

Appendix A

Complex Numbers

The purpose of this short appendix is to review the basics of complex numbers and complex arithmetic. Complex function theory and calculus is the subject of Chapter 7.

A *complex number* is an expression of the form $z = x + iy$, where $x, y \in \mathbb{R}$ are real and $i = \sqrt{-1}$. The set of all complex numbers is denoted by \mathbb{C} . We call $x = \operatorname{Re} z$ the *real part* of z and $y = \operatorname{Im} z$ the *imaginary part* of $z = x + iy$. (Note: The imaginary part is the real number y , not iy .) A real number x is merely a complex number with zero imaginary part, $\operatorname{Im} z = 0$, and so we may regard $\mathbb{R} \subset \mathbb{C}$. Complex addition and multiplication are based on simple adaptations of the rules of real arithmetic to include the identity $i^2 = -1$, and so

$$\begin{aligned}(x + iy) + (u + iv) &= (x + u) + i(y + v), \\ (x + iy)(u + iv) &= (xu - yv) + i(xv + yu).\end{aligned}\tag{A.1}$$

Complex numbers enjoy all the usual laws of real addition and multiplication, *including commutativity*: $zw = wz$.

We can identify a complex number $x + iy$ with a vector $(x, y) \in \mathbb{R}^2$ in the real, two-dimensional plane. For this reason, \mathbb{C} is sometimes referred to as the *complex plane*. (Although keep in mind that, as a complex vector space, \mathbb{C} is only one-dimensional.) Based on this identification, we shall employ the standard terminology of planar vector calculus, e.g., domain, curve, etc., without alteration. Complex addition (A.1) corresponds to vector addition, but the vector interpretation of complex multiplication is more obscure.

The *complex conjugate* of $z = x + iy$ is $\bar{z} = x - iy$. Note that $\operatorname{Re} \bar{z} = \operatorname{Re} z$, while $\operatorname{Im} \bar{z} = -\operatorname{Im} z$. Geometrically, the complex conjugate of z is obtained by reflecting the corresponding vector through the real axis, as illustrated in Figure A.1. In particular $\bar{\bar{z}} = z$ if and only if z is real. In general,

$$\operatorname{Re} z = \frac{z + \bar{z}}{2}, \quad \operatorname{Im} z = \frac{z - \bar{z}}{2i}.\tag{A.2}$$

Complex conjugation is compatible with complex arithmetic:

$$\overline{z + w} = \bar{z} + \bar{w}, \quad \overline{zw} = \bar{z}\bar{w}.$$

In particular, the product of a complex number and its conjugate

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2\tag{A.3}$$

is real and non-negative. Its square root is known as the *modulus* or *norm* of the complex number $z = x + iy$, and written

$$|z| = \sqrt{x^2 + y^2}.\tag{A.4}$$

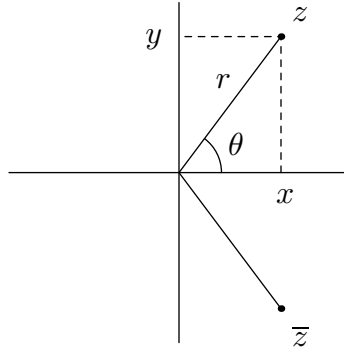


Figure A.1. Complex Numbers.

Note that $|z| \geq 0$, with $|z| = 0$ if and only if $z = 0$. The modulus $|z|$ generalizes the absolute value of a real number, and coincides with the standard Euclidean norm in the xy -plane. This implies the validity of the triangle inequality

$$|z + w| \leq |z| + |w|. \quad (\text{A.5})$$

Equation (A.3) can be rewritten in terms of the modulus as

$$z \bar{z} = |z|^2. \quad (\text{A.6})$$

Rearranging the factors, we deduce the formula for the reciprocal of a nonzero complex number:

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}, \quad z \neq 0, \quad \text{or, equivalently,} \quad \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}. \quad (\text{A.7})$$

The general formula for complex division,

$$\frac{w}{z} = \frac{w \bar{z}}{|z|^2} \quad \text{or} \quad \frac{u + iv}{x + iy} = \frac{(xu + yv) + i(xv - yu)}{x^2 + y^2}, \quad (\text{A.8})$$

is an immediate consequence.

The modulus of a complex number,

$$r = |z| = \sqrt{x^2 + y^2},$$

is one component of its polar coordinate representation

$$x = r \cos \theta, \quad y = r \sin \theta \quad \text{or} \quad z = r(\cos \theta + i \sin \theta). \quad (\text{A.9})$$

The polar angle θ , which measures the angle that the line connecting z to the origin makes with the horizontal axis, is known as the *phase*, and written

$$\theta = \text{ph } z. \quad (\text{A.10})$$

As such, the phase is only defined up to an integer multiple of 2π . The unique *principal value* of the phase is restricted to $-\pi < \text{ph } z \leq \pi$. A more common term for the polar angle is the *argument* of z , written $\arg z = \text{ph } z$. However, in conformity with [104, 105], we prefer to use “phase” throughout this text, in part to avoid confusion with the argument z of a function $f(z)$.

Euler’s celebrated formula for the complex exponential,

$$e^{i\theta} = \cos \theta + i \sin \theta, \quad (\text{A.11})$$

can be used to compactly rewrite the polar form (A.9) of a complex number as

$$z = r e^{i\theta} \quad \text{where} \quad r = |z|, \quad \theta = \text{ph } z. \quad (\text{A.12})$$

We note that the modulus and phase of a product of complex numbers can be readily computed:

$$|zw| = |z| |w|, \quad \text{ph}(zw) = \text{ph } z + \text{ph } w, \quad (\text{A.13})$$

the latter formula requiring that we allow multiply-valued phases; the formula does *not* hold as stated for all z, w when the principal value of the phase is used. Similarly, the modulus and phase of the reciprocal of a non-zero complex number are

$$\left| \frac{1}{z} \right| = \frac{1}{|z|}, \quad \text{ph} \left(\frac{1}{z} \right) = -\text{ph } z. \quad (\text{A.14})$$

On the other hand, complex conjugation preserves the modulus, but negates the phase:

$$|\bar{z}| = |z|, \quad \text{ph } \bar{z} = -\text{ph } z. \quad (\text{A.15})$$

The latter formula is not valid for the principal value of the phase when z lies on the negative real axis. (For this reason, the handbook [105] advocates retaining two principal values of $\pm\pi$ for the phase of points on the negative real axis; this refinement will not play a role in this text.)