcal{D}\theta\}\, dx, \tag{3.5}$$

determined by a unique $q \times q$ skew-adjoint matrix differential operator $\mathcal{D}$.

Warning: The space Λ_k^* of functional multi-vectors is *not* the dual space to the space Λ_*^k of functional k-forms. This is because the wedge product of two functional forms is not a well-defined functional form!

If $\mathbf{v}_Q$ is a generalized vector field or uni-vector, we can define its prolongation to act as a Lie derivative on the space of functional multi-vectors. The key formula is

$$\mathrm{pr}\, \mathbf{v}_Q(\theta) = \mathbf{D}_Q^* \cdot \theta,$$

where $\mathbf{D}_Q^*$ is the adjoint of the Fréchet derivative of Q, and θ denotes the column vector of basis uni-vectors θ^α, $\alpha = 1,\ldots,q$. (See [10], [12] for a justification of this formula.) The prolongation $\mathrm{pr}\, \mathbf{v}_Q$ acts on differential functions as before, and the action is extended to the entire space by the usual rules of derivation and commutation with the total derivatives. In particular, as the reader can check, this definition of the Lie derivative recovers the correct form of the Lie bracket (2.6) between generalized vector fields.

If Θ is a functional k-vector, and $\omega_I \equiv \omega_{i_1} \wedge \ldots \wedge \omega_{i_m}$, $m \le k$, a wedge product of functional one-forms, we define the *interior product* to be the functional $(k-m)$-vector $\omega_I \lrcorner \Theta$ determined by the formula

$$\langle \omega_I \lrcorner \Theta; \eta_1 \wedge \ldots \wedge \eta_{k-m} \rangle = \langle \Theta; \omega_I \wedge \eta_1 \wedge \ldots \wedge \eta_{k-m} \rangle . \tag{3.6}$$

In particular, if $m = k-1$, then $\omega_I \lrcorner \Theta \in \Lambda_1^*$, and hence can be viewed as an generalized vector field, as in (3.4).

The most important operation on functional multi-vectors is the Schouten bracket, which generalizes the Lie bracket between vector fields. If Φ is a k–vector and Ψ an ℓ–vector, then their Schouten bracket $[\Phi, \Psi]$, is a $(k+\ell-1)$–vector. It is uniquely defined by the following formula:

$$\langle [\Phi,\Psi];\, \delta\mathcal{L}_1 \wedge \ldots \wedge \delta\mathcal{L}_{k+\ell-1} \rangle =$$

$$\frac{(-1)^{k\cdot\ell+\ell}}{\ell} \sum_I (\mathrm{sgn}\, I)\, \langle \Phi;\, \{\delta\mathcal{L}_I \lrcorner \Psi\}\delta\mathcal{L}_{I'} \rangle + \frac{(-1)^k}{k} \sum_J (\mathrm{sgn}\, J)\, \langle \Psi;\, \{\delta\mathcal{L}_J \lrcorner \Phi\}\delta\mathcal{L}_{J'} \rangle. \tag{3.7}$$

which must hold for every set of functionals $\mathcal{L}_1, \ldots, \mathcal{L}_{k+\ell-1}$, with variational derivatives $\delta\mathcal{L}_i$ given by (3.3). In (3.7), the first sum is over all multi-indices $I = (i_1, \ldots, i_{\ell-1})$, $1 \le i_1 < \ldots < i_{\ell-1} \le k+\ell-1$, with $I' = (i'_1, \ldots, i'_k)$ being the complementary multi-index, so $I \cup I' = \pi(1, \ldots, k+\ell-1)$ for some permutation π, and $\mathrm{sgn}\, I$ denoting the sign of the permutation π. Similarly, the second sum is over all multi-indices $J = (j_1, \ldots, j_{k-1})$, $1 \le j_1 < \ldots < j_{k-1} \le k+\ell-1$, with J' and $\mathrm{sgn}\, J$ being defined analogously. Note also that, according to the remark in the previous paragraph, the terms $\delta\mathcal{L}_I \lrcorner \Psi$ and $\delta\mathcal{L}_J \lrcorner \Phi$ are in $\bigwedge_1^*$, and hence determine generalized vector fields, which act on the remaining wedge products $\delta\mathcal{L}_{I'}$ and $\delta\mathcal{L}_{J'}$ as Lie derivatives. (This definition, first proposed in [10], has the advantage of being the only one I know of which works equally well in both finite and infinite dimensions.)

The Schouten bracket satisfies the following properties. Let $\Phi \in \bigwedge_k^*$, $\Psi \in \bigwedge_\ell^*$, $\Theta \in \bigwedge_m^*$ be functional multi-vectors.

i) *Bilinearity*: $[\Phi, \Psi]$ is an $\mathbb{R}$–bilinear function of Φ and Ψ.

ii) *Super–symmetry*: $$[\Phi, \Psi] = (-1)^{k\cdot\ell}\, [\Psi, \Phi].$$

iii) *Super–Jacobi Identity*:

$$(-1)^{k\cdot m}\, [[\Phi,\Psi],\Theta] + (-1)^{\ell\cdot m}\, [[\Theta, \Phi],\Psi] + (-1)^{k\cdot\ell}\, [[\Psi,\Theta], \Phi] = 0.$$

iv) *Lie Derivative*: If $\mathbf{v}_Q$ is a generalized vector field or functional uni-vector, then the Schouten bracket $[\mathbf{v}_Q, \Phi]$ is also a functional k–vector, and coincides with the Lie derivative of Φ with respect to $\mathrm{pr}\, \mathbf{v}_Q$. In particular, the Schouten bracket of two generalized vector fields is the same as their Lie bracket (2.6).

Each functional bi–vector $\Theta_{\mathcal{D}}$ determines an alternating, bilinear map on the space of one–forms, and hence a bilinear, skew–symmetric "bracket" on the space of real–valued function(al)s:

$$\{\mathcal{F}, \mathcal{H}\} \equiv \langle \Theta_{\mathcal{D}}; \delta\mathcal{F}, \delta\mathcal{H} \rangle.$$

Explicitly, using (3.5), we see that this bracket is given by the standard formula

$$\{\mathcal{F}, \mathcal{H}\} = \int E(F) \cdot \mathcal{D} \cdot E(H) \; dx, \tag{3.8}$$

where F and H are the integrands of the functionals $\mathcal{F}$, $\mathcal{H}$. The bracket automatically satisfies the Leibniz rule, and hence to be a genuine Poisson bracket must only satisfy the additional restriction imposed by the Jacobi identity. This can be easily expressed in invariant form using the tri–vector $[\Theta_{\mathcal{D}}, \Theta_{\mathcal{D}}]$ obtained by bracketing $\Theta_{\mathcal{D}}$ with itself:

$$\{\{F, H\}, P\} + \{\{P, F\}, H\} + \{\{H, P\}, F\} = \frac{2}{3} \langle [\Theta_{\mathcal{D}}, \Theta_{\mathcal{D}}]; dF, dH, dP \rangle.$$

Therefore a functional bi-vector $\Theta_{\mathcal{D}}$ determines a Poisson bracket if and only if it satisfies the extra condition

$$[\Theta_{\mathcal{D}}, \Theta_{\mathcal{D}}] = 0. \tag{3.9}$$

This condition is a nonlinear condition on the underlying differential operator $\mathcal{D}$. Any functional bi-vector satisfying (3.9) is called a Hamiltonian bi-vector; similarly, any skew-adjoint differential operator coming from a Hamiltonian bi-vector is called a *Hamiltonian operator*.

Given a Hamiltonian bi-vector, let $\vartheta = \vartheta_\Theta$ be the map taking functional k–vectors to functional (k+1)–vectors defined by bracketing with the Poisson bivector Θ:

$$\vartheta(\Psi) = [\Theta, \Psi]. \tag{3.10}$$

The determining property (3.9) along with the super Jacobi identity for the Schouten bracket immediately implies that

$$\vartheta(\vartheta(\Psi)) = [\Theta, [\Theta, \Psi]] = 0$$

for any multi–vector Ψ. Therefore the maps ϑ determine a complex, called the *Poisson complex* corresponding to the Poisson bivector Θ:

$$0 \longrightarrow \Lambda_0^* \xrightarrow{\vartheta} \Lambda_1^* \xrightarrow{\vartheta} \Lambda_2^* \xrightarrow{\vartheta} \Lambda_3^* \xrightarrow{\vartheta} \cdots .$$

The composition of two successive maps is always trivial: $\vartheta \circ \vartheta = 0$.

The first stage of the Poisson complex, $\vartheta : \Lambda_0^* \to \Lambda_1^*$, maps functionals to generalized vector fields. Specifically, if $\mathcal{H}[u] = \int H[u]\, dx$ is a (Hamiltonian) functional, the corresponding generalized vector field

$$\hat{v}_{\mathcal{H}} = \vartheta(\mathcal{H}) = [\Theta, \mathcal{H}] \tag{3.11}$$

is called the *Hamiltonian vector field* determined by $\mathcal{H}$. Explicitly, it is readily seen that $\hat{v}_{\mathcal{H}}$ has characteristic $Q = \mathcal{D}\cdot E(H)$, where $\mathcal{D}$ is the Hamiltonian operator determined by Θ. The corresponding Hamiltonian flow, cf. (2.3), is governed by the Hamiltonian system of evolution equations

$$u_t = \mathcal{D}\cdot E(H). \tag{3.12}$$

We note the standard formula

$$\{\mathcal{H}, \mathcal{F}\} = -\operatorname{pr} \hat{v}_{\mathcal{H}}(\mathcal{F}) = \operatorname{pr} \hat{v}_{\mathcal{F}}(\mathcal{H}) \tag{3.13}$$

for any pair of functionals $\mathcal{H}, \mathcal{F}$, which proves that a functional $\mathcal{F}$ determines a conserved density for the Hamiltonian system (3.12) if and only if $\{\mathcal{H}, \mathcal{F}\} = 0$. Therefore, every conserved density $\mathcal{F}$ determines a generalized (Hamiltonian) symmetry $\hat{v}_{\mathcal{F}}$ of the Hamiltonian system (3.12).

Conversely, if an generalized vector field v_Q is a Hamiltonian vector field, so $Q = \mathcal{D}\cdot E(H)$ for some differential function H, then closure of the Poisson complex at the Λ_1^*–stage implies that

$$\vartheta(v_Q) = [\Theta_{\mathcal{D}}, v_Q] = \operatorname{pr} v_Q(\Theta_{\mathcal{D}}) = 0. \tag{3.14}$$

If the Poisson complex is *exact* at the Λ_1^*–stage, then (3.14) is both necessary and sufficient for v_Q to be a Hamiltonian vector field. In this case, (3.14) provides a simple and readily verifiable condition that will tell whether or not a given vector field is Hamiltonian with respect to the given Poisson bracket. Writing out (3.14) explicitly leads to the following characterization of Hamiltonian vector fields, [12]; see also [5] for a similar result.