On the Algebra of Differential Invariants of a Lie Pseudo-Group

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Abstract. In this paper we prove some basic theoretical results underlying the moving frame theory of pseudo-groups developed in the first two papers in this series. The first result demonstrates that a pseudo-group that acts locally freely at some sufficiently high order acts locally freely at all subsequent orders, and thus can be completely analyzed with the moving frame method. The second result is an algorithmic version of the Tresse–Kumpera theorem that states that a locally freely acting pseudo-group admits a finite generating system of differential invariants. The results will be based on moving frame methods and Gröbner basis algorithms.

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The goal of this paper is to prove some of the basic theoretical results underlying the general theory and practical algorithms that were presented in the first two papers, [23, 24]. Specifically, we demonstrate that a pseudo-group that acts locally freely on a jet space of sufficiently high order acts locally freely on all higher order jet spaces. Freeness quantifies the growth conditions on the dimensions of the jet subgroupoids generated by the pseudo-group. Freeness is a key requirement for the existence of a moving frame map, and thereby underlies the consequent construction of differential invariants and invariant differential forms.

The second main goal is to prove a constructive version of the Tresse-Kumpera theorem, [31, 17], that states that, under suitable hypotheses, all higher order differential invariants of a pseudo-group action are generated by application of the operators of invariant differentiation to a finite system of generating differential invariants. Our proof relies on local freeness of the pseudo-group action. Furthermore, with the finiteness theorem in hand, the recurrence formulae relating the differentiated invariants and, also, their syzygies are constructed from the moving frame normalizations using the methods introduced in [12, 16]. A key innovation is that our algorithm for finding the generating system of invariants is completely constructive, as it relies entirely on Gröbner basis methods. We shall not require the more sophisticated algebraic involutivity conditions based on involutive divisions, [5, 25], or on differential Gröbner bases, [6, 19, 20], here.

We refer to Seiler's forthcoming book [29, 30] for a full discussion of involutivity and its connections with Gröbner basis theory. See also [13] for the more detailed theory of involutive bases of polynomial ideals.

Our results will apply to pseudo-group actions on smooth m -dimensional manifolds. However, as all of the constructions are local, throughout this paper we shall restrict our attention to the case when the manifold $M \subset \mathbb{R}^m$ is an open subset of Euclidean space with a fixed system of local coordinates $z = (z^1, \dots, z^m)$.

Certain key applications, including gauge groups, [15], are not free, but do satisfy a more general condition that we call “projectively free”, that serve the same purpose. While (projective) freeness is not necessary for the existence of differential invariants, it is required for a complete system of invariant differential forms and a moving frame, and so we will defer consider any extensions of this theory to non-free actions.

Our key innovation is to apply methods from modern constructive algebra, particularly Gröbner basis techniques, to analyze the *symbol module* generated by the symbols of all the equations in the linear determining system for a given Lie pseudo-group. All of the objects are parametrized by the coordinates $z \in M$, so the Gröbner basis theory needs to be carefully developed so as to maintain linearity of the polynomials in R and smoothness (analyticity) in z . The theory of comprehensive Gröbner bases developed by Weispfenning, [32], covers the parametric situation.

It is worth noting that the modern theory of Gröbner bases has its foundations in early research on the integrability of systems of partial differential equations. Intimations can be found in [11, 27, 28]; this culminates in the as yet obscure work of Gjunter, cf. [14], which summarizes his earlier papers from 1910–1913, in which he anticipates Gröbner basis

methods and the Buchberger algorithm, [4]. This work was rediscovered by Renschuch and his students and collaborators, and an English translation of the summary of Gjunter's results recently appeared, [26]. In our applications, we shall use the basic terminology in [1; §3.4–5] and [10].

The goal of this paper is to prove a constructive version of the Tresse Basis Theorem — that, just as in the finite-dimensional Lie group version, the differential invariant algebra is generated, through invariant differentiation, by a finite number of independent differential invariants $I_1, \dots, I_\ell$. This will be a consequence of the recurrence relations that allow us to express differentiated invariants in terms of the normalized differential invariants (7.1). Since we already know the latter form a complete system of differential invariants, we need only prove that they can be expressed as functions of the differentiated invariants.

Remark: by “differential invariant” we mean a local invariant for the prolonged pseudo-group action on submanifolds. global invariance under all pseudo-group transformations is more delicate, and cannot be studied by the infinitesimal methods used throughout this paper.

Let $\mathcal{I}(\mathcal{G})$ denote the function algebra of differential invariants of the pseudo-group $\mathcal{G}$. Tresse's Theorem is a version of the Hilbert Basis theorem for algebras differential invariants; a more modern version based on the Spencer machinery can be found in [17].

Bear in mind that we are working in the function space category, and so classify differential invariants up to functional independence — *not* in the polynomial category. Our primary goal is to give a constructive proof of the Tresse Theorem.

Tresse

Theorem 1.1. *Suppose $\mathcal{G}$ acts locally freely on an open subset some finite order jet bundle $J^n(M, p)$. Then there are a finite number of differential invariants $I_1, \dots, I_\ell$ and p invariant differential operators $\mathcal{D}_1, \dots, \mathcal{D}_p$ that generate its algebra of differential invariants $\mathcal{I}(\mathcal{G})$, meaning that every differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives: $\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_\kappa$*

At its core, our theory is based on rather simple ideas combining linear algebra and polynomial algebra. However, exposing this simple core is not an easy task, and the intervening constructions can appear much more complicated than necessary.

ap 2. Algebraic Preliminaries.

Let $\mathbb{R}[r, R]$ denote the algebra of real polynomials in the variables $r = (r_1, \dots, r_m)$, $R = (R_1, \dots, R_m)$. The subspace

$$\mathcal{R} = \left\{ P(r, R) = \sum_{a=1}^m P_a(r) R_a \right\} \simeq \mathbb{R}[r] \otimes \mathbb{R}^m \subset \mathbb{R}[r, R] \quad (2.1)$$

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of homogeneous linear polynomials in the R 's forms a free module over the polynomial algebra $\mathbb{R}[r]$, isomorphic to the module of vector-valued polynomials $\mathbf{P}: \mathbb{R}^m \rightarrow \mathbb{R}^m$. We grade $\mathcal{R} = \bigoplus_{n \geq 0} \mathcal{R}^n$, where $\mathcal{R}^n$ consists of the homogeneous polynomials of *degree* n in r , and set $\mathcal{R}^{\leq n} = \bigoplus_{k=0}^n \mathcal{R}^k$ to be the space of polynomials of degree $\leq n$. We shall also use the notation $\mathcal{R}^{\geq n} = \bigoplus_{k=n}^{\infty} \mathcal{R}^k$. In order to unambiguously specify Gröbner bases of

submodules, we shall fix a convenient graded monomial ordering, $[\mathbf{1}, \mathbf{10}]$, on the module $\mathcal{R}$ from the outset.

Given a subspace $\mathcal{S} \subset \mathcal{R}$, we set $\mathcal{S}^n = \mathcal{S} \cap \mathcal{R}^n$, $\mathcal{S}^{\leq n} = \mathcal{S} \cap \mathcal{R}^{\leq n}$ and $\mathcal{S}^{\geq n} = \mathcal{S} \cap \mathcal{R}^{\geq n}$. The subspace is *graded* if $\mathcal{S} = \bigoplus_{n \geq 0} \mathcal{S}^n$ is the sum of its homogeneous constituents. A subspace $\mathcal{S} \subset \mathcal{R}$ is called a *submodule* if the product $Q(r)P(r, R) \in \mathcal{S}$ whenever $P(r, R) \in \mathcal{S}$ and $Q(r) \in \mathbb{R}[r]$. In the one-dimensional case, $\mathcal{R} \simeq \mathbb{R}[r]$, and a submodule is the same as an *ideal*. In this paper, all submodules (but not all subspaces) are graded. A subspace $\mathcal{S} \subset \mathcal{R}$ spanned by monomials

$$r^A R_b = r_{a_1} \cdots r_{a_n} R_b, \quad \text{where} \quad 1 \leq a_1, \dots, a_n, b \leq m,$$

is called a *monomial subspace*; all monomial subspaces are graded. In particular, a *monomial submodule* is a submodule that is spanned by monomials.

A polynomial $P \in \mathcal{R}^{\leq n}$ has *degree* $n = \deg P$ and *leading term* $0 \neq P^* \in \mathcal{R}^n$ provided $P = P^* + Q$ where $Q \in \mathcal{R}^{\leq n-1}$ is of lower degree. By convention, only the zero polynomial has zero leading term. It is important to note that the map $\lambda: P \rightarrow P^*$ that produces the leading term of a polynomial is not, in general, linear; indeed, $\lambda(P + Q) = \lambda(P)$ if $\deg P > \deg Q$, while if $\deg P = \deg Q$ then $\lambda(P + Q) = \lambda(P) + \lambda(Q)$ if and only if either $\lambda(P) + \lambda(Q) \neq 0$ or $P + Q = 0$.

sy 3. The Symbol Module.

Many of the properties of systems of partial differential equations, both linear and nonlinear, are dependent upon the structure of their symbols. In local coordinates, the symbols of an involutive system of differential equations generate a submodule of the space of vector-valued polynomials, and hence can be analyzed by modern computational algebra — in particular the method of Gröbner bases, $[\mathbf{1}, \mathbf{10}]$. The symbol polynomials are intrinsically realized as elements of the dual space to the jets of solutions, and we will show how to exploit this duality in a fundamental way.

We begin with a review of the construction of the symbol module for linear systems of partial differential equations involving m independent variables and m dependent variables. Although the constructions work for arbitrary numbers of independent and dependent variables, in our applications the system in question consists of the determining equations for a Lie pseudo-group on an m -dimensional manifold M , and so we shall only need the particular case when the number of independent variables coincides with the number of dependent variables.

Let $M \subset \mathbb{R}^m$ be an open subset, or, more generally a local coordinate chart on an m -dimensional manifold, with coordinates $z = (z^1, \dots, z^m)$. Let $\mathcal{X}(M)$ denote the space of smooth locally defined vector fields

$$\mathbf{v} = \sum_{a=1}^m \zeta^a(z) \frac{\partial}{\partial z^a}, \quad (3.1)$$

i.e., smooth local sections of the tangent bundle TM . For $0 \leq n \leq \infty$, let $J^n TM$ denote

the associated n^{th} order jet bundle, whose fiber coordinates $j_n \mathbf{v} = (\dots \zeta_A^b \dots)$ where

$$\text{zcfbA} \quad \zeta_A^b = \frac{\partial^{\#A} \zeta^b}{\partial z^A} = \frac{\partial^k \zeta^b}{\partial z^{a_1} \dots \partial z^{a_k}}, \quad 1 \leq a_\nu \leq m, \quad 0 \leq k = \#A \leq n, \quad (3.2)$$

represent partial derivatives of the vector field coefficients with respect to the base coordinates on M .

We will identify the dual bundle[†] $(J^\infty TM)^* \simeq M \times \mathcal{R}$ by use of the pairing

$$\langle j_\infty \mathbf{v}|_z; r^A R_b \rangle = \zeta_A^b.$$

A section of $(J^\infty TM)^*$ is thereby identified with a smoothly parametrized polynomial

$$\text{pdpol} \quad P(z; r, R) = \sum_{b=1}^m \sum_{\#A \leq n} h_A^b(z) r^A R_b, \quad (3.3)$$

of some, necessarily finite, degree $n < \infty$, and represents the linear differential polynomial

$$\text{ldpol} \quad L(z, \zeta^{(n)}) = \langle j_n \mathbf{v}|_z; P(z; r, R) \rangle = \sum_{b=1}^m \sum_{\#A \leq n} h_A^b(z) \zeta_A^b \quad (3.4)$$

on the space of vector fields $\mathcal{X}(M)$.

symbol **Definition 3.1.** The *symbol* of the linear differential polynomial (3.4) is the leading term of its defining polynomial (3.3):

$$\text{symbL} \quad \sigma[L(z, \zeta^{(n)})] = \lambda[p(z; r, R)] = \sum_{b=1}^m \sum_{\#A=n} h_A^b(z) r^A R_b. \quad (3.5)$$

Let $\mathcal{G}$ be a regular, tame Lie pseudo-group acting smoothly on M . (See [23] for complete details on the basic set-up.) For $0 \leq n \leq \infty$, let $\mathcal{G}^{(n)} \subset J^n(M, M)$ denote the subbundle (or, more accurately, groupoid) consisting of all n -jets of pseudo-group diffeomorphisms. Let $\mathfrak{g} \subset \mathcal{X}(M)$ be the space[‡] spanned by its infinitesimal generators. Let $J^n \mathfrak{g} \subset J^n TM$ denote the subbundle (or, more accurately, Lie algebroid) spanned by their n^{th} order jets. In view of our regularity assumption, $J^\infty \mathfrak{g}$ is prescribed by the *linear determining system*

$$\text{ldn3} \quad L^{(\infty)}(z, \zeta^{(\infty)}) = L^{(\infty)}(\dots z^a \dots \zeta_A^b \dots) = 0, \quad (3.6)$$

which, by assumption, forms an involutive, locally solvable system of homogeneous linear partial differential equations, [23, 30]. Each equation in the determining system (3.6) is

[†] *Warning:* $(J^\infty TM)^* \neq J^\infty T^*M$ is *not* the same as the infinite jet bundle of the cotangent bundle!

[‡] $\mathfrak{g}$ would be a Lie algebra if all the infinitesimal generators $\mathbf{v} \in \mathfrak{g}$ were globally defined vector fields, but this need not be the case. It does have the structure of a “local Lie algebra”.

represented by a parametrized polynomial (3.3), which cumulatively carve out the *annihilator subbundle*

$$\text{LV} \quad \mathcal{L} = (J^\infty \mathfrak{g})^\perp \subset (J^\infty TM)^*. \quad (3.7)$$

Involutivity requires that the linear determining system be closed under application of the total differentiation operators

$$\text{DDza} \quad \mathbb{D}_{z^a} = \frac{\partial}{\partial z^a} + \sum_{c=1}^m \sum_{\#B \geq 0} \zeta_{B,c}^a \frac{\partial}{\partial \zeta_B^a}, \quad a = 1, \dots, m. \quad (3.8)$$

In applications to symmetry groups of differential equations, (3.6) represents the involutive completion, under total differentiation, of the usual determining equations obtained by the usual Lie algorithm, [21].

Let

$$\text{SI} \quad \mathcal{S} = \lambda(\mathcal{L}) \subset (J^\infty TM)^* \quad (3.9)$$

denote the *symbol subbundle* for the linear determining system (3.6). Total differentiation corresponds to multiplication of the symbol polynomials:

$$\text{Dsymb} \quad \sigma(\mathbb{D}_{z^a} L) = r_a \sigma(L), \quad a = 1, \dots, m. \quad (3.10)$$

Thus, involutivity implies that, at each point $z \in M$, the fiber $\mathcal{S}|_z \subset \mathcal{R}$ forms a graded submodule, known as the *symbol module* of the pseudo-group at the point z . (On the other hand, unless we are dealing with constant coefficient differential equations, the annihilator $\mathcal{L}|_z \subset \mathcal{R}$ is *not* a submodule.)

Let $\gamma_1(z; r, R), \dots, \gamma_k(z; r, R)$ be the reduced comprehensive[†] Gröbner basis, [2, 32], for the parametrized symbol modules $\mathcal{S}|_z$ under the prescribed graded monomial ordering. The constant rank conditions imply that, on a possibly smaller open subset, each symbol polynomial $S(z; r, R) \in \mathcal{S}|_z$ can be written as a smoothly varying linear combination

$$\text{Gbe} \quad S(z; r, R) = \sum_{\nu=1}^k P_\nu(z; r) \gamma_\nu(z; r, R) \quad (3.11)$$

of the Gröbner basis elements.

In general, we restrict our attention to the open submanifold of M where the comprehensive Gröbner basis is “generic”, and shall usually drop the adjectives “reduced” and “comprehensive” in the ensuing discussion. For specificity, we choose particular representatives in the determining system $\mathcal{L}$ to represent the Gröbner basis symbols:

$$\text{Gde} \quad L_\nu \in \mathcal{L} \quad \text{such that} \quad \sigma(L_\nu) = \gamma_\nu, \quad \text{for} \quad \nu = 1, \dots, k, \quad (3.12)$$

Let us prove that the Gröbner basis system (3.12) generates the entire determining system through prolongation. As a result, a vector field $\mathbf{v} \in \mathfrak{g}$ is a pseudo-group generator if and only if its coefficients solve the system of partial differential equations prescribed by (3.12).

[†] Becker and Weispfenning restrict to polynomially dependent modules, but the extension to the smoothly parametrized category is straightforward.

Lemma 3.2. *A differential polynomial L belongs to the linear determining system $\mathcal{L}$ if and only if $L = \sum_{\nu} \mathcal{D}_{\nu} L_{\nu}$ for some total differential operators $\mathcal{D}_1, \dots, \mathcal{D}_k$.*

Proof: Indeed, the specification of a Gröbner basis provides an computational algorithm for checking when a given linear differential polynomial $L(z, \zeta^{(n)})$ belongs to the linearized determining system. We first determine, by the standard Gröbner basis test [10], whether its symbol $\sigma(L)$ belongs to the symbol module. If not, the involutivity hypothesis implies that the differential polynomial does not belong to the determining system. Suppose, on the other hand, that

$$\sigma(L) = \sum_{\nu} P_{\nu}(z; r) \gamma_{\nu}(z; r, R) \in \mathcal{S}.$$

Since the symbols of the terms on the right hand side sum up to zero, the linear differential polynomial

$$\tilde{L}(z, \zeta^{(n)}) = L(z, \zeta^{(n)}) - \sum_{\nu=1}^k P_{\nu}(z; \mathbb{D}) L_{\nu}(z, \zeta^{(n)})$$

is of lower order than L . Repeating the process, we eventually express

$$L = \sum_{\nu=1}^k \mathcal{D}_{\nu} L_{\nu} + K,$$

where the $\mathcal{D}_{\nu}$ are certain linear differential operators. Moreover, either $\sigma(K) \notin \mathcal{S}$, which means $L \notin \mathcal{L}$ does not belong to the determining system, or $K = 0$ and so $L \in \mathcal{L}$. We call $\sigma(K)$ the *reduced symbol* of the differential polynomial L with respect to the linear determining system, and write[†] $\tilde{\sigma}(L) = \sigma(K)$. In particular, a differential polynomial $L \in \mathcal{S}$ if and only if it has reduced symbol $\tilde{\sigma}(L) = 0$, and hence L is a linear combination of derivatives of the differential equations in the Gröbner basis equations. *Q.E.D.*

We define the *order* of the symbol module, and, by extension, the pseudo-group, to be the maximal degree, in r , of the polynomials in its Gröbner basis:

$$\text{order} \quad r = \text{order } \mathcal{G} = \text{order } \mathcal{S} = \max \deg \gamma_{\nu}. \quad (3.13)$$

As such, the order of the pseudo-group may depend upon the original choice of term ordering in $\mathcal{R}$. One way to avoid this is to adopt a *universal Gröbner basis*, [1; p. 50], although this will be unnecessary for our purposes.

Let $\mathcal{M}|_z \subset \mathcal{R}$ denote the monomial module generated by the leading (with respect to the specified term ordering) monomials of the polynomials in the symbol module $\mathcal{S}|_z$. We can assume, possibly by restricting to an open subset[‡] of the manifold M , that $\mathcal{M}|_z = \mathcal{M}$

[†] *Warning:* The reduced symbol is not intrinsically defined, but relies on the detailed implementation of the Gröbner basis reduction algorithm.

[‡] Using the methods of δ regularity, [30], or comprehensive Gröbner bases, ■ by suitable change of variables, this can be arranged anywhere on M .

does not depend upon z . Let

$$\text{SC} \quad \mathcal{C} = \text{Span}\{r^B R_c \notin \mathcal{M}\} \subset \mathcal{R} \quad (3.14)$$

be the *complementary monomial subspace* spanned by all monomials *not* in the monomial module $\mathcal{M}$. Thus, there is a linear basis for the space of symbol polynomials $\mathcal{S}|_z$ that has the form

$$r^A R_b + \sum_{r^B R_c \in \mathcal{C}^n} h_B^c(z) r^B R_c \quad \text{for all} \quad r^A R_b \in \mathcal{M}^n, \quad n \geq 0.$$

A similar statement holds for the subspace $\mathcal{L}|_z$, where the sum now runs over $r^C R_e \in \mathcal{C}^{\leq n}$. Therefore, $\mathcal{C}$ forms a fixed complement to the symbol module $\mathcal{S}|_z$ as well as the determining equation subspace $\mathcal{L}|_z$, so that

$$\mathcal{R} = \mathcal{C} \oplus \mathcal{M} = \mathcal{C} \oplus \mathcal{S}|_z = \mathcal{C} \oplus \mathcal{L}|_z$$

at each $z \in M$. Reinterpreting this decomposition back in terms of the linear determining system, we conclude that, locally, the differential equations can be rewritten in solved triangular form, [25],

$$\text{tri} \quad \zeta_A^b = - \sum_{r^B R_c \in \mathcal{C}^{\leq n}} h_B^c(z) \zeta_B^c \quad \text{for all} \quad r^A R_b \in \mathcal{M}^n, \quad n \geq 0. \quad (3.15)$$

The *parametric derivatives* ζ_B^c correspond to the complementary monomials $r^B R_c \in \mathcal{C}$ thus serve to uniquely parametrize the infinitesimal generator jets of the pseudo-group $\mathcal{G}$. Thus, in view of (3.7), we can identify the complementary subspace

$$\text{SCdual} \quad \mathcal{C} \simeq \mathcal{R} / (\text{J}^\infty \mathfrak{g}|_z)^\perp \simeq (\text{J}^\infty \mathfrak{g}|_z)^* \quad (3.16)$$

as the dual vector space to the infinitesimal generator jets at each point $z \in M$. In particular,

$$\text{dimker} \quad \dim \mathcal{C}^{\leq n} = \dim \text{J}^n \mathfrak{g}|_z = \dim \mathcal{G}^{(n)}|_z = r_n \quad (3.17)$$

is the same as the fiber dimension of the n^{th} order pseudo-group jet bundle, which corresponds to the number of free parameters required to represent the pseudo-group at order n .

Let

$$\tilde{H}(n) = \text{codim } \mathcal{S}^{\leq n}|_z = \text{codim } \mathcal{M}^{\leq n} = \dim \mathcal{C}^{\leq n}$$

denote the (affine) Hilbert function of the symbol module. According to [10; p. 453] (as adapted to modules) when n is sufficiently large, the Hilbert function coincides with a polynomial:

$$\text{HFHP} \quad \tilde{H}(n) = H(n) \quad \text{for all} \quad n \geq n^*, \quad (3.18)$$

where the *Hilbert polynomial* of the symbol module must have the form

$$\text{HP} \quad H(n) = \sum_{i=0}^d b_i \binom{n}{d-i} = b \frac{n^d}{d!} + O(n^{d-1}), \quad (3.19)$$

for certain integer coefficients $b = b_0, b_1, \dots, b_d \in \mathbb{Z}$. The integer $0 \leq d \leq m$ will be called the *dimension* of the symbol module. Unless $\mathcal{S} = \mathcal{R}$, in which case $\tilde{H}(n) \equiv 0$ and the pseudo-group consists only of the identity transformation, the leading coefficient is positive, $b = b_0 > 0$, and is known as the *degree* of the module, [10; p. 465].

The dimension d and degree b of the symbol module have an important interpretation for the determining system for the pseudo-group. Informally, the general solution, that is, the pseudo-group transformations, can be written in terms of b arbitrary functions $f_1, \dots, f_b$, each depending on d variables[†]. In particular, the system is of finite type — and hence $\mathcal{G}$ is, in fact, a b -dimensional Lie group action — if and only if the symbol module has dimension $d = 0$. This interpretation can be made precise if we assume that the system can be solved by the Cartan–Kähler Theorem, [3, 22]. Since this theorem will not be used, the proof is omitted. See Seiler, [30], for additional details.

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Theorem 3.3. *If the determining equations for the pseudo-group form a regular system in the sense of the Cartan–Kähler Theorem, then the last nonzero Cartan character is $c_d = b$.*

The smallest integer n^* for which (3.18) holds is called the *index of regularity* of the module, [10; p. 449]. The index of regularity of the symbol module appears to be related to the maximal order of the integrability constraints for the pseudo-group, but this remains to be completely clarified.

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4. Infinitesimal Generator Symbols.

Our primary object of study is the action of the pseudo-group $\mathcal{G}$ on submanifolds $N \subset M$. Let $J^n(M, p)$, for $0 \leq n \leq \infty$, denote the bundle of jets of p -dimensional submanifolds, [22, 24]. We use $\tilde{\pi}_n^k: J^k(M, p) \rightarrow J^n(M, p)$, $k \geq n$, to indicate the standard projection. Local coordinates on $J^n(M, p)$ are fixed by splitting the coordinates on M into independent and dependent variables: $(z^1, \dots, z^m) = (x^1, \dots, x^p, u^1, \dots, u^q)$, where $p + q = m = \dim M$. We let

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$$z^{(n)} = (x, u^{(n)}) = (\dots x^i \dots u_j^\alpha \dots) \quad (4.1)$$

be the induced local coordinates on $J^n(M, p)$. In many applications, $\mathcal{G}$ represents the symmetry (pseudo-)group of a system of differential equations $\{\Delta(x, u^{(n)}) = 0\} \subset J^n(M, p)$, with p denoting the number of independent variables, q the number of dependent variables, and the submanifolds $N = \{u = f(x)\} \subset M$ of interest are the graphs of candidate solutions.

Given a smooth vector field

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$$\mathbf{v} = \sum_{a=1}^m \zeta^a(z) \frac{\partial}{\partial z^a} = \sum_{i=1}^p \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \varphi_\alpha(x, u) \frac{\partial}{\partial u^\alpha} \in \mathcal{X}(M), \quad (4.2)$$

[†] The precise meaning of this remark was the principal subject of the famous Cartan–Einstein correspondence, [7]; see also [29].

let

$$\mathbf{v}^{(\infty)} = \sum_{i=1}^p \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \sum_{n=\#J \geq 0} \varphi_\alpha^J(x, u^{(k)}) \frac{\partial}{\partial u_J^\alpha} \in \mathcal{X}(\mathbf{J}^\infty(M, p)) \quad (4.3)$$

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denote its infinite prolongation. The prolonged vector field coefficients φ_α^J can be computed via Lie's recursive prolongation formula, [21; (2.44)],

$$\varphi_\alpha^{J,i} = D_{x^i} \varphi_\alpha^J - \sum_{k=1}^p u_{J,k}^\alpha D_{x^i} \xi^k, \quad \#J \geq 0, \quad i = 1, \dots, p, \quad \alpha = 1, \dots, q. \quad (4.4)$$

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Here

$$D_{x^i} = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \sum_{\#J \geq 0} u_{J,i}^\alpha \frac{\partial}{\partial u_J^\alpha} \quad (4.5)$$

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denotes the total derivative with respect to the independent variable x^i . Consequently, each prolonged vector field coefficient

$$\varphi_\alpha^J = \Phi_\alpha^J(u^{(n)}; \zeta^{(n)}) = \Phi_\alpha^J(\dots u_K^\beta \dots; \dots \xi_A^i \dots \varphi_A^\alpha \dots)$$

is a particular linear combination of the jets, i.e., the *partial* derivatives ζ_A^b (with respect to both the independent and dependent variables), (3.2), of the vector field coefficients up to order $n = \#J$, whose coefficients are polynomial functions of the jet coordinates u_K^β , $1 \leq \#K \leq n$, $\beta = 1, \dots, q$. Thus, the prolongation of vector fields factors through the vector field jet bundle. Given $0 \leq n \leq \infty$ and a point $z^{(n)} \in \mathbf{J}^n(M, p)|_z$, we let

$$\mathbf{p}^{(n)}: \mathbf{J}^n TM|_z \longrightarrow T\mathbf{J}^n|_{z^{(n)}}, \quad \mathbf{p}^{(n)}(\mathbf{j}_n \mathbf{v}|_z) = \mathbf{v}^{(n)}|_{z^{(n)}} \quad (4.6)$$

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denote the associated *prolongation map* that takes the n -jet of a vector field at the point $z = \tilde{\pi}_0^n(z^{(n)}) \in M$ to the tangent vector $\mathbf{v}^{(n)} \in T\mathbf{J}^n|_{z^{(n)}}$ prescribed by the prolongation formula (4.3), (4.4).

We now introduce a symbol calculus for the infinitesimal generators for a prolonged pseudo-group action. We introduce variables $s = (s_1, \dots, s_p)$, $S = (S_1, \dots, S_q)$, and let

$$\mathcal{Q} = \left\{ T(s, S) = \sum_{\alpha=1}^q T_\alpha(s) S_\alpha \right\} \simeq \mathbb{R}[s] \otimes \mathbb{R}^q \subset \mathbb{R}[s, S], \quad (4.7)$$

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be the module consisting of polynomials which are linear in S , isomorphis to the module of vector-valued polynomials $\mathbf{T}: \mathbb{R}^p \rightarrow \mathbb{R}^q$. Further, define

$$\widehat{\mathcal{Q}} = \mathbb{R}^p \oplus \mathcal{Q} = \bigoplus_{n=-1}^{\infty} \widehat{\mathcal{Q}}^n, \quad (4.8)$$

QV

whose summands $\widehat{\mathcal{Q}}^n = \mathcal{Q}^n$ for $n \geq 0$ contain all polynomials $T(s, S) \in \mathcal{Q}$ that are homogeneous of degree n in s , while, by convention,

$$\widehat{\mathcal{Q}}^{-1} = \mathcal{Q}^{-1} = \mathbb{R}^p = \{ c_1 \tilde{s}_1 + \dots + c_p \tilde{s}_p \mid c_i \in \mathbb{R} \};$$

we place tildes on $\tilde{s} = (\tilde{s}_1, \dots, \tilde{s}_p) \in \mathbb{R}^p$ to avoid confusion with the polynomial variables $(s_1, \dots, s_p)$. Thus, an element $H \in \widehat{\mathcal{Q}}$ has the form

$$H(s, S, \tilde{s}) = \sum_{i=1}^p c_i \tilde{s}_i + T(s, S), \quad \text{where} \quad T(s, S) = \sum_{\alpha=1}^q T_\alpha(s) S_\alpha \in \mathcal{Q}, \quad (4.9)$$

$c_1, \dots, c_p \in \mathbb{R}.$

That

We define the leading term map $\lambda: \widehat{\mathcal{Q}} \rightarrow \mathcal{Q}$ so that

$$\lambda[H(s, S, \tilde{s})] = \lambda[T(s, S)] \quad (4.10)$$

ltQV

whenever H has the form (4.9); thus, by convention, all elements of $\widehat{\mathcal{Q}}^{-1}$ have zero leading term.

In coordinates, we can identify the cotangent bundle

$$T^*J^\infty(M, p) = (TJ^\infty(M, p))^* \simeq J^\infty(M, p) \times \widehat{\mathcal{Q}} \quad (4.11)$$

TUJis

by the explicit pairing

$$\langle \mathbf{v}|_z; \tilde{s}_i \rangle = \xi^i, \quad \langle \mathbf{v}|_z; S_\alpha \rangle = Q_\alpha \equiv \varphi_\alpha - \sum_{i=1}^p u_i^\alpha \xi^i, \quad (4.12)$$

TUJi

$$\langle \mathbf{v}|_z; s^J S_\alpha \rangle = \varphi_\alpha^J, \quad \text{for} \quad \#J \geq 1.$$

Here $Q = (Q_1, \dots, Q_q)$ is the usual *characteristic* of the vector field, [21]; the reason for its appearance at order 0 will become clear shortly. Every one-form on $J^\infty(M, p)$, i.e., section of $T^*J^\infty(M, p)$, is then represented in local coordinates by a smoothly parametrized polynomial

$$H(x, u^{(n)}; s, S, \tilde{s}) = \sum_{i=1}^p \tilde{T}_i(x, u^{(n)}) \tilde{s}_i + \sum_{\alpha=1}^q T_\alpha(x, u^{(n)}; s) S_\alpha,$$

which is linear in the variables $(\tilde{s}, S) \in \mathbb{R}^m$, polynomial in the variables $s \in \mathbb{R}^p$, and depends smoothly on the jet coordinates $(x, u^{(n)})$ of some finite order $n < \infty$. (And note that the first set of summands does not depend on s .)

Using their identification as the dual spaces to those appearing in (4.6), we let

$$\mathbf{p}^* = (\mathbf{p}^{(\infty)})^*: J^\infty(M, p) \times \widehat{\mathcal{Q}} \longrightarrow M \times \mathcal{R},$$

be the linear *dual prolongation map*, as defined on the polynomial modules:

$$\langle j_\infty \mathbf{v}; \mathbf{p}^*(H) \rangle = \langle \mathbf{p}^{(\infty)}(j_\infty \mathbf{v}); H \rangle = \langle \mathbf{v}^{(\infty)}; H \rangle \quad \text{for all} \quad j_\infty \mathbf{v} \in J^\infty TM, \quad H \in \widehat{\mathcal{Q}}, \quad (4.13)$$

prU

In general, $\mathbf{p}^*$ is *not* a module morphism. However, on the symbol level it essentially is, as we now demonstrate.

At a prescribed a 1-jet $(x, u^{(1)}) \in J^1$, we introduce the particular linear polynomials

$$\beta_i(r) = r_i + \sum_{\alpha=1}^q u_i^\alpha r_{p+\alpha}, \quad i = 1, \dots, p, \quad (4.14)$$

$$B_\alpha(R) = R_{p+\alpha} - \sum_{i=1}^p u_i^\alpha R_i, \quad \alpha = 1, \dots, q,$$

lk

where $u_i^\alpha = \partial u^\alpha / \partial x^i$. Note that $B_\alpha(R)$ is the symbol of Q_α , the α^{th} component of the characteristic of $\mathbf{v}$, cf. (4.12). Furthermore, it is not hard to see that $\beta_i(r)$ represents the symbol of the total derivative (4.5), so

$$\sigma[\mathbf{D}_{x^i} L(x, u^{(n+1)}; \zeta^{(n+1)})] = \beta_i(r) \sigma[L(x, u^{(n)}; \zeta^{(n)})] \quad (4.15)$$

for any linear differential polynomial

$$L(x, u^{(n)}; \zeta^{(n)}) = \sum_{b=1}^m \sum_{\#A \leq n} h_A^b(x, u^{(n)}) \zeta_A^b. \quad (4.16)$$

ldpoln

Keep in mind that total differentiation in (4.15) acts on both the submanifold jet variables $(x, u^{(n)})$ and the vector field jets ζ_A^b ; see [24] for complete details.

The functions (4.14) serve to define a linear map

$$\beta: \mathbb{R}^{2m} \longrightarrow \mathbb{R}^m, \quad \text{so that} \quad s_i = \beta_i(r), \quad S_\alpha = B_\alpha(R). \quad (4.17)$$

svr

Since β has maximal rank, the pull-back map $\beta^*: \mathcal{Q} \rightarrow \mathcal{R}$, given by

$$\beta^*[P(s_1, \dots, s_p, S_1, \dots, S_q)] = P(\beta_1(r), \dots, \beta_p(r), B_1(R), \dots, B_q(R)) \quad (4.18)$$

svrU

defines an injection. As a direct consequence of (4.4) and (4.15), we have established:

Lemma 4.1. *The symbols of the prolonged vector field coefficients (4.4) are*

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$$\begin{aligned} \sigma(\xi^i) &= R_i, & i &= 1, \dots, p, \\ \sigma(\varphi_\alpha) &= R_{\alpha+p}, & \alpha &= 1, \dots, q, \\ \sigma(Q_\alpha) &= \beta^*(S_\alpha) = B_\alpha(R), & n &= \#J \geq 1. \\ \sigma(\varphi_\alpha^J) &= \beta^*(s^J S_\alpha) = \beta_{j_1}(r) \cdots \beta_{j_n}(r) B_\alpha(R), \end{aligned} \quad (4.19)$$

svn

In general, given a polynomial $T(s, S) \in \mathcal{Q} \subset \widehat{\mathcal{Q}}$, we have

$$\lambda[\mathbf{p}^*(T(s, S))] = \beta^*[\lambda(T(s, S))]. \quad (4.20)$$

svIU

Now let $\mathcal{G}$ be a Lie pseudo-group acting on M , and $\mathfrak{g}^{(\infty)} = \mathbf{p}^{(\infty)}(J^\infty \mathfrak{g}) \subset TJ^\infty$ denote the subbundle spanned by its prolonged infinitesimal generators. Let

$$\mathcal{Z} = (\mathfrak{g}^{(\infty)})^\perp = (\mathbf{p}^*)^{-1} \mathcal{L} \subset J^\infty(M, p) \times \widehat{\mathcal{Q}} \quad (4.21)$$

SZ

denote the subbundle containing those polynomials (4.9) that annihilate all prolonged infinitesimal generators $\mathbf{v}^{(\infty)} \in \mathfrak{g}^{(\infty)}$. Further, define the *prolonged symbol bundle*

$$\text{SU} \quad \mathcal{U} = \lambda(\mathcal{Z}) \subset J^\infty(M, p) \times \widehat{\mathcal{Q}}, \quad (4.22)$$

spanned by the leading terms of the annihilators. In general, $\mathcal{U}$ is *not* a submodule, although it inherits a lot of algebraic structure that we intend to exploit.

At each $z^{(\infty)} \in J^\infty(M, p)|_z$, let us introduce the submodule

$$\text{SJz} \quad \mathcal{J}|_{z^{(\infty)}} = (\beta^*)^{-1}(\mathcal{S}|_z) = \{ T(s, S) \mid T(\beta(r), B(R)) \in \mathcal{S}|_z \} \subset \mathcal{Q}, \quad (4.23)$$

that is, the inverse image of the symbol module under the polynomial pull-back morphism (4.18). In view of (4.20), (4.21),

$$\text{SUSJ} \quad \mathcal{U}|_{z^{(\infty)}} \subset \mathcal{J}|_{z^{(\infty)}}. \quad (4.24)$$

However, in general, the two subspaces *are not* equal, $\mathcal{U}|_{z^{(\infty)}} \neq \mathcal{J}|_{z^{(\infty)}}$, as the following example makes clear.

Example 4.2. Consider the Lie pseudo-group $\mathcal{G}$ acting on $M = \mathbb{R}^3$ consisting of all local diffeomorphisms

$$\text{ps3f} \quad X = a(x), \quad Y = a'(x)y + b(x), \quad U = u + \frac{a''(x)y + b'(x)}{a'(x)}, \quad (4.25)$$

where $a(x) \in \mathcal{D}(\mathbb{R})$ is an arbitrary local diffeomorphism of $\mathbb{R}$, while $b(x) \in C^\infty(\mathbb{R})$ is an arbitrary smooth function, cf. [23, 24]. Its infinitesimal generators are of the form

$$\text{ps3ig} \quad \mathbf{v} = \xi \partial_x + \eta \partial_y + \varphi \partial_u = f(x) \partial_x + [f'(x)y + g(x)] \partial_y + [f''(x)y + g'(x)] \partial_u, \quad (4.26)$$

where $f(x), g(x)$ are arbitrary functions of a single variable. The linearized determining equations are obtained by prolonging the first order system

$$\text{ps3ide1} \quad \xi_y = \xi_u = 0, \quad \eta_x = \varphi, \quad \eta_y = \xi_x, \quad \eta_u = 0, \quad \varphi_u = 0. \quad (4.27)$$

In particular, the additional second order determining equations are

$$\text{ps3ide2} \quad \begin{aligned} \xi_{xx} = \varphi_y, \quad \xi_{xy} = \xi_{yy} = \xi_{xu} = \xi_{yu} = \xi_{uu} = 0, \quad \eta_{xx} = \varphi_x, \quad \eta_{xy} = \varphi_y, \\ \eta_{yy} = \eta_{xu} = \eta_{yu} = \eta_{uu} = 0, \quad \varphi_{yy} = \varphi_{xu} = \varphi_{yu} = \varphi_{uu} = 0. \end{aligned} \quad (4.28)$$

Let r, s, t, R, S, T be the polynomial variables corresponding to $x, y, u, \xi, \eta, \varphi$, respectively. To second order, then, the annihilator $\mathcal{L}|_z$ at any $z = (t, x, u) \in M$ is spanned by

$$\begin{aligned} sR, \quad tR, \quad rS - T, \quad sS - rR, \quad tS, \quad tT, \\ r^2R - sT, \quad rsR, \quad s^2R, \quad rtR, \quad stR, \quad t^2R, \\ r^2S - rT, \quad rsS - sT, \quad s^2S, \quad rtS, \quad stS, \quad t^2S, \quad s^2T, \quad rtT, \quad stT, \quad t^2T, \end{aligned}$$

Using the orderings $r < s < t, R < S < T$, the symbol module $\mathcal{S}|_z$ is generated by the Gröbner basis

$$\text{ps3Gba} \quad sR, \quad tR, \quad r^2R, \quad rS, \quad sS - rR, \quad tS, \quad tT, \quad s^2T. \quad (4.29)$$

The complementary monomial basis

$$R, \quad S, \quad T, \quad rR, \quad rT, \quad sT, \quad r^2T, \quad rsT, \quad r^3T, \quad r^2sT, \quad \dots,$$

indexes the free derivatives

$$\xi, \quad \eta, \quad \varphi, \quad \xi_x, \quad \varphi_x, \quad \varphi_y, \quad \varphi_{xx}, \quad \varphi_{xy}, \quad \varphi_{xx}, \quad \varphi_{xy}, \quad \dots,$$

which correspond to the independent pseudo-group parameters.

We consider the action of this pseudo-group on surfaces $u = f(x, y)$, and hence prolong to $J^\infty(M, 2)$. The general prolonged infinitesimal generator is

$$\begin{aligned} \mathbf{v}^{(\infty)} = & \xi \partial_x + \eta \partial_y + \varphi \partial_u + (\varphi_x - u_x \xi_x - u_y \varphi) \partial_{u_x} + (\varphi_y - u_y \xi_x) \partial_{u_y} + \\ & + (\varphi_{xx} - u_x \varphi_y - u_y \varphi_x - 2u_{xx} \xi_x - 2u_y \varphi) \partial_{u_{xx}} + \\ & + (\varphi_{xy} - u_y \varphi_y - 2u_{xy} \xi_x - u_{yy} \varphi) \partial_{u_{xy}} - 2u_{yy} \xi_x \partial_{u_{yy}} + \dots, \end{aligned} \quad (4.30)$$

ps3vi

where we have explicitly implemented the linearized determining equations (4.27), (4.28) to keep the expressions short. We employ w, z, Z to designate the polynomial variables in $\mathcal{Q}$ representing x, y, φ respectively, so that

$$\langle \mathbf{v}^{(\infty)}; w^j z^k Z \rangle = \varphi^{jk}, \quad \text{when} \quad \mathbf{v}^{(\infty)} = \sum_{i=1}^p \xi^i \frac{\partial}{\partial x^i} + \sum_{j,k \geq 0} \varphi^{jk} \frac{\partial}{\partial u_{jk}},$$

where $u_{jk} = \partial^{j+k} u / \partial t^j \partial x^k$. Continuing to prolong the infinitesimal generators (4.30), we discover that, at a jet $z^{(\infty)} = (x, u^{(\infty)})$, the prolonged annihilator subbundle $\mathcal{Z}|_{z^{(\infty)}}$, as defined in (4.21), is spanned by the polynomials

$$z^3 Z, \quad (u_{yy} w z^2 - (\frac{3}{2} u_{xyy} + u_y u_{yy}) z^2 + 2u_{yy}^2 z) Z, \quad w z^3 Z, \quad z^4 Z, \quad \dots$$

The prolonged symbol space $\mathcal{U}|_{z^{(\infty)}}$, (4.22), is spanned by their leading terms:

$$z^3 Z, \quad u_{yy} w z^2 Z, \quad w z^3 Z, \quad z^4 Z, \quad \dots,$$

and so, provided $u_{yy} \neq 0$, is, in fact, the submodule generated by $z^3 Z, w z^2 Z$.

On the other hand, since the linear polynomial map β^* is defined by

$$\begin{aligned} w &= \beta_1(r, s, t) = r + u_x t, & z &= \beta_2(r, s, t) = s + u_y t, \\ Z &= B(R, S, T) = T - u_x R - u_y S, \end{aligned}$$

the submodule $\mathcal{J}|_{z^{(\infty)}}$, which, according to (4.23), contains all polynomials $Q(w, z) Z$ such that

$$\beta^*[Q(w, z) Z] = Q(r + u_x t, s + u_y t) (T - u_x R - u_y S) \in \mathcal{U}|_{z^{(\infty)}},$$

is easily seen to be generated by the single monomial $z^2 Z$, and hence strictly contains the prolonged symbol space: $\mathcal{J}|_{z^{(\infty)}} \supsetneq \mathcal{U}|_{z^{(\infty)}}$. The problem is that the symbol of the prolonged infinitesimal generator coefficient

$$\sigma(\varphi^{yy}) = \beta^*(z^2 Z) = (s + u_y t)^2 (T - u_x R - u_y S)$$

belongs to the symbol module of the linear determining equations, but φ^{yy} does not since the “subsymbol” term $-2u_{yy} r R$ remains nonzero on the space of infinitesimal generators.

free 5. Freeness of Prolonged Pseudo-group Actions.

We are now in a position to study the local freeness of the prolonged pseudo-group action. Let us begin with the infinitesimal version of the basic definition; see [24] for full details.

lfree **Definition 5.1.** A pseudo-group acts *locally freely* at $z^{(n)} \in J^n(M, p)|_z$ if and only if the prolongation map $\mathbf{p}^{(n)}: J^n \mathfrak{g}|_z \xrightarrow{\sim} \mathfrak{g}^{(n)}|_{z^{(n)}}$ is a linear isomorphism.

Local freeness at $z^{(n)}$ implies local freeness in a neighborhood of $z^{(n)}$ — and, in the analytic category, on a dense open subset of $J^n(M, p)$. In particular, Definition 5.1 implies that *all* pseudo-group actions are locally free at order 0, and so the condition has nontrivial implications only at the jet level.

Warning: According to the standard definition, [12], any locally free action of a finite-dimensional Lie group satisfies the local freeness condition of Definition 5.1, but not necessarily conversely. For instance, the four-dimensional group with infinitesimal generators $\partial_x, \partial_u, x\partial_u, x^2\partial_u$ acting on $M = \mathbb{R}^2$ has locally free action on $J^n(M, p)$ for all $n \geq 0$ according to Definition 5.1; while, in the usual Lie group terminology, the action is only locally free when $n \geq 2$. In this paper, even for finite-dimensional Lie group actions, we will use the more general terminology introduced in Definition 5.1.

Local freeness requires that the bundle of pseudo-group jets $\mathcal{G}^{(n)}$ (or, equivalently, the infinitesimal generator jet bundle $J^n \mathfrak{g}$) not be too large. Specifically, local freeness at a point of $z^{(n)} \in J^n(M, p)|_z$ requires

$$\text{lfbound} \quad \dim \mathcal{G}^{(n)}|_z = \dim J^n \mathfrak{g}|_z \leq \dim J^n(M, p)|_z = q \binom{p+n}{p}. \quad (5.1)$$

According to Section 3, for sufficiently large $n \gg 0$,

$$\dim J^n \mathfrak{g}|_z = \dim \mathcal{C}^{\leq n} = H(n)$$

is characterized by the symbol module's Hilbert polynomial. Thus, by comparing (3.19) and (5.1), we deduce:

nonfree **Proposition 5.2.** *If the symbol module $\mathcal{S}$ has dimension $d > p$, or $d = p$ and degree $b > q$, then the pseudo-group cannot act freely on $J^n(M, p)$ for $n \gg 0$ sufficiently large.*

Remark: Theorem 5.4 below will allow us to replace $n \gg 0$ by simply $n > 0$.

In other words, if the pseudo-group is to act locally freely, as required for our moving frame constructions to be valid, the rate of growth of its jet subgroups cannot be too rapid. Intuitively, Proposition 5.2 says that, since we are acting on p -dimensional submanifolds of an $m = p + q$ dimensional space, which are thus parametrized by q functions of p variables, a locally freely acting pseudo-group can itself depend upon arbitrary functions of at most p variables, and only at most q functions of exactly p variables. In Kumpera's approach, [17], assumptions on the dimensions of $\mathcal{G}^{(n)}$ are encoded in terms of Spencer cohomology groups. Here, the required growth rate assumptions are simply stated in algebraic terms — specifically the dimension and degree of the symbol module.

Proposition 5.2 only gives a rough, preliminary dimension bound that is required for local freeness. In order to fully characterize a locally free action, we must understand the detailed structure of its prolonged infinitesimal generators (4.3). We begin by fixing a point $z^{(\infty)} \in J^\infty(M, p)|_z$. To keep the notation uncluttered, we use $\mathbf{p}^*: \widehat{\mathcal{Q}} \rightarrow \mathcal{R}$ to denote the restriction of the dual prolongation map (4.13) to the fiber $T^*J^\infty(M, p)|_{z^{(\infty)}} \simeq \widehat{\mathcal{Q}}$.

Lemma 5.3. *The pseudo-group $\mathcal{G}$ acts locally freely at $z^{(n)} \in J^n(M, p)$ if and only if*

$$\mathbf{p}^*(\widehat{\mathcal{Q}}^{\leq n}) \oplus \mathcal{L}^{\leq n}|_z = \mathcal{R}^{\leq n}. \quad (5.2)$$

Proof: In view of (3.7), the condition of local freeness in Definition 5.1 can be restated as

$$\{0\} = \ker \mathbf{p}^{(n)} \cap J^n \mathfrak{g}|_z = (\text{rng } (\mathbf{p}^{(n)})^*)^\perp \cap (\mathcal{L}^{\leq n}|_z)^\perp = (\mathbf{p}^*(\widehat{\mathcal{Q}}^{\leq n}) + \mathcal{L}^{\leq n}|_z)^\perp,$$

from which (5.2) immediately follows. Q.E.D.

We are now in a position to prove a key result that guarantees local freeness of higher order prolonged pseudo-group actions.

Theorem 5.4. *Let $\mathcal{G}$ be a regular pseudo-group acting on an m -dimensional manifold M . If $\mathcal{G}$ acts locally freely at $z^{(n)} \in J^n(M, p)$ for some $n > 0$, then it acts locally freely at any $z^{(k)} \in J^k(M, p)$ with $\tilde{\pi}_n^k(z^{(k)}) = z^{(n)}$, for $k \geq n$.*

Proof: The proof is by induction on the order $k \geq n \geq 1$. According to Lemma 5.3, local freeness at order k implies that

$$\mathbf{p}^*(\widehat{\mathcal{Q}}^{\leq k}) + \mathcal{L}^{\leq k}|_z = \mathcal{R}^{\leq k}. \quad (5.3)$$

Taking leading terms of both sides and using (4.20), we find that

$$\beta^*(\mathcal{Q}^k) + \mathcal{S}^k|_z = \mathcal{R}^k. \quad (5.4)$$

We claim that, to prove freeness at order $k+1$, we just need to show

$$\beta^*(\mathcal{Q}^{k+1}) + \mathcal{S}^{k+1}|_z = \mathcal{R}^{k+1}. \quad (5.5)$$

Indeed, if this holds, then any $P \in \mathcal{R}^{k+1}$ can be written as

$$P = \beta^*(Q) + S \quad \text{for} \quad Q \in \mathcal{Q}^{k+1}, \quad S \in \mathcal{S}^{k+1}|_z. \quad (5.6)$$

But then there exists $L \in \mathcal{L}^{\leq k+1}|_z$ with leading term $\lambda(L) = S$, and so

$$\mathbf{p}^*(Q) = \beta^*(Q) + U, \quad L = S + T, \quad \text{for some} \quad U, T \in \mathcal{R}^{\leq k}.$$

We use the induction hypothesis (5.3) to write

$$T + U = \beta^*(V) + W \quad \text{for} \quad V \in \widehat{\mathcal{Q}}^{\leq k}, \quad W \in \mathcal{L}^{\leq k}|_z.$$

Thus, comparing with (5.6),

$$P = \mathbf{p}^*(Q) + L - (T + U) = \mathbf{p}^*(Q - V) + (L - W) \in \mathbf{p}^*(\mathcal{Q}^{k+1}) + \mathcal{S}^{k+1}|_z,$$

which completes the induction step.

To prove the claim (5.5), first observe that (5.4) implies that any polynomial $P \in \mathcal{R}^k$ can be written in the form

$$\text{Pk} \quad P(r, R) = \beta^*(T(s, S)) + H(r, R), \quad \text{where} \quad T(s, S) \in \mathcal{Q}^k, \quad H \in \mathcal{S}^k|_z. \quad (5.7)$$

Since $k \geq 1$, we can (non-uniquely) write

$$T(s, S) = \sum_{i=1}^p s_i T_i(s, S) \quad \text{for some} \quad T_i \in \mathcal{Q}^{k-1},$$

and hence

$$P(r, R) = \sum_{i=1}^p \beta_i(r) Q_i(r, R) + V(r, R), \quad \text{where} \quad Q_i = \beta^*(T_i) \in \mathcal{R}^{k-1}, \quad V \in \mathcal{S}^k|_z. \quad (5.8)$$

Pdec

Every polynomial of degree $k+1$ can be written as

$$\hat{P}(r, R) = \sum_{j=1}^p r_j P_j(r, R) \in \mathcal{R}^{k+1}, \quad \text{for some} \quad P_1, \dots, P_p \in \mathcal{R}^k.$$

Hence, applying (5.8) to each summand $P_j(r, R)$ and using the fact that $\mathcal{S}|_z$ is a submodule,

$$\text{Pko} \quad \hat{P}(r, R) = \sum_{i=1}^p \beta_i(r) \hat{Q}_i(r, R) + \hat{V}(r, R), \quad \text{where} \quad \hat{Q}_i \in \mathcal{R}^k, \quad \hat{V} \in \mathcal{S}^{k+1}|_z. \quad (5.9)$$

On the other hand, by (5.4), we can write each

$$\hat{Q}_i(r, R) = \beta^*(A_i(s, S)) + B_i(r, R), \quad \text{where} \quad A_i \in \mathcal{Q}^k, \quad B_i \in \mathcal{S}^k|_z.$$

Then, substituting back into (5.9), we find

$$\hat{P}(r, R) = \beta^* \left(\sum_{i=1}^p s_i A_i(s, S) \right) + \left(\sum_{i=1}^p \beta_i(r) B_i(r, R) + \hat{V}(r, R) \right) \in \beta^*(\mathcal{Q}^{k+1}) + \mathcal{S}^{k+1}|_z.$$

This validates (5.5), and hence the induction step.

Q.E.D.

amf 6. Algebraic Moving Frames.

We now turn to the construction of a moving frame for the prolonged pseudo-group action on the submanifold jet spaces $J^n(M, p)$. As described in [24], just as with finite-dimensional Lie group actions, the process relies on the choice of a local cross-section, that is, a submanifold $K^n \subset J^n(M, p)$ that transversally intersects each prolonged pseudo-group orbit, and so

$$\text{TKn} \quad TK^n|_{z^{(n)}} \oplus g^{(n)}|_{z^{(n)}} = TJ^n(M, p)|_{z^{(n)}}, \quad \text{for all} \quad z^{(n)} \in K^n. \quad (6.1)$$

In practice, it suffices to construct a *coordinate cross-section*, that is, a submanifold parallel to the local coordinate axes, and so prescribed by setting a specified number of the jet coordinates $(x, u^{(n)})$ to assume constant values.

Existence of a cross-section passing through $z_0^{(n)} \in J^n(M, p)$ requires that prolonged the pseudo-group orbits form a regular foliation in a neighborhood of $z_0^{(n)}$. Assuming regularity, a submanifold $z_0^{(n)} \in K^n \subset J^n(M, p)$ defines a cross-section in a neighborhood of $z_0^{(n)}$ if and only if the annihilator of its tangent space defines a complementary subspace to the annihilator subspace (4.21), so

$$(TK^n|_{z_0^{(n)}})^\perp \oplus \mathcal{Z}^{\leq n}|_{z_0^{(n)}} = \widehat{\mathcal{Q}}^{\leq n}.$$

In particular, coordinate cross-sections passing through $z_0^{(n)}$ are in one-to-one correspondence with monomial complements; thus,

$$\mathcal{K}^{\leq n} = \text{Span}\{\tilde{s}_j, s^K S_\beta\}, \quad \text{with} \quad \mathcal{K}^{\leq n} \oplus \mathcal{Z}^{\leq n}|_{z_0^{(n)}} = \widehat{\mathcal{Q}}^{\leq n},$$

corresponds to the coordinate cross-section

$$\text{ccss} \quad x^j = c_j, \quad u_K^\beta = c_K^\beta, \quad \text{for all} \quad \tilde{s}_j, s^K S_\beta \in \mathcal{K}^{\leq n}, \quad (6.2)$$

where the constants c_j, c_K^β are fixed by the coordinate values at the base point $z_0^{(n)}$. From now on, we only deal with coordinate cross-sections, although our methods can, with some additional work, be adapted to more general cross-sections, cf. [18].

Now, let[†]

$$\text{gpn} \quad \widehat{U}_J^\alpha = \widehat{F}_J^\alpha(x, u^{(n)}, g^{(n)}), \quad \alpha = 1, \dots, q, \quad \#J \leq n, \quad g^{(n)} \in \mathcal{G}^{(n)}|_{z=(x,u)}, \quad (6.3)$$

be the explicit formulas the prolonged pseudo-group action on $J^n(M, p)$, obtained by implicit total differentiation of the pseudo-group action

$$\text{gp0} \quad X^i = \widehat{F}^i(x, u, g^{(0)}), \quad U^\alpha = \widehat{F}^\alpha(x, u, g^{(0)}), \quad (6.4)$$

on M ; see [24] for details. The normalization equations corresponding to the coordinate cross-section (6.2) are

$$\text{ncss} \quad \widehat{F}^j(x, u, g^{(0)}) = c_j, \quad \widehat{F}_K^\beta(x, u^{(n)}, g^{(n)}) = c_K^\beta, \quad \text{for} \quad \tilde{s}_j, s^K S_\beta \in \mathcal{K}^{\leq n}. \quad (6.5)$$

Suppose $\mathcal{G}$ acts locally freely at a point $z_0^{(n)} \in J^n(M, p)$. Since we are assuming that K^n is a cross-section, we can locally solve the normalization equations for the pseudo-group parameters

$$\text{mf n} \quad g^{(n)} = \rho^{(n)}(x, u^{(n)}), \quad (6.6)$$

which prescribes a locally equivariant moving frame. See [24, 9] for additional details and explicit examples.

[†] We place hats over the transformed submanifold jet coordinates to avoid confusion with the pseudo-group jet coordinates U_A^b .

We use

$$\text{iota} \quad \iota: F(x, u^{(n)}) \longmapsto I(x, u^{(n)}) = F(\rho^{(n)}(x, u^{(n)}) \cdot (x, u^{(n)})) \quad (6.7)$$

to denote the invariantization process on differential invariants induced by the moving frame. Invariantization is implemented by replacing the pseudo-group parameters in a transformed function (or form, operator, ...) by their moving frame normalizations (6.6). Invariantization defines a morphism that projects the algebra of differential functions to the algebra of differential invariants, and so $\iota \circ \iota = \text{id}$. In particular, we let

$$\text{ndin} \quad \iota(x^i) = H^i \quad \iota(u_J^\alpha) = I_J^\alpha, \quad i = 1, \dots, n, \quad \alpha = 1, \dots, q, \quad 0 \leq \#J \leq n, \quad (6.8)$$

denote the normalized differential invariants obtained by invariantizing the standard jet space coordinates. Each normalized differential invariant is identified by a monomial $\tilde{s}_i, s^J S_\alpha \in \widehat{\mathcal{Q}}^{\leq n}$. The monomials $\tilde{s}_i, s^K S_\beta \in \mathcal{K}^{\leq n}$ corresponding to the cross-section normalization equations (6.2) correspond to the *phantom differential invariants*, that reduce to the normalization constants c_i, c_K^b . The remaining monomials, i.e., those in the complementary subspace

$$\text{SNIn} \quad \mathcal{N}^{\leq n} = \text{span} \{ \tilde{s}_i, s^J S_\alpha \in \widehat{\mathcal{Q}}^{\leq n} \setminus \mathcal{K}^{\leq n} \}, \quad \text{so} \quad \mathcal{N}^{\leq n} \oplus \mathcal{K}^{\leq n} = \widehat{\mathcal{Q}}^{\leq n}, \quad (6.9)$$

correspond to a complete system of functionally independent normalized differential invariants

$$\text{gdin} \quad H^i, I_J^\alpha \quad \text{for} \quad \tilde{s}_i, s^J S_\alpha \in \mathcal{N}^{\leq n}, \quad (6.10)$$

with the property that any other differential invariant of order $\leq n$ can be locally written as a function of these generating differential invariants. In this manner, the space of differential invariants of order $\leq n$ is completely characterized by the monomial subspace $\mathcal{N}^{\leq n}$.

According to Theorem 5.4, if $\mathcal{G}$ acts locally freely at $z_0^{(n)}$, it also acts locally freely at any $z_0^{(n+1)}$ with $\tilde{\pi}_n^{n+1}(z_0^{(n+1)}) = z_0^{(n)}$. We always arrange that our choice of cross-section K^{n+1} through $z_0^{(n+1)}$ is *compatible* with that through $z_0^{(n)}$, so that $\tilde{\pi}_n^{n+1}(K^{n+1}) = K^n$. On the algebraic level, this requires that the annihilator complements satisfy $\mathcal{K}^{\leq n} \subset \mathcal{K}^{\leq n+1}$. Clearly, this will be the case if and only if $\mathcal{K}^{n+1} = \mathcal{K}^{\leq n+1} \cap \mathcal{Q}^{n+1}$ forms a complementary monomial subspace to the prolonged symbol space $\mathcal{U}^{n+1}|_{z_0^{(n+1)}}$. Now comes the crucial fact: local freeness allows us to replace (4.24) by an equality and identify the prolonged symbol space with a submodule, thereby bringing some remarkable algebraic structure into the picture.

Lemma 6.1. *Suppose $\mathcal{G}^{(n)}$ acts locally freely at $z^{(n)} \in J^n(M, p)$. Then*

$$\text{SUSJe} \quad \mathcal{U}^{n+1}|_{z^{(n+1)}} = \mathcal{J}^{n+1}|_{z^{(n+1)}} \quad (6.11)$$

for any $z^{(n+1)}$ with $\tilde{\pi}_n^{n+1}(z^{(n+1)}) = z^{(n)}$.

Proof: In view of (4.24), we need to show that if $T \in \mathcal{J}^{n+1}|_{z^{(n+1)}}$, then there exists $U \in \mathcal{Q}^{\leq n}$ such that $T + U \in \mathcal{Z}^{\leq n+1}|_{z^{(n+1)}}$, which implies $T = \lambda(T + U) \in \mathcal{U}^{n+1}|_{z^{(n+1)}}$. Let $P = \mathbf{p}^*(T)$. Our hypothesis implies $\beta^*(T) = \lambda(P) \in \mathcal{S}^{n+1}|_z$, and hence there exists $Q \in \mathcal{R}^{\leq n}$ such that $P + Q \in \mathcal{L}^{\leq n+1}|_z$. Now, by Lemma 5.3, local freeness implies that $Q = \mathbf{p}^*(U) + V$ where $U \in \widehat{\mathcal{Q}}^{\leq n}$ and $V \in \mathcal{L}^{\leq n}|_z$. Thus, $\mathbf{p}^*(T + U) = (P + Q) - V \in \mathcal{L}^{\leq n+1}|_z$, which implies that $T + U \in \mathcal{Z}^{\leq n+1}|_{z^{(n+1)}}$. Q.E.D.

Let now fix a degree term ordering[†] on the polynomial module $\mathcal{Q}$. Let $z_0^{(\infty)} \in J^\infty(M, p)$ be a fixed jet such that $\mathcal{G}$ acts locally freely at $z_0^{(n)} = \tilde{\pi}_n^\infty(z_0^{(\infty)})$ for all $n \geq n^*$. We will call n^* the *order of freeness* of the prolonged pseudo-group action (although we allow the possibility that n^* not be the minimal such order). Let $\mathcal{J}|_{z^{(\infty)}}$ be the module defined in (4.23), whence, by Lemma 6.1, $\mathcal{U}^n|_{z^{(n)}} = \mathcal{J}^n|_{z^{(n)}}$ for all $n > n^*$.

We fix a coordinate cross-section K^{n^*} through $z_0^{(n^*)}$ by selecting a monomial complement $\mathcal{K}^{\leq n^*}$ to the annihilator $\mathcal{Z}^{\leq n^*}|_{z_0^{(n^*)}}$. We shall complete the cross-section to $z_0^{(\infty)}$ by applying Gröbner basis techniques to the symbol module $\mathcal{J}|_{z^{(\infty)}}$. Specifically, for each $z^{(\infty)}$ near $z_0^{(\infty)}$, we let $\mathcal{W}|_{z^{(\infty)}}$ be the monomial submodule generated by the leading monomials of the polynomials in $\mathcal{U}|_{z^{(\infty)}}$. As before, by restricting to an open subset (■ possibly changing to δ regular coordinates ■) we can assume that $\mathcal{W}|_{z^{(\infty)}} = \mathcal{W}$ does not depend upon $z^{(\infty)}$. We define the monomial subspace

$$\text{SN} \quad \mathcal{N} = \mathcal{N}^{\leq n^*} \oplus \mathcal{W}^{> n^*}. \quad (6.12)$$

In general, $\mathcal{N}$ forms an “eventual submodule” meaning that beyond a certain finite order, $\mathcal{N}^{> n^*} = \mathcal{W}^{> n^*}$ is a submodule, but lower order terms may preclude it from being a bona fide module.

Let

$$\text{SK} \quad \mathcal{K} = \text{Span}\{\tilde{s}j, s^K S_\beta \notin \mathcal{N}\} \subset \hat{\mathcal{Q}} \quad (6.13)$$

be the *complementary monomial subspace* spanned by all monomials *not* in $\mathcal{N}$. In view of Lemma 6.1, it also forms a complement to the prolonged annihilator subspace $\mathcal{Z}|_{z^{(\infty)}}$, so

$$\mathcal{Q} = \mathcal{K} \oplus \mathcal{N} = \mathcal{K} \oplus \mathcal{Z}|_{z^{(\infty)}} \quad \text{and} \quad \mathcal{K} \simeq \hat{\mathcal{Q}}/\mathcal{Z}|_{z^{(\infty)}},$$

whenever $z^{(\infty)}$ is sufficiently near $z_0^{(\infty)}$. We will call the moving frame fixed by such an algebraic cross-section an *algebraic moving frame*. From now on, we exclusively work with such algebraic moving frames.

As noted earlier, we can identify the phantom differential invariants with the monomials in the complementary subspace $\mathcal{K}$, while the complete system of functionally independent normalized differential invariants is in one-to-one correspondence with the monomials in $\mathcal{N}$. Thus, while the differential invariants are not, in general, in correspondence with a monomial module, those of sufficiently high order are. In particular, if $\mathcal{G}$ represents the action of a finite-dimensional Lie group, then, for $n \gg 0$ sufficiently large (more than the classical order of freeness), $\mathcal{N}^n = \mathcal{Q}^n$ and $\mathcal{K}^n = \{0\}$, since there are only finitely many phantom differential invariants. On the other hand, for infinite-dimensional pseudo-groups, phantom differential invariants will occur at all orders, but, assuming we choose an algebraic moving frame, those of sufficiently high order are indexed by the elements (or, rather, their leading monomials) of the prolonged symbol module.

[†] In practical implementations of the algorithm, this ordering should be “compatible” ■ with the chosen term ordering on $\mathcal{R}$.

di 7. Differential Invariants.

We are now ready to present the proof of Theorem 1.1. We begin by fixing an algebraic moving frame, as in the preceding section. Let

$$\text{ndi} \quad H^i, I_J^\alpha \quad \text{for} \quad \tilde{s}_i, s^J S_\alpha \in \mathcal{N}, \quad (7.1)$$

be the induced system of functionally independent normalized differential invariants that serve to generate the algebra of differential invariants $\mathcal{I}(\mathcal{G})$. Further, let

$$\text{ih} \quad \varpi^i = \iota(dx^i), \quad \mathcal{D}_i = \iota(D_{x^i}), \quad i = 1, \dots, p, \quad (7.2)$$

be the invariantized horizontal coframe and associated dual invariant differential operators.

The goal of this section is to prove that, just as in the finite-dimensional Lie group version, the differential invariant algebra is generated, through invariant differentiation, by a finite number of independent differential invariants $I_1, \dots, I_l$. This will be a consequence of the recurrence relations that allow us to express differentiated invariants in terms of the normalized differential invariants (7.1). Since we already know the latter form a complete system of differential invariants, we need only prove that they can be expressed as functions of the differentiated invariants.

The key to understanding the structure of the differential invariant algebra $\mathcal{I}(\mathcal{G})$ is the fundamental recurrence formulae relating the normalized and differentiated differential invariants, which were established in [24], with further improvements in [9]. We begin by extending the invariantization process to include the derivatives (jets) of vector field coefficients[†] by setting

$$\text{ihmc} \quad \iota(\zeta_A^b) = \gamma_A^b, \quad a = 1, \dots, m, \quad \#A \geq 0, \quad (7.3)$$

where γ_A^b are the horizontal[‡] components of the pull-backs of the Maurer–Cartan forms for the pseudo-group. Since the final algorithm for constructing (7.3) will follow from the recurrence formulae, we will not take the time to recall how they are constructed, referring the curious reader to [9, 24]. For our purposes, there are just two key facts that are needed, from which everything else will follow. First, each invariantized Maurer–Cartan form

$$\text{hmc} \quad \gamma_A^b = \sum_{i=1}^p J_{A,i}^b \varpi^i \quad (7.4)$$

is a certain linear combination of the invariant horizontal forms (7.2), whose coefficients $J_{K,i}^b$ are certain differential invariants uniquely prescribed by the moving frame. As we will see, these coefficients can be explicitly determined from the recurrence formulae, and

[†] Each derivative ζ_A^b serves to define a linear function on the space of vector fields $\mathcal{X}(M)$, and so should properly be viewed as a differential form. Thus, the fact that its invariantization is another differential form should not come as a complete surprise.

[‡] Technically, [24], one should also include the contact components in the invariantization, but since these are not required here, by a slight abuse of terminology, we use “invariantization” to refer to the horizontal components alone.

so, while we do rely on the general constructions of [24] to establish the representation (7.4), the details of the derivation is irrelevant for what we are studying here.

More generally, the *invariantization* of any linear differential function

$$L(x, u^{(n)}, \zeta^{(n)}) = \sum_{b=1}^m \sum_{\#A \leq n} h_A^b(x, u^{(n)}) \zeta_A^b$$

depending on the submanifold jet coordinates $(x, u^{(n)}) \in J^n(M, p)$ and linear on the vector field coefficient jets (3.2) will be defined as the corresponding invariant linear combination

$$L(H, I^{(n)}, \gamma^n) = \sum_{b=1}^m \sum_{\#A \leq n} h_A^b(H, I^{(n)}) \gamma_A^b$$

of invariantized Maurer–Cartan forms. Then the second key fact, [24, 9], is that the invariantized Maurer–Cartan forms (7.3) are subject to the linear constraints

$$\text{ide} \quad L^{(\infty)}(\dots, H^i, \dots, I^\alpha, \dots, \gamma_A^b, \dots) = 0. \quad (7.5)$$

obtained by invariantizing the linear determining equations (3.6) for our pseudo-group. For computations, it is preferable to work with the invariantization of the solved triangular form (3.15) of the linear determining equations, which implies that we can write each invariantized Maurer–Cartan form as a linear combination

$$\gamma_A^b = - \sum_{r^B R_c \in \mathcal{C} \leq n} h_B^c(\dots, H^i, \dots, I^\alpha, \dots) \gamma_B^c \quad \text{for all} \quad r^A R_b \in \mathcal{M}^n, \quad n \geq 0, \quad (7.6)$$

of the one-forms corresponding to the monomial symbol module $\mathcal{N}$. By a slight abuse of terminology, we will refer to the latter forms, which are indexed by the complementary monomials in $\mathcal{C}$, as a “basis” for the space of invariantized Maurer–Cartan form, although we are ignoring the fact that additional linear dependencies amongst the forms may arise owing to their formulae (7.4) in terms of the invariant horizontal coframe. But once we have found these formulae for the basis invariantized Maurer–Cartan forms, the remaining invariantized Maurer–Cartan forms, corresponding to monomials in the monomial symbol module $\mathcal{M}$, are all then completely specified by the invariantized determining equations (7.6).

According to [9, 24], the horizontal recurrence formulae for the normalized differential invariants (6.8) assume the explicit form

$$\text{rf} \quad d_H H^i = \iota(dx^i + \xi^i) = \varpi^i + \sigma^i, \quad d_H I_J^\alpha = \iota(du_J^\alpha + \varphi_J^\alpha) = \sum_{i=1}^p I_{J,i}^\alpha \varpi^i + \psi_J^\alpha, \quad (7.7)$$

where

$$\text{igx} \quad \sigma^i = \iota(\xi^i), \quad \psi_J^\alpha = \iota(\varphi_J^\alpha), \quad (7.8)$$

denote the (horizontal) invariantizations of the prolonged infinitesimal generator coefficients (4.3). Or, to state this another way, we can obtain the formulae for ψ_J^α by invariantizing the prolongation map (4.6), by setting, for any $0 \leq n \leq \infty$,

$$\text{iprm} \quad \mathbf{P}^{(n)}(\gamma) = \psi, \quad \text{where} \quad \gamma = (\dots \gamma_A^b \dots), \quad \psi = (\dots \sigma^i \dots \psi_J^\alpha \dots), \quad (7.9)$$

represent, respectively, the horizontal invariantized Maurer–Cartan forms and the invariantized prolonged vector field coefficients. In coordinates, the invariantized prolongation map $\mathbf{P}$ is obtained from the ordinary vector field prolongation map $\mathbf{p} = \mathbf{p}^{(\infty)}$ by replacing all jet coordinates u_J^α by the corresponding normalized differential invariants I_J^α .

Remark: Since invariantization is implemented by restricting to the cross-section and then using invariance to recalibrate the object, the invariantized and ordinary maps *co-incide* when evaluated at any point $z^{(\infty)}$ on the chosen cross-section. Thus, each of the following facts is an immediate consequence of its non-invariantized counterpart.

Thus, the invariantized prolongation map (7.9) retains its algebraic structure at the symbol level. The dual map $\mathbf{P}^*: \hat{\mathcal{Q}} \rightarrow \mathcal{R}$ satisfies

$$\text{isvrU} \quad \lambda[\mathbf{P}^*(T(s, S))] = \hat{\beta}^*[\lambda(T(s, S))], \quad (7.10)$$

where $\hat{\beta}^*$ is the invariantized version of the pull-back map (4.18). Therefore,

$$\hat{\beta}^*[P(s_1, \dots, s_p, S_1, \dots, S_q)] = P(\hat{\beta}_1(r), \dots, \hat{\beta}_p(r), \hat{B}_1(R), \dots, \hat{B}_q(R)),$$

where

$$\text{ilk} \quad \hat{\beta}_i(r) = r_i + \sum_{\alpha=1}^q I_i^\alpha r_{p+\alpha}, \quad \hat{B}_\alpha(R) = R_{p+\alpha} - \sum_{i=1}^p I_i^\alpha R_i, \quad (7.11)$$

in which $I_i^\alpha = \iota(u_i^\alpha)$ indicate the order 1 normalized differential invariants specified by the moving frame.

Taking into account our choice of cross-section, the recurrence formulae (7.7) naturally split into two species. The *phantom recurrence formulae* take the form

$$\text{phrf} \quad 0 = \varpi^j + \sigma^j, \quad 0 = \sum_{i=1}^p I_{K,i}^\beta \varpi^i + \psi_K^\beta, \quad \tilde{s}_j, s^K S_\beta \in \mathcal{K}, \quad (7.12)$$

whose left hand sides vanish since the phantom differential invariants are constant. As we shall see, once expanded out, the phantom recurrence formulae can be uniquely solved for the basis invariantized Maurer–Cartan forms, and hence we deduce, via the invariantized determining equations, the desired formulas (7.4). We can now use the now known formulae for the invariantized Maurer–Cartan forms in the remaining *fundamental recurrence formulae*

$$\text{frf} \quad d_H H^i = \varpi^i + \sigma^i, \quad d_H I_J^\alpha = \sum_{i=1}^p I_{J,i}^\alpha \varpi^i + \psi_J^\alpha, \quad \tilde{s}_i, s^J S_\alpha \in \mathcal{N}, \quad (7.13)$$

which are then used to completely characterize the differential invariant algebra of the pseudo-group. The details and illustrative examples will appear below.

The individual coefficients of the horizontal one-forms $\varpi^1, \dots, \varpi^p$ in (7.13) form the system of *recurrence relations*, which are $p q^{(n)}$ equations relating the $q^{(n+1)}$ normalized differential invariants $I_{J,i}^\alpha = \iota(u_{J,i}^\alpha)$ of order $n+1$ to the differentiated n^{th} order invariants:

$$\text{rHI} \quad \mathcal{D}_i H^j = \delta_i^j + M_i^j, \quad \mathcal{D}_i I_J^\alpha = I_{J,i}^\alpha + M_{J,i}^\alpha, \quad (7.14)$$

where the *correction terms*, which are certain differential invariants, are the coefficients of the invariant horizontal forms in

$$\sigma^i = \sum_{j=1}^p M_i^j \varpi^j, \quad \psi_J^\alpha = \sum_{i=1}^p M_{J,i}^\alpha \varpi^i. \quad (7.15)$$

cterms

Thus, each correction term turns out to be a certain invariant linear combination of the invariantized Maurer–Cartan form coefficients $J_{A,i}^b$, cf. (7.4), which are completely prescribed by the invariantized prolongation map (7.9).

We are assuming that, at the base point $z_0^{(\infty)}$, the prolonged pseudo-group action is locally free at order n^* . Thus, by invariantization of the criterion in Definition 5.1, the invariantized prolongation map

$$\mathbf{P}^{(n^*)}: \gamma^{(n^*)} \longmapsto \psi^{(n^*)}$$

defines a linear monomorphism from the space of invariantized Maurer–Cartan forms of order $\leq n^*$ to the invariantized vector field coefficients up to order n^* . We can thus uniquely solve the recurrence formulae (7.7) of order $\leq n^*$ for the invariantized Maurer–Cartan forms γ_A^b for all $\#A \leq n^*$, so as to determine their formulae (7.4) in terms of the invariant horizontal coframe. The fact that we can uniquely solve the phantom recurrence formulae for the basis invariantized Maurer–Cartan forms follows immediately from that fact that we have normalized using a cross-section to the pseudo-group orbits. The remaining, non-phantom terms, corresponding to monomials in $\mathcal{N}^{\leq n^*}$, will give recurrence formulae relating the normalized and differentiated differential invariants of order $\leq n^*$. However, owing to the lack of algebraic structure before we reach the order of freeness of the pseudo-group, the newly computed invariantized Maurer–Cartan forms may depend upon *all* of these differential invariants (and not just those of lower order than the order of the Maurer–Cartan form) in an unpredictable, possibly nonlinear manner, and thus it seems hard to make any a priori predictions on how they may be effectively utilized to reduce the number of differential invariants required to generate the lower order differential invariants. Nevertheless, in all practical examples, these recurrence relations do play a significant role in simplifying and reducing the basis differential invariants. An intriguing open question, therefore, is how much this can be predicted in advance.

Moving on, we now know a (reduced) generating system of fundamental differential invariants of order $\leq n^*$. In order to determine the remaining generating differential invariants of higher order $n > n^*$, we will exploit the algebraic structure of the prolonged symbol module. The key lemma is:

keylemma

Lemma 7.1. *If $n > n^*$, the system of recurrence formulae (7.7) at order $n = \#J$ can be simultaneously uniquely solved for the $(n+1)^{\text{st}}$ order fundamental differential invariants I_K^β corresponding to monomials $s^K S_\beta \in \mathcal{N}^{n+1}$ and the basis n^{th} order pulled-back Maurer–Cartan forms γ_A^b corresponding to monomials $r^A R_b \in \mathcal{K}^n$.*

Proof: We already know that a solution to the recurrence formulae exists — since they are indeed satisfied by the normalized differential invariants. Thus, we need only show that the solution is unique, which we do by induction on the order n . To this end, we

assume that $n > n^*$ and that we have established the formulae (7.4) for the invariantized Maurer–Cartan forms of order $< n$, and hence, by substitution, the formulae (7.15) for the invariantized vector field coefficients of order $< n$. We can write the order n recurrence formulae, both phantom and fundamental, in the form

$$d_H I_J^\alpha = \sum_{i=1}^p I_{J,i}^\alpha \varpi^i + \psi_J^\alpha, \quad \#J = n. \quad (7.16)$$

rfn

We rewrite these formulae in vectorial form

$$\psi_n = \mathbf{P}_n(\gamma^{(n)}) = \widehat{\beta}(\gamma_n) + \delta^{(n-1)},$$

where

- ψ_n contains the invariantized prolonged vector field coefficients ψ_J^α of order $\#J = n$;
- $\gamma^{(n)}$ contains the invariantized Maurer–Cartan forms γ_A^b of orders $\#A \leq n$;
- $\mathbf{P}_n$ denotes the n^{th} order terms in the invariantized prolongation map (7.9);
- γ_n contains only the invariantized Maurer–Cartan forms γ_A^b of order $\#A = n$;
- $\widehat{\beta}(\gamma_n)$ indicates the leading terms in the invariantized prolonged vector field coefficients ψ_n , i.e., those containing only the invariantized Maurer–Cartan forms of order n ; its connection with (7.10) will be made clear shortly;
- $\delta^{(n-1)}$ indicates the linear combination of all invariantized Maurer–Cartan forms $\gamma^{(n-1)}$ of order $\leq n-1$ contained in ψ_n , which, by the induction hypothesis, have all been explicitly computed, so

$$\delta^{(n-1)} = \sum_{i=1}^p W_{J,i}^\alpha \varpi^i$$

is a known linear combination, with differential invariant coefficients, of the horizontal invariant coframe.

Further, let us write

$$\mathbf{I}_{n+1} = (\dots I_K^\alpha \dots), \quad \#K = n+1,$$

for the vector whose components are all the normalized differential invariants of order $n+1$, and

$$d_H \mathbf{I}_n = (\dots d_H I_J^\alpha \dots), \quad \#J = n,$$

for the vector containing the horizontal differentials of all the n^{th} order normalized differential invariants. We further split

$$\mathbf{I}_{n+1} = \mathbf{J}_{n+1} + \mathbf{c}_{n+1}$$

into fundamental and phantom components, whose respective entries

$$J_K^\alpha = \begin{cases} I_K^\alpha, & s^K S_\alpha \in \mathcal{N}^{n+1}, \\ 0, & s^K S_\alpha \in \mathcal{K}^{n+1}, \end{cases} \quad c_K^\alpha = \begin{cases} 0, & s^K S_\alpha \in \mathcal{N}^{n+1}, \\ I_K^\alpha, & s^K S_\alpha \in \mathcal{K}^{n+1}. \end{cases}$$

Note that the entries of the phantom part $\mathbf{c}_{n+1}$ are the constant, phantom differential invariants whose values are fixed by the normalization constants (6.2), while those of $\mathbf{J}_{n+1}$ are the as yet unknown non-constant normalized differential invariants of order $n + 1$.

Let

$$\alpha(\mathbf{J}_{n+1}) + \alpha(\mathbf{c}_{n+1}) = \alpha(\mathbf{I}_{n+1}) = \left(\dots \sum_{i=1}^p I_{J,i}^\alpha \varpi^i \dots \right), \quad \#J = n, \quad (7.17)$$

indicate the linear map that produces the first set of forms on the right hand side of (7.16), which we rewrite in vectorial form

$$d_H \mathbf{I}_n = \alpha(\mathbf{I}_{n+1}) + \psi_n = \alpha(\mathbf{J}_{n+1}) + \widehat{\beta}(\gamma_n) + \alpha(\mathbf{c}_{n+1}) + \delta^{(n-1)}. \quad (7.18)$$

In this identity, our induction hypothesis implies that the left hand side and the third and fourth terms on the right hand side are known. Our goal is to prove that we can simultaneously solve this linear system *uniquely* for the non-phantom differential invariants $\mathbf{J}_{n+1}$ and the invariantized Maurer–Cartan forms γ_n , subject to the invariantized determining equations (7.5). Since we are assured of the existence of a solution – since the differential invariants are already specified by the moving frame normalizations, and so necessarily satisfy (7.18) — our only task is to check uniqueness, and so we need to prove that the associated homogeneous linear system

$$\alpha(\mathbf{J}_{n+1}) + \widehat{\beta}(\gamma_n) = 0 \quad (7.19)$$

has only the trivial solution: $\mathbf{J}_{n+1} = 0$, $\gamma_n = 0$.

To reveal the underlying algebraic structure, we identify the first three vectors of one-forms in (7.18) as elements of the dual vector space to the polynomial module summand $\mathcal{Q}^n \otimes \mathbb{R}^p$ via the pairings[†]

$$\langle d_H \mathbf{I}_n; s^J S_\alpha t_i \rangle = \mathcal{D}_i I_J^\alpha, \quad \langle \alpha(\mathbf{I}_{n+1}); s^J S_\alpha t_i \rangle = I_{J,i}^\alpha, \quad \langle \psi_n; s^J S_\alpha t_i \rangle = M_{J,i}^\alpha,$$

whenever $\#J = n$, where $M_{J,i}^\alpha$ are the correction terms defined in (7.15). We further identify $\mathbf{I}_{n+1} \in (\mathcal{Q}^{n+1})^*$ by adapting the pairing (4.12):

$$\langle \mathbf{I}_{n+1}; s^J S_\alpha \rangle = I_K^\alpha, \quad \#K = n + 1.$$

Finally, we identify γ_n as belonging to the dual space to $\mathcal{R}^n \otimes \mathbb{R}^p$ via the pairing

$$\langle \gamma_n; r^A R_b t_i \rangle = J_{A,i}^b, \quad \#A = n,$$

where $J_{A,i}^b$ are the (as yet unknown) coefficients in the representation (7.4). Then the linear maps

$$\alpha: (\mathcal{Q}^{n+1})^* \longrightarrow (\mathcal{Q}^n \otimes \mathbb{R}^p)^*, \quad \widehat{\beta}: (\mathcal{R}^n \otimes \mathbb{R}^p)^* \longrightarrow (\mathcal{Q}^n \otimes \mathbb{R}^p)^*,$$

[†] We use $(t_1, \dots, t_p) \in \mathbb{R}^p$ to indicate the polynomial variables dual to the one-forms $\varpi^1, \dots, \varpi^p$.

appearing in the recurrence formulae (7.18) are the duals (or, in matrix terms, transposes) to the polynomial maps

$$\begin{aligned} \alpha^*: \mathcal{Q}^n \otimes \mathbb{R}^p &\longrightarrow \mathcal{Q}^{n+1}, & \alpha^* \left[\sum_{i=1}^p T_i(s, S) t_i \right] &= \sum_{i=1}^p s_i T_i(s, S), \\ \widehat{\beta}^*: \mathcal{Q}^n \otimes \mathbb{R}^p &\longrightarrow \mathcal{R}^n \otimes \mathbb{R}^p, & \widehat{\beta}^* \left[\sum_{i=1}^p T_i(s, S) t_i \right] &= \sum_{i=1}^p T_i(\widehat{\beta}(r), \widehat{B}(R)) t_i, \end{aligned} \quad (7.20)$$

AB

where the linear polynomials $\widehat{\beta}_i(r), \widehat{B}_\alpha(R)$ were defined in (7.11).

Thus, to prove Lemma 7.1 we need to show that the direct sum map

$$\begin{aligned} \alpha \oplus \widehat{\beta}: (\mathcal{N}^{n+1})^* \oplus (\mathcal{C}^n \otimes \mathbb{R}^p)^* &\longrightarrow (\mathcal{Q}^n \otimes \mathbb{R}^p)^*, \\ [\alpha \oplus \widehat{\beta}](\mathbf{J}_{n+1}, \gamma_n) &= \alpha(\mathbf{J}_{n+1}) + \widehat{\beta}(\gamma_n), \end{aligned}$$

has trivial kernel: $\ker(\alpha \oplus \widehat{\beta}) = \{0\}$. By duality, this holds if and only if the dual map

$$\alpha^* \oplus \widehat{\beta}^*: \mathcal{Q}^n \otimes \mathbb{R}^p \longrightarrow (\mathcal{N}^n \otimes \mathbb{R}^p) \oplus [(\mathcal{R}^n / \mathcal{S}^n) \otimes \mathbb{R}^p] \simeq \mathcal{N}^{n+1} \oplus (\mathcal{C}^n \otimes \mathbb{R}^p)$$

is onto. To see why this is the case, we split the domain space using the splitting of monomials induced by the prolonged symbol module:

$$\mathcal{Q}^n \otimes \mathbb{R}^p = (\mathcal{N}^n \otimes \mathbb{R}^p) \oplus (\mathcal{K}^n \otimes \mathbb{R}^p).$$

By local freeness,

$$\widehat{\beta}^*: \mathcal{K}^n \otimes \mathbb{R}^p \longrightarrow (\mathcal{R}^n / \mathcal{S}^n) \otimes \mathbb{R}^p \simeq \mathcal{C}^n \otimes \mathbb{R}^p$$

is onto provided $n > n^*$. On the other hand, the multiplication map

$$\alpha^*: \mathcal{N}^n \otimes \mathbb{R}^p \longrightarrow \mathcal{N}^{n+1}$$

is onto if and only if every monomial in $\mathcal{N}^{n+1}$ can be expressed as the product of one of the variables s_i with a monomial in $\mathcal{N}^n$. This occurs if and only if $n \geq n_{\max}$, where $n_{\max}$ is the maximal degree of the Gröbner basis monomials in $\mathcal{N}$. Indeed, every monomial in $\mathcal{N}$ has this product form *except* for the Gröbner basis monomials, and so the corresponding normalized differential invariants cannot (at least a priori) be determined by solving the recurrence formulae.

This completes the proof of Lemma 7.1. The Constructive Basis Theorem for the differential invariant algebra follows immediately.

Tressec

Theorem 7.2. *There exists an order n^* such that a generating system of differential invariants consists of*

- (a) *a finite collection of differential invariants of order $< n^*$, and*
- (b) *the differential invariants $I_\kappa = I_{J_\kappa}^{\alpha_\kappa}$ corresponding to a Gröbner basis $m_\kappa = s^{J_\kappa} S_{ka_\kappa}$ for the monomial module $\mathcal{N}^{>n^*} = \mathcal{W}^{>n^*}$.*

Remark: A priori, one might need to include all differential invariants of order $\leq n^*$ and the Gröbner basis differential invariants to form the generating system. However, in practice, many of these (including, possibly, from the Gröbner basis) are redundant, as they can be written as invariant derivatives of lower order differential invariants in the system. The explicit recurrence formulae for the differentiated invariants, as discussed below, will allow one to systematically eliminate redundant differential invariants from the generating system, and lead to a minimal generating system of differential invariants.

x23 **Example 7.3.** Examples ■

syz **8. Syzygies.**

to be done ■

zzz **9. Further directions.**

Some further directions and applications to be considered:

One question is whether the order of the Gröbner basis for $\mathcal{K}$ can be chosen to have order equal to the order of freeness for the pseudo-group.

Explicit connections between the Gröbner basis approach and the more usual Cartan–Kähler and Spencer sequence methods needs to be worked out.

Nonfree actions.

Recursive construction of differential invariants as in [12].

We have not tried to precisely correlate our bounds with those in Kumpera’s work.

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