

Chapter 3

Fourier Series

Just before 1800, the French mathematician/physicist/engineer Jean Baptiste Joseph Fourier made an astonishing discovery. Through his deep analytical investigations into the partial differential equations modeling heat propagation in bodies, Fourier was led to claim that “every” function could be represented by an infinite series of elementary trigonometric functions: sines and cosines. For example, consider the sound produced by a musical instrument, e.g., piano, violin, trumpet, or drum. Decomposing the signal into its trigonometric constituents reveals the fundamental frequencies (tones, overtones, etc.) that combine to produce the instrument’s distinctive timbre. This Fourier decomposition lies at the heart of modern electronic music; a synthesizer combines pure sine and cosine tones to reproduce the diverse sounds of instruments, both natural and artificial, according to Fourier’s general prescription.

Fourier’s claim was so remarkable and counter-intuitive that most of the leading mathematicians of the time did not believe him. Nevertheless, it was not long before scientists came to appreciate the power and far-ranging applicability of Fourier’s method, thereby opening up vast new realms of mathematics, physics, engineering, and elsewhere. Indeed, Fourier’s discovery easily ranks in the “top ten” mathematical advances of all time, a list that would also include Newton’s invention of the calculus, and Gauss and Riemann’s differential geometry that, 70 years later, became the foundation of Einstein’s general relativity. Fourier analysis is an essential component of much of modern applied (and pure) mathematics. It forms an exceptionally powerful analytical tool for solving a broad range of partial differential equations. Applications in pure mathematics, physics and engineering are almost too numerous to catalogue: typing the word “Fourier” in the subject index of a modern science library will dramatically demonstrate just how ubiquitous these methods are. Fourier analysis lies at the heart of signal processing, including audio, speech, images, videos, seismic data, radio transmissions, and so on. Many modern technological advances, including television, music CD’s and DVD’s, movies, computer graphics, image processing, and fingerprint analysis and storage, are, in one way or another, founded upon the many ramifications of Fourier theory. In your career as a mathematician, scientist or engineer, you will find that Fourier theory, like calculus and linear algebra, is one of the most basic and essential weapons in your mathematical arsenal. Mastery of the subject is unavoidable.

Furthermore, a remarkably large fraction of modern mathematics rests on subsequent attempts to place Fourier series on a firm mathematical foundation. Thus, all of modern analysis’ most basic analytical tools, including the definition of a function, the ε - δ definition of limit and continuity, convergence properties in function space, the modern

theory of integration and measure, generalized functions such as the delta function, and many others, all owe a profound debt to the prolonged struggle to establish a rigorous framework for Fourier analysis. Even more remarkably, modern set theory, and, thus, the foundations of modern mathematics and logic, can be traced directly back to the nineteenth century German mathematician Georg Cantor's attempts to understand the sets upon which Fourier series converge!

We begin our development of Fourier methods by explaining why Fourier series naturally appear when we try to solve the one-dimensional heat equation. The reader uninterested in such motivations can safely omit this initial section as the same material reappears in Chapter 4, when we apply Fourier methods to solve several important linear partial differential equations. Beginning in Section 3.2, we shall introduce the most basic computational techniques for Fourier series. The final section is an abbreviated introduction to the analytical background required to develop a rigorous foundation for Fourier series methods. While it requires more analytical sophistication than what has appeared earlier, the reader is strongly encouraged to go through it, for additional insight and further results of importance in applications.

3.1. Eigensolutions to Linear Evolution Equations.

The next important partial differential equation to merit study is the second order linear equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad (3.1)$$

known as the *heat equation* since it models (among other diffusion processes) heat flow in a one-dimensional medium, e.g., a metal bar. For simplicity, we have set the physical parameters equal to 1 in order to focus on the solution techniques. A more complete discussion, including a brief derivation from physical principles, will appear in Chapter 4. Unlike the wave equation considered in Chapter 2, there is no comparably elementary formula for the general solution to the heat equation. Instead, we will write solutions as infinite series in certain simple, explicit solutions. This solution method, pioneered by Fourier, will lead us immediately to the definition of a Fourier series. The remainder of this chapter will be devoted to developing the basic properties and calculus of Fourier series. Once we have mastered these essential mathematical techniques, we will apply them to solving partial differential equations in Chapter 4.

Let us begin by writing the heat equation (3.1) in a more abstract, but suggestive linear *evolutionary form*

$$\frac{\partial u}{\partial t} = L[u], \quad (3.2)$$

in which

$$L[u] = \frac{\partial^2 u}{\partial x^2} \quad (3.3)$$

is a linear second order differential operator. Recall, (1.11), that linearity imposes two requirements on the operator L :

$$L[u + v] = L[u] + L[v], \quad L[cu] = cL[u], \quad (3.4)$$

for any functions[†] u, v and any constant c . Moreover, since L only involves differentiation with respect to x , it also satisfies

$$L[c(t)u] = c(t)L[u] \quad (3.5)$$

for any function $c(t)$ that does not depend on x .

Of course, there are many other possible linear differential operators, and so our abstract linear evolution equation (3.2) can represent a wide range of linear partial differential equations. For example, if

$$L[u] = -c(x) \frac{\partial u}{\partial x}, \quad (3.6)$$

where $c(x)$ is a function representing the wave speed in a nonuniform medium, then (3.2) becomes the transport equation

$$\frac{\partial u}{\partial t} = -c(x) \frac{\partial u}{\partial x} \quad (3.7)$$

that we studied in Chapter 2. If

$$L[u] = \frac{1}{\sigma(x)} \frac{\partial}{\partial x} \left(\kappa(x) \frac{\partial u}{\partial x} \right), \quad (3.8)$$

where $\sigma(x) > 0$ represents heat capacity, and $\kappa(x) > 0$ *thermal conductivity*, then (3.2) becomes the *generalized heat equation*

$$\frac{\partial u}{\partial t} = \frac{1}{\sigma(x)} \frac{\partial}{\partial x} \left(\kappa(x) \frac{\partial u}{\partial x} \right), \quad (3.9)$$

governing the diffusion of heat in a non-uniform bar. If

$$L[u] = \frac{\partial^2 u}{\partial x^2} - \gamma u, \quad (3.10)$$

where $\gamma > 0$ is a positive constant, then (3.2) becomes the *damped heat equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \gamma u \quad (3.11)$$

that models the temperature of a bar that is cooling off due to radiation of heat energy. We can even take u to be a function of more than one space variables, e.g., $u(t, x, y)$ or $u(t, x, y, z)$, in which case (3.2) includes higher dimensional versions of the heat equation for plates and solid bodies that we will study in due course. In all cases, the key requirements on the operator L are (a) linearity, and (b) only differentiation with respect to the spatial variables is allowed.

Fourier's inspired idea for solving such linear evolution equations, is a direct adaptation of the eigensolution method for first order linear systems of ordinary differential equations,

[†] We assume throughout that the functions are sufficiently smooth so that the indicated derivatives are well defined.

[17, 23, 108], which we now recall. The starting point is the elementary scalar ordinary differential equation

$$\frac{du}{dt} = \lambda u. \quad (3.12)$$

The general solution is an exponential function

$$u(t) = c e^{\lambda t}, \quad (3.13)$$

whose coefficient c is an arbitrary constant.

This elementary observation motivates the solution method for a first order homogeneous, linear system of ordinary differential equations

$$\frac{d\mathbf{u}}{dt} = A \mathbf{u}, \quad (3.14)$$

in which A is a constant $n \times n$ matrix. Working by analogy, we will seek solutions of exponential form

$$\mathbf{u}(t) = e^{\lambda t} \mathbf{v}, \quad (3.15)$$

where $\mathbf{v} \in \mathbb{R}^n$ is a constant vector. We substitute this *ansatz*[†] into the equation. First,

$$\frac{d\mathbf{u}}{dt} = \frac{d}{dt} (e^{\lambda t} \mathbf{v}) = \lambda e^{\lambda t} \mathbf{v}.$$

On the other hand, since $e^{\lambda t}$ is a scalar, it commutes with matrix multiplication, and so

$$A \mathbf{u} = A e^{\lambda t} \mathbf{v} = e^{\lambda t} A \mathbf{v}.$$

Therefore, $\mathbf{u}(t)$ will solve the system (3.14) if and only if \mathbf{v} satisfies

$$A \mathbf{v} = \lambda \mathbf{v}. \quad (3.16)$$

We recognize this as the *eigen-equation* that determines the eigenvalues of the matrix A . Namely, (3.16) has a non-zero solution $\mathbf{v} \neq \mathbf{0}$ if and only if λ is an *eigenvalue* and \mathbf{v} a corresponding *eigenvector*. Each eigenvalue λ and eigenvector \mathbf{v} produces an exponentially varying *eigensolution* (3.15) to the linear system of ordinary differential equations.

Remark: Any nonzero scalar multiple of an eigenvector $\hat{\mathbf{v}} = c \mathbf{v}$, for $c \neq 0$, is automatically another eigenvector for the same eigenvalue λ . However, the only effect is to multiply the eigensolution by the scalar c . Thus, to obtain a complete system of independent solutions, we only need use the independent eigenvectors.

[†] The German word *ansatz* (plural *ansätze*) refers to the method of finding a solution to a complicated equation by postulating that it be of a special form. Usually, an *ansatz* will depend on one or more free parameters — in this case the entries of the vector \mathbf{v} along with the scalar λ — that, with some luck, can be adjusted to fulfill the requirements imposed by the equation. Thus, a reasonable English translation of “*ansatz*” is “inspired guess”.

For simplicity — and also because *all* of the linear partial differential equations we will treat will have the analogous property — suppose that the $n \times n$ matrix A has a *complete* system of real eigenvalues $\lambda_1, \dots, \lambda_n$ and corresponding real, linearly independent eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, which therefore form an *eigenvector basis* of the underlying space \mathbb{R}^n . (We allow the possibility of repeated eigenvalues, but require that all eigenvectors be independent to avoid superfluous solutions.) For example, according to Theorem B.26 (see also [108; Theorem 8.20]), all real, symmetric matrices, $A = A^T$, are complete. Complex eigenvalues lead to complex exponential solutions, whose real and imaginary parts can be used to construct the associated real solutions. Incomplete matrices, having an insufficient number of eigenvectors, are more tricky, and the solution to the corresponding linear system requires use of the Jordan canonical form, [108; Section 8.6]. Fortunately, we do not have to deal with the latter, technically annoying cases here.

Using our completeness assumption, we can produce n independent real exponential eigensolutions

$$u_1(t) = e^{\lambda_1 t} \mathbf{v}_1, \quad \dots \quad u_n(t) = e^{\lambda_n t} \mathbf{v}_n,$$

to the linear system (3.14). The Linear Superposition Principle of Theorem 1.4 tells us that, for any choice of scalars c_1, \dots, c_n , the linear combination

$$c_1 u_1(t) + \dots + c_n u_n(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + \dots + c_n e^{\lambda_n t} \mathbf{v}_n \quad (3.17)$$

is also a solution. The basic Existence and Uniqueness Theorems for first order systems of ordinary differential equations, [23, 108] implies that (3.17) forms the *general solution* to the original linear system, and so the eigensolutions form a basis for the solution space.

Let us now adapt this seminal idea to construct exponentially varying solutions to the heat equation (3.1) or, for that matter, any linear evolution equation in the form (3.2). To this end, we introduce an analogous exponential ansatz:

$$u(t, x) = e^{\lambda t} v(x), \quad (3.18)$$

in which we replace the vector \mathbf{v} in (3.15) by a function $v(x)$. We substitute the expression (3.18) into the dynamical equations (3.2). First, the time derivative of such a function is

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} [e^{\lambda t} v(x)] = \lambda e^{\lambda t} v(x).$$

On the other hand, in view of (3.5),

$$L[u] = L[e^{\lambda t} v(x)] = e^{\lambda t} L[v].$$

Equating these two expressions and canceling the common exponential factor, we conclude that $v(x)$ must satisfy the *eigen-equation*

$$L[v] = \lambda v \quad (3.19)$$

for the linear differential operator L , in which λ is the *eigenvalue*, while $v(x)$ is the corresponding *eigenfunction*. Each eigenvalue and eigenfunction pair will produce an exponentially varying *eigensolution* (3.18) to the partial differential equation (3.2). We will then appeal to Linear Superposition to combine the resulting eigensolutions to form additional

solutions. The key complication is that partial differential equations admit an infinite number of independent eigensolutions, and thus one cannot hope to write the general solution as a finite linear combination thereof. Rather, one is led to try constructing solutions as *infinite series* in the eigensolutions. However, justifying such series solution formulae requires a quantum leap in our analytical prowess. Not every infinite series converges to a bona fide function. Moreover, a convergent series of differentiable functions need not converge to a differentiable function, and hence the series may not represent a (classical) solution to the partial differential equation. We are being reminded, yet again, that partial differential equations are far wilder than their relatively tame cousins, ordinary differential equations.

Let's, for specificity, focus our attention on the heat equation, for which the linear operator L is given by (3.3). If $v(x)$ is a function of x alone,

$$L[v] = v''(x).$$

Thus, our eigen-equation (3.19) becomes

$$v'' = \lambda v. \tag{3.20}$$

This is a linear, second order ordinary differential equation for $v(x)$, and so has two linearly independent solutions. The explicit solution formulas depend on the sign of the eigenvalue λ , and can be found in any basic text on ordinary differential equations, e.g., [17, 23, 40]. The following table summarizes the results for real eigenvalues λ ; the case of complex λ is left as Exercise ■ for the reader.

Real Eigensolutions of the Heat Equation

λ	Eigenfunctions $v(x)$	Eigensolutions $u(t, x) = e^{\lambda t} v(x)$
$\lambda = -\omega^2 < 0$	$\cos \omega x, \sin \omega x$	$e^{-\omega^2 t} \cos \omega x, e^{-\omega^2 t} \sin \omega x$
$\lambda = 0$	$1, x$	$1, x$
$\lambda = \omega^2 > 0$	$e^{-\omega x}, e^{\omega x}$	$e^{\omega^2 t - \omega x}, e^{\omega^2 t + \omega x}$

The resulting exponential eigensolutions are also referred to as *separable solutions* to indicate that they are the product of a function of t alone times a function of x alone. The general method of separation of variables will be one of our main tools for solving linear partial differential equations, and developed in detail starting in Chapter 4.

Remark: Thus, in the absence of boundary conditions, each real number λ qualifies as an eigenvalue of the linear differential operator (3.3), possessing two linearly independent eigenfunctions, and thus two linearly independent eigensolutions to the heat equation. As with eigenvectors, any (non-zero) linear combination of eigenfunctions (eigensolutions)

with the same eigenvalue is also an eigenfunction (eigensolution). Thus, the table only lists independent eigenfunctions and eigensolutions.

As noted above, any *finite* linear combination of these basic eigensolutions is automatically a solution. Thus, for example,

$$u(t, x) = c_1 e^{-t} \cos x + c_2 e^{-4t} \sin 2x + c_3 x + c_4$$

is a solution to the heat equation for any choice of constants c_1, c_2, c_3, c_4 , as you can easily check. But, since there are infinitely many independent eigensolutions, we cannot expect to be able to represent *every* solution to the heat equation as a finite linear combination of eigensolutions. And so, we must learn how to deal with *infinite series* of eigensolutions.

Remark: The first class of eigensolutions, where $\lambda < 0$, are exponentially decaying, which is in accord with our physical intuition as to how the temperature of a body should behave. The second class are constant in time — also physically reasonable. However, the third class, corresponding to positive eigenvalues $\lambda > 0$, are exponentially growing in time. In the absence of external heat sources, physical bodies should approach some sort of thermal equilibrium, and certainly not an exponentially growing temperature! However, notice that the latter eigensolutions (as well as the solution x) are not bounded in space, and so include an infinite amount of heat energy being supplied to the system from infinity. As we will come to appreciate, physically relevant boundary conditions — either posed on a bounded interval, or by specifying the asymptotics of the solutions at large distances — will serve to separate out the physically reasonable solutions from the mathematically valid but physically irrelevant cases.

The Heated Ring

So far, we have not paid any attention to boundary conditions. As noted above, these will serve to eliminate non-physical eigensolutions, and thereby reduce them to a manageable, albeit still infinite number. In this subsection, we will discuss a particularly important case, that, following Fourier's line of reasoning, leads us directly into the heart of Fourier series.

Consider the heat equation on the interval $-\pi \leq x \leq \pi$, subject to the *periodic boundary conditions*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(t, -\pi) = u(t, \pi), \quad \frac{\partial u}{\partial x}(t, -\pi) = \frac{\partial u}{\partial x}(t, \pi). \quad (3.21)$$

The physical problem being modeled is the thermodynamical behavior of an insulated circular ring, in which x represents the angular coordinate. The boundary conditions ensure that the temperature remains continuously differentiable at the junction point where the angle switches over from $-\pi$ to π . Given the ring's initial temperature distribution

$$u(0, x) = f(x), \quad -\pi \leq x \leq \pi, \quad (3.22)$$

our task is to determine the temperature of the ring $u(t, x)$ at each subsequent time $t > 0$.

Let us find out which of the preceding eigensolutions respect the boundary conditions. Substituting our exponential ansatz (3.18) into the differential equation and boundary

conditions (3.21), we find that the eigenfunction $v(x)$ must satisfy the periodic boundary value problem

$$v'' = \lambda v, \quad v(-\pi) = v(\pi), \quad v'(-\pi) = v'(\pi). \quad (3.23)$$

Our task is to find those values of λ for which (3.23) has a non-zero solution $v(x) \not\equiv 0$. These are the eigenvalues and eigenfunctions.

As noted above, there are three cases, depending on the sign of λ . First, suppose $\lambda = \omega^2 > 0$. Then the general solution to the ordinary differential equation is

$$v(x) = ae^{\omega x} + be^{-\omega x},$$

where a, b are arbitrary constants. Substituting into the boundary conditions, we find that a, b must satisfy the pair of linear equations

$$ae^{-\omega\pi} + be^{\omega\pi} = ae^{\omega\pi} + be^{-\omega\pi}, \quad a\omega e^{-\omega\pi} - b\omega e^{\omega\pi} = a\omega e^{\omega\pi} - b\omega e^{-\omega\pi}.$$

Since $\omega \neq 0$, the first equation implies that $a = b$, while the second requires $a = -b$. So, the only way to satisfy both boundary conditions is to take $a = b = 0$, and so $v(x) \equiv 0$ is a trivial solution. We conclude that there are no positive eigenvalues.

Second, if $\lambda = 0$, then the ordinary differential equation reduces to $v'' = 0$, with solution

$$v(x) = a + bx.$$

Substituting into the boundary conditions requires

$$a - b\pi = a + b\pi, \quad b = b.$$

The first equation implies that $b = 0$, but this is the only condition. Therefore, any constant function, $v(x) = a$, solves the boundary value problem, and hence $\lambda = 0$ is an eigenvalue. We take $v_0(x) \equiv 1$ as the unique independent eigenfunction, bearing in mind that any constant multiple of an eigenfunction is automatically also an eigenfunction. We will call 1 a *null eigenfunction*, indicating that it is associated with the zero eigenvalue $\lambda = 0$. The corresponding eigensolution (3.18) is $u(t, x) = e^{0t}v_0(x) = 1$, a constant solution to the heat equation.

Finally, we must deal with the case $\lambda = -\omega^2 < 0$. Now, the general solution to the differential equation in (3.23) is a trigonometric function:

$$v(x) = a \cos \omega x + b \sin \omega x. \quad (3.24)$$

Since

$$v'(x) = -a\omega \sin \omega x + b\omega \cos \omega x,$$

when we substitute into the boundary conditions, we find

$$\begin{aligned} a \cos \omega \pi - b \sin \omega \pi &= a \cos \omega \pi + b \sin \omega \pi, \\ a \sin \omega \pi + b \cos \omega \pi &= -a \sin \omega \pi + b \cos \omega \pi, \end{aligned}$$

where we canceled out a common factor of ω in the second equation. These simplify to

$$2b \sin \omega \pi = 0, \quad 2a \sin \omega \pi = 0.$$

If $\sin \omega \pi \neq 0$, then $a = b = 0$, and so we only have the trivial solution $v(x) \equiv 0$. Thus, to obtain a non-zero eigenfunction, we must have

$$\sin \omega \pi = 0,$$

which requires that $\omega = 1, 2, 3, \dots$ be a positive integer. For such ω , every solution

$$v(x) = a \cos kx + b \sin kx, \quad k = 1, 2, 3, \dots,$$

satisfies both boundary conditions, and hence (unless identically zero) qualifies as an eigenfunction of the boundary value problem. Thus, the eigenvalue $\lambda_k = -k^2$ admits a two-dimensional space of eigenfunctions, with basis $v_k(x) = \cos kx$ and $\tilde{v}_k(x) = \sin kx$.

Consequently, the basic trigonometric functions

$$1, \quad \cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x, \quad \cos 3x, \quad \dots \quad (3.25)$$

form a system of independent eigenfunctions for the periodic boundary value problem (3.23). The corresponding exponentially varying eigensolutions are

$$u_k(x) = e^{-k^2 t} \cos kx, \quad \tilde{u}_k(x) = e^{-k^2 t} \sin kx, \quad k = 0, 1, 2, 3, \dots, \quad (3.26)$$

each of which, by design, is a solution to the heat equation (3.21) and satisfies the periodic boundary conditions. Note that we subsumed the case $\lambda_0 = 0$ in (3.26), keeping in mind that, when $k = 0$, the sine function is trivial, $\sin 0x \equiv 0$, and hence not needed. So the null eigenvalue $\lambda_0 = 0$ provides (up to constant multiple) only one eigensolution, whereas the strictly negative eigenvalues $\lambda_k = -k^2 < 0$ each provide two independent eigensolutions.

One should also deal with the possibility of complex eigenvalues. If $\lambda = \omega^2 \neq 0$, where ω is now allowed to be complex, then all solutions to the differential equation (3.23) are of the form

$$v(x) = a e^{\omega x} + b e^{-\omega x}.$$

The periodic boundary conditions require

$$a e^{-\omega \pi} + b e^{\omega \pi} = a e^{\omega \pi} + b e^{-\omega \pi}, \quad a \omega e^{-\omega \pi} - b \omega e^{\omega \pi} = a \omega e^{\omega \pi} - b \omega e^{-\omega \pi}.$$

If $e^{\omega \pi} \neq e^{-\omega \pi}$, or, equivalently, $e^{2\omega \pi} \neq 1$, then the first condition implies $a = b$, but then the second implies $a = b = 0$, and so $\lambda = \omega^2$ is not an eigenvalue. Thus, the only eigenvalues are when $e^{2\omega \pi} = 1$. This implies $\omega = ki$ where k is an integer, and so $\lambda = -k^2$, leading to the trigonometric solutions. Later, in Section 10.5, we will learn that the “self-adjoint” structure of the underlying boundary value problem implies, a priori, that all its eigenvalues are necessarily real and non-positive. So a good part of the preceding analysis was, in fact, superfluous.

We conclude that there are an infinite number of independent eigensolutions (3.26) to the periodic heat equation (3.21). Linear Superposition, as described in Theorem 1.4, tells us that any *finite* linear combination of the eigensolutions is automatically a solution to the periodic heat equation. However, only solutions whose initial data $u(0, x) = f(x)$ happens to be a finite linear combination of the trigonometric eigenfunctions (a trigonometric polynomial), can be so represented. Fourier’s brilliant idea was to propose taking infinite

“linear combinations” of the eigensolutions in an attempt to solve the general initial value problem. Thus, we try representing a general solution to the periodic heat equation as an infinite series of the form[†]

$$u(t, x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k e^{-k^2 t} \cos kx + b_k e^{-k^2 t} \sin kx]. \quad (3.27)$$

The coefficients $a_0, a_1, a_2, \dots, b_1, b_2, \dots$, are constants, to be fixed by the initial condition. Indeed, substituting our proposed solution formula (3.27) into (3.22), we find

$$f(x) = u(0, x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos kx + b_k \sin kx]. \quad (3.28)$$

Thus, we must represent the initial temperature distribution $f(x)$ as an infinite *Fourier series* in the elementary trigonometric eigenfunctions. Once we have prescribed the *Fourier coefficients* $a_0, a_1, a_2, \dots, b_1, b_2, \dots$, we expect that the corresponding eigensolution series (3.27) will provide an explicit formula for the solution to the periodic initial-boundary value problem for the heat equation.

However, infinite series are much more delicate than finite sums, and so this formal construction requires some serious mathematical analysis to place it on a rigorous foundation. The key questions are:

- When does an infinite trigonometric Fourier series converge?
- What kinds of functions $f(x)$ can be represented by a convergent Fourier series?
- Given such a function, how do we determine its Fourier coefficients a_k, b_k ?
- Are we allowed to differentiate a Fourier series?
- Does the result actually form a solution to the initial-boundary value problem for the heat equation?

These are the basic issues in Fourier analysis, which must be properly addressed before we can make any serious progress towards actually solving the heat equation. Thus, we will leave partial differential equations aside for the time being, and start a detailed investigation into the mathematics of Fourier series.

3.2. Fourier Series.

The preceding section served to motivate the development of Fourier series as a tool for solving partial differential equations. Fourier analysis requires more analytical sophistication than basic calculus, and we need to exercise some care when learning its subtleties. Our immediate goal is to represent a given function $f(x)$ as a convergent series in the elementary trigonometric functions:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos kx + b_k \sin kx]. \quad (3.29)$$

[†] For technical reasons, one takes the basic null eigenfunction to be $\frac{1}{2}$ instead of 1. The explanation for this choice will be revealed in the following section.

The first order of business is to determine the formulae for the Fourier coefficients a_k, b_k ; only then will we deal with convergence issues.

The key that unlocks the Fourier treasure chest is orthogonality. Recall, that two vectors in Euclidean space are called *orthogonal* if they meet at a right angle. More explicitly, \mathbf{v}, \mathbf{w} are orthogonal if and only if their dot product is zero: $\mathbf{v} \cdot \mathbf{w} = 0$. Orthogonality, and particularly orthogonal bases, has profound consequences that underpin many modern computational techniques. See Section B.4 for the basics, and [108] for full details on finite-dimensional developments. In infinite-dimensional function space, were it not for orthogonality, Fourier theory would be vastly more complicated, if not completely impractical for applications.

The starting point is the introduction of a suitable inner product on function space, to assume the role played by the dot product in the finite-dimensional context. For classical Fourier series, we use the rescaled L^2 inner product

$$\langle f; g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x) dx \quad (3.30)$$

on the space of continuous functions defined on the interval[†] $[-\pi, \pi]$. It is not hard to show that (3.30) satisfies the basic inner product axioms listed in Definition B.10. The associated norm is

$$\|f\| = \sqrt{\langle f; f \rangle} = \sqrt{\frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx}. \quad (3.31)$$

Lemma 3.1. *Under the rescaled L^2 inner product (3.30), the trigonometric functions $1, \cos x, \sin x, \cos 2x, \sin 2x, \dots$, satisfy the following orthogonality relations:*

$$\begin{aligned} \langle \cos kx; \cos lx \rangle &= \langle \sin kx; \sin lx \rangle = 0, & \text{for } k \neq l, \\ \langle \cos kx; \sin lx \rangle &= 0, & \text{for all } k, l, \\ \|1\| &= \sqrt{2}, \quad \|\cos kx\| = \|\sin kx\| = 1, & \text{for } k \neq 0, \end{aligned} \quad (3.32)$$

where k and l indicate non-negative integers.

Proof: The formulas follow immediately from the elementary integration identities

$$\int_{-\pi}^{\pi} \cos kx \cos lx dx = \begin{cases} 0, & k \neq l, \\ 2\pi, & k = l = 0, \\ \pi, & k = l \neq 0, \end{cases} \quad \int_{-\pi}^{\pi} \sin kx \sin lx dx = \begin{cases} 0, & k \neq l, \\ \pi, & k = l \neq 0, \end{cases} \quad (3.33)$$

$$\int_{-\pi}^{\pi} \cos kx \sin lx dx = 0,$$

which are valid for all nonnegative integers $k, l \geq 0$. *Q.E.D.*

[†] Extensions to more general spaces of functions are discussed in depth below. We have chosen to use the interval $[-\pi, \pi]$ for convenience. A common alternative is to develop Fourier series on the interval $[0, 2\pi]$. In fact, since the basic trigonometric functions are 2π periodic, any interval of length 2π will serve equally well. Adapting Fourier series to other intervals will be discussed in Section 3.4.

Lemma 3.1 implies that the elementary trigonometric functions form an *orthogonal system*, meaning that any distinct pair are orthogonal under the chosen inner product. If we were to replace the constant function 1 by $\frac{1}{\sqrt{2}}$, then the resulting functions would form an *orthonormal system* meaning that, in addition, they all have norm 1. However, the extra $\sqrt{2}$ is utterly annoying, and best omitted.

Remark: As with all essential mathematical facts, the orthogonality of the trigonometric functions is not an accident, but indicates something deeper is at work. Indeed, the underlying reason is that the trigonometric functions are the eigenfunctions for the self-adjoint boundary value problem (3.23). Orthogonality of eigenfunctions is the function space counterpart to the orthogonality of eigenvectors of symmetric matrices, as explained in Theorem B.26. This theory will be developed in Section 10.5, and then applied to the more complicated systems of eigenfunctions we will encounter when dealing with higher dimensional partial differential equations.

If we ignore convergence issues, then the trigonometric orthogonality relations serve to prescribe the Fourier coefficients: Taking the inner product of both sides of (3.29) with $\cos lx$ for $l > 0$, and invoking linearity of the inner product, yields

$$\begin{aligned}\langle f; \cos lx \rangle &= \frac{a_0}{2} \langle 1; \cos lx \rangle + \sum_{k=1}^{\infty} [a_k \langle \cos kx; \cos lx \rangle + b_k \langle \sin kx; \cos lx \rangle] \\ &= a_l \langle \cos lx; \cos lx \rangle = a_l,\end{aligned}$$

since, by the orthogonality relations (3.32), all terms but the l^{th} vanish. This serves to prescribe the Fourier coefficient a_l . A similar manipulation with $\sin lx$ fixes $b_l = \langle f; \sin lx \rangle$, while taking the inner product with the constant function 1 gives

$$\langle f; 1 \rangle = \frac{a_0}{2} \langle 1; 1 \rangle + \sum_{k=1}^{\infty} [a_k \langle \cos kx; 1 \rangle + b_k \langle \sin kx; 1 \rangle] = \frac{a_0}{2} \|1\|^2 = a_0,$$

which agrees with the preceding formula for a_l when $l = 0$, and explains why we include the extra factor of $\frac{1}{2}$ in the constant term. Thus, *if the Fourier series converges to the function $f(x)$, then its coefficients are prescribed by taking inner products with the basic trigonometric functions.*

Definition 3.2. The *Fourier series* of a function $f(x)$ defined on $-\pi \leq x \leq \pi$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos kx + b_k \sin kx], \quad (3.34)$$

whose coefficients are given by the inner product formulae

$$\begin{aligned}a_k &= \langle f; \cos kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx, & k = 0, 1, 2, 3, \dots, \\ b_k &= \langle f; \sin kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx, & k = 1, 2, 3, \dots\end{aligned} \quad (3.35)$$

The function $f(x)$ cannot be completely arbitrary, since, at the very least, the integrals in the coefficient formulae must be well defined and finite. Even if the coefficients (3.35)

are finite, there is no guarantee that the resulting infinite series converges, and, even if it converges, no guarantee that it converges to the original function $f(x)$. For these reasons, we will tend to use the \sim symbol instead of an equals sign when writing down a Fourier series. Before tackling these critical issues, let us work through an elementary example.

Example 3.3. Consider the function $f(x) = x$. We may compute its Fourier coefficients directly, employing integration by parts to evaluate the integrals:

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \, dx = 0, & a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos kx \, dx = \frac{1}{\pi} \left[\frac{x \sin kx}{k} + \frac{\cos kx}{k^2} \right] \Big|_{x=-\pi}^{\pi} = 0, \\ b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin kx \, dx = \frac{1}{\pi} \left[-\frac{x \cos kx}{k} + \frac{\sin kx}{k^2} \right] \Big|_{x=-\pi}^{\pi} = \frac{2}{k} (-1)^{k+1}. \end{aligned} \quad (3.36)$$

The resulting Fourier series is

$$x \sim 2 \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots \right). \quad (3.37)$$

Establishing convergence of this infinite series is far from elementary. Standard calculus criteria, including the ratio and root tests, are inconclusive. Even if we know that the series converges (which it does — for all x), it is certainly not obvious what function it converges to. Indeed, it *cannot* converge to the function $f(x) = x$ everywhere! For instance, if $x = \pi$, then every term in the Fourier series is zero, and so it converges to 0 — which is not the same as $f(\pi) = \pi$.

Recall that the convergence of an infinite series is based on the convergence of its sequence of partial sums, which, in this case, are

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^n [a_k \cos kx + b_k \sin kx]. \quad (3.38)$$

By definition, the Fourier series *converges* at a point x if and only if its partial sums have a limit:

$$\lim_{n \rightarrow \infty} s_n(x) = \tilde{f}(x), \quad (3.39)$$

which may or may not equal the value of the original function $f(x)$. Thus, a key requirement is to find conditions on the function $f(x)$ that guarantee that the Fourier series converges, and, even more importantly, the limiting sum reproduces the original function: $\tilde{f}(x) = f(x)$. This will all be done in detail below.

Remark: A finite Fourier sum, of the form (3.38), is also known as a *trigonometric polynomial*. This is because, by trigonometric identities, it can be re-expressed as a polynomial $P(\cos x, \sin x)$ in the cosine and sine functions; vice versa, every such polynomial can be uniquely written as such a sum; see [108] for details.

The passage from trigonometric polynomials to Fourier series might be viewed as analogous to the passage from polynomials to power series. Recall that the *Taylor series*

of an infinitely differentiable function $f(x)$ at the point $x = 0$ is

$$f(x) \sim c_0 + c_1 x + \cdots + c_n x^n + \cdots = \sum_{k=0}^{\infty} c_k x^k,$$

where, according to Taylor's formula, the coefficients $c_k = \frac{f^{(k)}(0)}{k!}$ are expressed in terms of its derivatives at the origin, *not* by an inner product. The partial sums

$$s_n(x) = c_0 + c_1 x + \cdots + c_n x^n = \sum_{k=0}^n c_k x^k$$

of a power series are ordinary polynomials, and the same basic convergence issues arise.

Although superficially similar, in actuality the two theories are profoundly different. Indeed, while the theory of power series was well established in the early days of the calculus, there remain, to this day, unresolved foundational issues in Fourier theory. A power series either converges everywhere, or on an interval centered at 0, or nowhere except at 0. On the other hand, a Fourier series can converge on quite bizarre sets. Secondly, when a power series converges, it converges to an analytic function, whose derivatives are represented by the differentiated power series. Fourier series may converge, not only to continuous functions, but also to a wide variety of discontinuous functions and even more general objects. Therefore, term-wise differentiation of a Fourier series is a nontrivial issue.

Once one appreciates how radically different the two subjects are, one begins to understand why Fourier's astonishing claims were initially widely disbelieved. Before that time, all functions were taken to be analytic. The fact that Fourier series might converge to a non-analytic, even discontinuous function was extremely disconcerting, resulting in a profound re-evaluation of the foundations of function theory and the calculus, culminating in the modern definitions of function and convergence that you now learn in your first courses in analysis. Only through the combined efforts of many of the leading mathematicians of the nineteenth century was a rigorous theory of Fourier series firmly established. Section 3.5 contains the most important details, while more comprehensive treatments can be found in the advanced texts [44, 153].

Periodic Extensions

The trigonometric constituents (3.25) of a Fourier series are all periodic functions of period 2π . Therefore, if the series converges, the limiting function $\tilde{f}(x)$ must also be periodic of period 2π :

$$\tilde{f}(x + 2\pi) = \tilde{f}(x) \quad \text{for all } x \in \mathbb{R}.$$

A Fourier series can only converge to a 2π periodic function. So it was unreasonable to expect the Fourier series (3.37) to converge to the non-periodic function $f(x) = x$ everywhere. Rather, it should converge to its "periodic extension", as we now define.

Lemma 3.4. *If $f(x)$ is any function defined for $-\pi < x \leq \pi$, then there is a unique 2π periodic function \tilde{f} , known as the 2π periodic extension of f , that satisfies $\tilde{f}(x) = f(x)$ for all $-\pi < x \leq \pi$.*

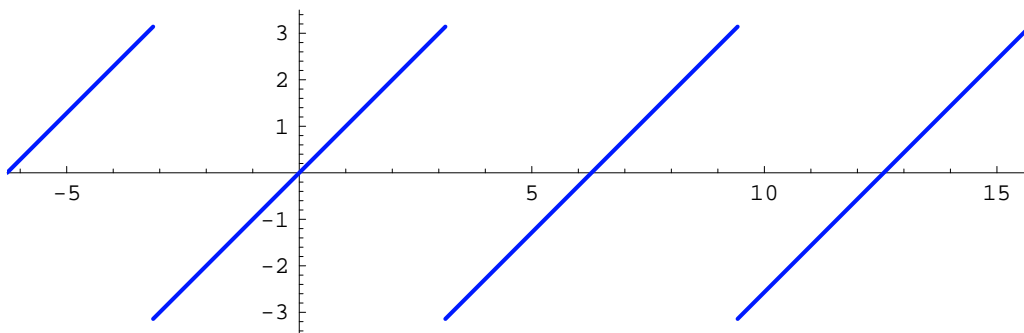


Figure 3.1. 2π periodic extension of x .

Proof: Pictorially, the graph of the periodic extension of a function $f(x)$ is obtained by repeatedly copying the part of its graph between $-\pi$ and π to adjacent intervals of length 2π ; Figure 3.1 shows a simple example. More formally, given $x \in \mathbb{R}$, there is a unique integer m so that $(2m - 1)\pi < x \leq (2m + 1)\pi$. Periodicity of \tilde{f} leads us to define

$$\tilde{f}(x) = \tilde{f}(x - 2m\pi) = f(x - 2m\pi). \quad (3.40)$$

In particular, if $-\pi < x \leq \pi$, then $m = 0$ and hence $\tilde{f}(x) = f(x)$ for such x . The proof that the resulting function \tilde{f} is 2π periodic is left as Exercise ■. *Q.E.D.*

Remark: The construction of the periodic extension in Lemma 3.4 uses the value $f(\pi)$ at the right endpoint and requires $\tilde{f}(-\pi) = \tilde{f}(\pi) = f(\pi)$. One could, alternatively, require $\tilde{f}(\pi) = \tilde{f}(-\pi) = f(-\pi)$, which, if $f(-\pi) \neq f(\pi)$, leads to a slightly different 2π periodic extension of the function. There is no *a priori* reason to prefer one over the other. In fact, as we shall discover, the preferred Fourier periodic extension $\tilde{f}(x)$ takes the average of the two values:

$$\tilde{f}(\pi) = \tilde{f}(-\pi) = \frac{1}{2} [f(\pi) + f(-\pi)], \quad (3.41)$$

which then fixes its values at the odd multiples of π .

Example 3.5. The 2π periodic extension of $f(x) = x$ is the “sawtooth” function $\tilde{f}(x)$ graphed in Figure 3.1. It agrees with x between $-\pi$ and π . Since $f(\pi) = \pi$, $f(-\pi) = -\pi$, the Fourier extension (3.41) sets $\tilde{f}(k\pi) = 0$ for any odd integer k . Explicitly,

$$\tilde{f}(x) = \begin{cases} x - 2m\pi, & (2m - 1)\pi < x < (2m + 1)\pi, \\ 0, & x = (2m - 1)\pi, \end{cases} \quad \text{where } m \text{ is any integer.}$$

With this convention, it can be proved that the Fourier series (3.37) converges everywhere to the 2π periodic extension $\tilde{f}(x)$. In particular,

$$2 \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin kx}{k} = \begin{cases} x, & -\pi < x < \pi, \\ 0, & x = \pm\pi. \end{cases} \quad (3.42)$$

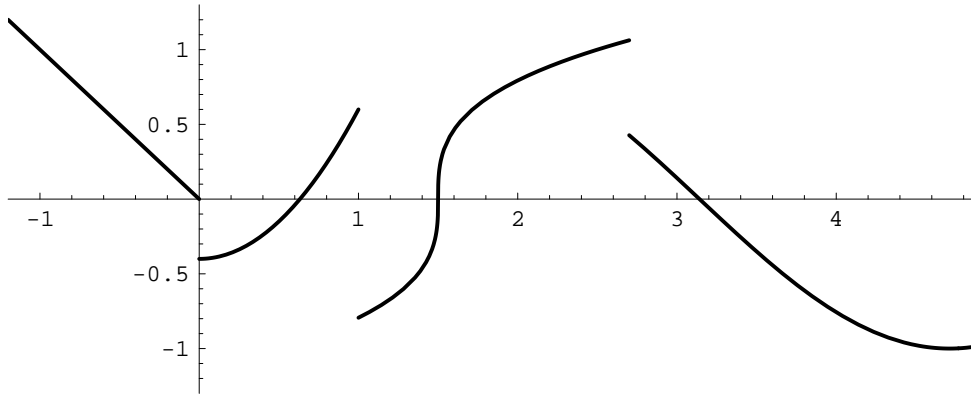


Figure 3.2. Piecewise Continuous Function.

Even this very simple example has remarkable and nontrivial consequences. For instance, if we substitute $x = \frac{1}{2}\pi$ in (3.42) and divide by 2, we obtain *Gregory's series*

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots . \quad (3.43)$$

While this striking formula predates Fourier theory — it was, in fact, first discovered by Leibniz — a direct proof is not easy.

Remark: While numerologically fascinating, Gregory's series is of scant practical use for actually computing π , since its rate of convergence is painfully slow. The reader may wish to try adding up terms to see how far out one needs to go to accurately compute even the first two decimal digits of π . Round-off errors will eventually interfere with any attempt to compute the complete summation with any reasonable degree of accuracy.

Piecewise Continuous Functions

As we shall see, all continuously differentiable, 2π periodic functions can be represented as convergent Fourier series. More generally, we can allow functions that have simple discontinuities.

Definition 3.6. A function $f(x)$ is said to be *piecewise continuous* on an interval $[a, b]$ if it is defined and continuous except possibly at a finite number of points $a \leq x_1 < x_2 < \dots < x_n \leq b$. At each point of discontinuity, the left and right hand limits[†]

$$f(x_k^-) = \lim_{x \rightarrow x_k^-} f(x), \quad f(x_k^+) = \lim_{x \rightarrow x_k^+} f(x), \quad (3.44)$$

exist. Note that we do not require that $f(x)$ be defined at x_k . Even if $f(x_k)$ is defined, it does not necessarily equal either the left or the right hand limit.

[†] At the endpoints a, b we only require one of the limits, namely $f(a^+)$ and $f(b^-)$, to exist.

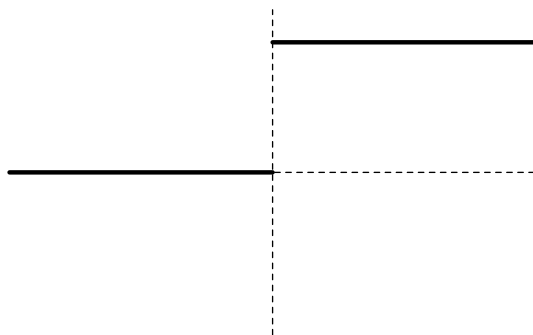


Figure 3.3. The Unit Step Function.

A representative graph of a piecewise continuous function appears in Figure 3.2. The points x_k are known as *jump discontinuities* of $f(x)$ and the difference

$$\beta_k = f(x_k^+) - f(x_k^-) = \lim_{x \rightarrow x_k^+} f(x) - \lim_{x \rightarrow x_k^-} f(x) \quad (3.45)$$

between the left and right hand limits is the *magnitude* of the jump. Note the value of the function at the discontinuity, namely $f(x_k)$ — which may not even be defined — plays no role in the specification of the jump magnitude. The jump magnitude is positive if the function jumps up (when moving from left to right) at x_k and negative if it jumps down. If the jump magnitude vanishes, $\beta_k = 0$, the right and left hand limits agree, and the discontinuity is *removable* since redefining $f(x_k) = f(x_k^+) = f(x_k^-)$ makes $f(x)$ continuous at $x = x_k$. Since removable discontinuities have no effect in either the theory or applications, they can always be removed without penalty.

The simplest example of a piecewise continuous function is the *unit step function*

$$\sigma(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases} \quad (3.46)$$

graphed in Figure 3.3. It has a single jump discontinuity at $x = 0$ of magnitude 1:

$$\sigma(0^+) - \sigma(0^-) = 1 - 0 = 1,$$

and is continuous — indeed, locally constant — everywhere else. If we translate and scale the step function, we obtain a function

$$h(x) = \beta \sigma(x - \xi) = \begin{cases} \beta, & x > \xi, \\ 0, & x < \xi, \end{cases} \quad (3.47)$$

with a single jump discontinuity of magnitude β at the point $x = \xi$.

If $f(x)$ is any piecewise continuous function on $[-\pi, \pi]$, then its Fourier coefficients are well-defined — the integrals (3.35) exist and are finite. Continuity, however, is not enough to ensure convergence of the associated Fourier series.

Definition 3.7. A function $f(x)$ is called *piecewise* C^1 on an interval $[a, b]$ if it is defined, continuous and continuously differentiable except at a finite number of points

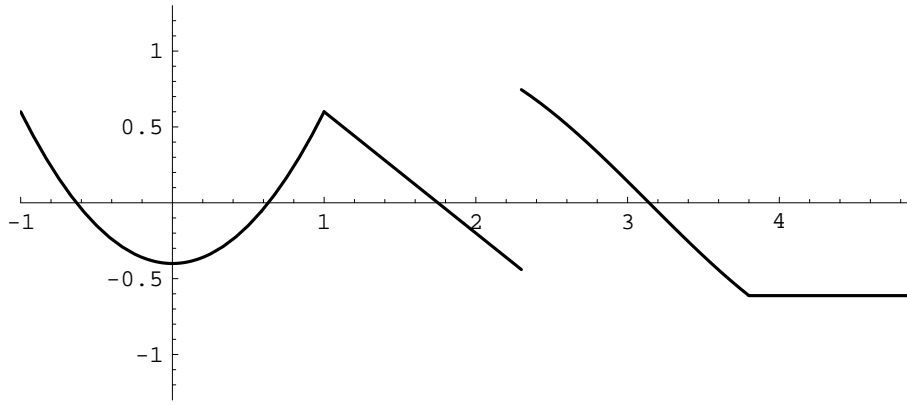


Figure 3.4. Piecewise C^1 Function.

$a \leq x_1 < x_2 < \dots < x_n \leq b$. At each exceptional point, the left and right hand limits[†] exist:

$$\begin{aligned} f(x_k^-) &= \lim_{x \rightarrow x_k^-} f(x), & f(x_k^+) &= \lim_{x \rightarrow x_k^+} f(x), \\ f'(x_k^-) &= \lim_{x \rightarrow x_k^-} f'(x), & f'(x_k^+) &= \lim_{x \rightarrow x_k^+} f'(x). \end{aligned}$$

See Figure 3.4 for a representative graph. For a piecewise C^1 function, an exceptional point x_k is either

- a *jump discontinuity* where the left and right hand derivatives exist, or
- a *corner*, meaning a point where f is continuous, so $f(x_k^-) = f(x_k^+)$, but has different left and right hand derivatives: $f'(x_k^-) \neq f'(x_k^+)$.

Thus, at each point, including jump discontinuities, the graph of $f(x)$ has well-defined right and left tangent lines. For example, the function $f(x) = |x|$ is piecewise C^1 since it is continuous everywhere and has a corner at $x = 0$, with $f'(0^+) = +1$, $f'(0^-) = -1$.

There is an analogous definition of piecewise C^n functions. One requires that the function has n continuous derivatives, except at a finite number of points. Moreover, at every point, the function has well-defined right and left hand limits of all its derivatives up to order n .

Finally, a function $f(x)$ defined for all $x \in \mathbb{R}$ is piecewise continuous (or C^1 or C^n) provided it is piecewise continuous (or C^1 or C^n) on any bounded interval. Thus, a piecewise continuous function on \mathbb{R} can have an infinite number of discontinuities, but they are not allowed to accumulate at any finite limit point. In particular, a 2π periodic function $\tilde{f}(x)$ is piecewise continuous if and only if it is piecewise continuous on the interval $[-\pi, \pi]$.

The Convergence Theorem

We are now able to state the fundamental convergence theorem for Fourier series. But we will defer a discussion of its proof until the end of Section 3.5.

[†] As before, at the endpoints we only require the appropriate one-sided limits, namely $f(a^+)$, $f'(a^+)$, and $f(b^-)$, $f'(b^-)$, to exist.

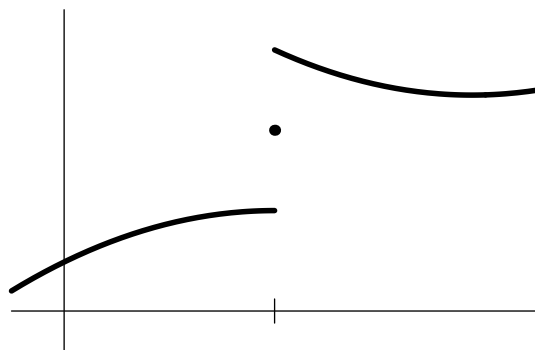


Figure 3.5. Splitting the Difference.

Theorem 3.8. If $\tilde{f}(x)$ is any 2π periodic, piecewise C^1 function, then, at any $x \in \mathbb{R}$, its Fourier series converges to

$$\begin{aligned} \tilde{f}(x), & \quad \text{if } \tilde{f} \text{ is continuous at } x, \\ \frac{1}{2} [\tilde{f}(x^+) + \tilde{f}(x^-)], & \quad \text{if } x \text{ is a jump discontinuity.} \end{aligned}$$

Thus, the Fourier series converges, as expected, to $\tilde{f}(x)$ at all points of continuity. At discontinuities, it apparently can't decide whether to converge to the right or left hand limit, and so ends up "splitting the difference" by converging to their average; see Figure 3.5. If we redefine $\tilde{f}(x)$ at its jump discontinuities to have the average limiting value, so

$$\tilde{f}(x) = \frac{1}{2} [\tilde{f}(x^+) + \tilde{f}(x^-)], \quad (3.48)$$

— an equation that automatically holds at all points of continuity — then Theorem 3.8 would say that the Fourier series converges to the 2π periodic function $\tilde{f}(x)$ everywhere.

Example 3.9. Let $\sigma(x)$ denote the step function (3.46). Its Fourier coefficients are easily computed:

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sigma(x) dx = \frac{1}{\pi} \int_0^{\pi} dx = 1, \\ a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sigma(x) \cos kx dx = \frac{1}{\pi} \int_0^{\pi} \cos kx dx = 0, \\ b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sigma(x) \sin kx dx = \frac{1}{\pi} \int_0^{\pi} \sin kx dx = \begin{cases} \frac{2}{k\pi}, & k = 2l + 1 \text{ odd,} \\ 0, & k = 2l \text{ even.} \end{cases} \end{aligned}$$

Therefore, the Fourier series for the step function is

$$\sigma(x) \sim \frac{1}{2} + \frac{2}{\pi} \left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \dots \right). \quad (3.49)$$

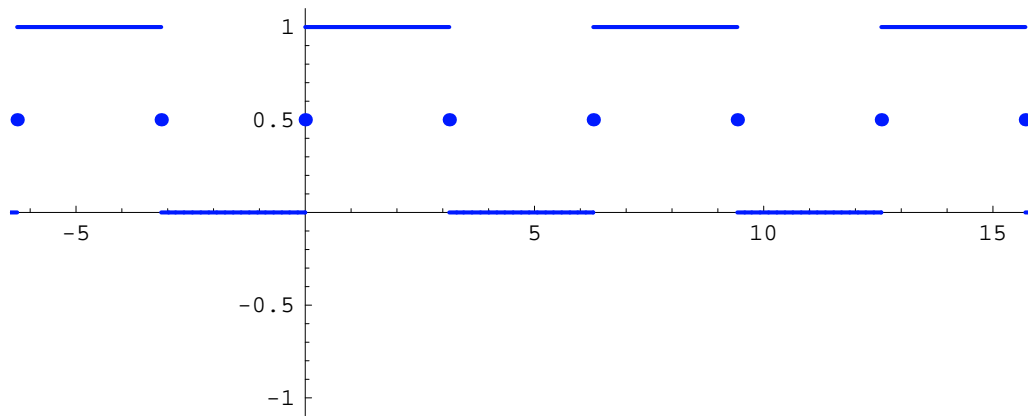


Figure 3.6. Periodic Step Function.

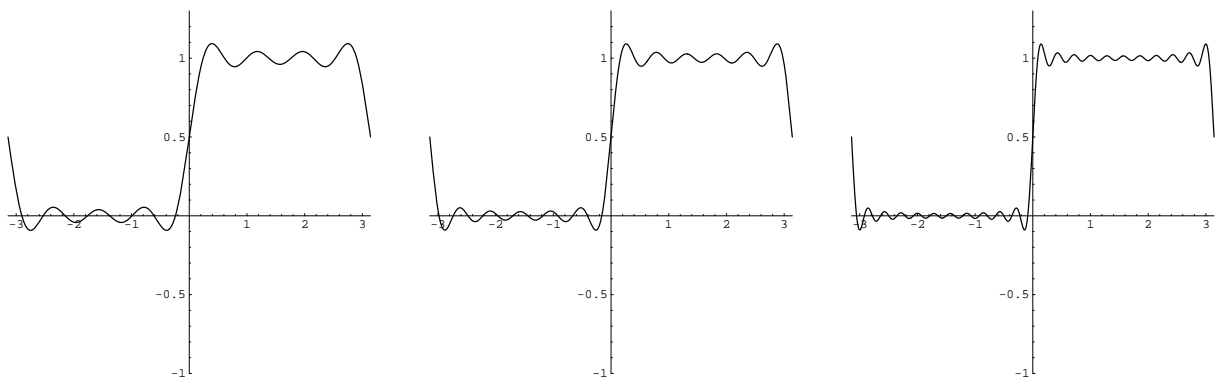


Figure 3.7. Gibbs Phenomenon.

According to Theorem 3.8, the Fourier series will converge to the 2π periodic extension of the step function:

$$\tilde{\sigma}(x) = \begin{cases} 0, & (2m-1)\pi < x < 2m\pi, \\ 1, & 2m\pi < x < (2m+1)\pi, \\ \frac{1}{2}, & x = m\pi, \end{cases} \quad \text{where } m \text{ is any integer,}$$

which is plotted in Figure 3.6. Observe that, in accordance with Theorem 3.8, $\tilde{\sigma}(x)$ takes the midpoint value $\frac{1}{2}$ at the jump discontinuities $0, \pm\pi, \pm2\pi, \dots$.

It is instructive to investigate the convergence of this particular Fourier series in some detail. Figure 3.7 displays a graph of the first few partial sums, taking, respectively, $n = 3, 5,$ and 10 terms. The reader will notice that away from the discontinuities, the series does appear to be converging, albeit slowly. However, near the jumps there is a consistent overshoot of about 9% of the jump magnitude. The region where the overshoot occurs becomes narrower and narrower as the number of terms increases, but the actual amount of overshoot persists no matter how many terms are summed up. This was first noted by the American physicist Josiah Gibbs, and is now known as the *Gibbs phenomenon* in his honor. The Gibbs overshoot is a manifestation of the subtle non-uniform convergence of the Fourier series.

Even and Odd Functions

We already noted that the Fourier cosine coefficients of the function $f(x) = x$ are all 0. This is not an accident, but rather a consequence of the fact that x is an odd function. Recall first the basic definition:

Definition 3.10. A function is called *even* if $f(-x) = f(x)$. A function is *odd* if $f(-x) = -f(x)$.

For example, the functions 1, $\cos kx$, and x^2 are all even, whereas x , $\sin kx$, and $\text{sign } x$ are odd. Note that an odd function necessarily has $f(0) = 0$. We require three elementary lemmas, whose proofs are left to the reader.

Lemma 3.11. *The sum, $f(x) + g(x)$, of two even functions is even; the sum of two odd functions is odd.*

Remark: Every function can be represented as the sum of an even and an odd function; see Exercise ■.

Lemma 3.12. *The product $f(x)g(x)$ of two even functions, or of two odd functions, is an even function. The product of an even and an odd function is odd.*

Lemma 3.13. *If $f(x)$ is odd and integrable on the symmetric interval $[-a, a]$, then $\int_{-a}^a f(x) dx = 0$. If $f(x)$ is even and integrable, then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$.*

The next result is an immediate consequence of applying Lemmas 3.12 and 3.13 to the Fourier integrals (3.35).

Proposition 3.14. *If $f(x)$ is even, then its Fourier sine coefficients all vanish, $b_k = 0$, and so $f(x)$ can be represented by a Fourier cosine series*

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx, \quad (3.50)$$

where

$$a_k = \frac{2}{\pi} \int_0^{\pi} f(x) \cos kx dx, \quad k = 0, 1, 2, 3, \dots \quad (3.51)$$

If $f(x)$ is odd, then its Fourier cosine coefficients vanish, $a_k = 0$, and so $f(x)$ can be represented by a Fourier sine series

$$f(x) \sim \sum_{k=1}^{\infty} b_k \sin kx, \quad (3.52)$$

where

$$b_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin kx dx, \quad k = 1, 2, 3, \dots \quad (3.53)$$

Conversely, a convergent Fourier cosine (respectively, sine) series always represents an even (respectively, odd) function.

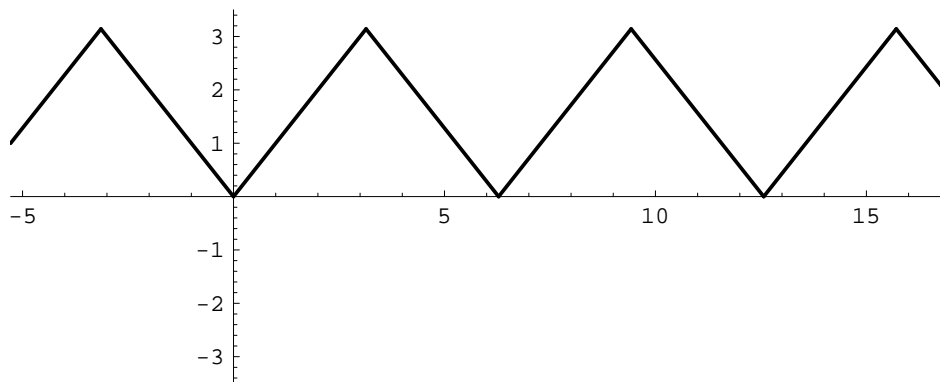


Figure 3.8. Periodic extension of $|x|$.

Example 3.15. The absolute value $f(x) = |x|$ is an even function, and hence has a Fourier cosine series. The coefficients are

$$a_0 = \frac{2}{\pi} \int_0^\pi x \, dx = \pi, \quad (3.54)$$

$$a_k = \frac{2}{\pi} \int_0^\pi x \cos kx \, dx = \frac{2}{\pi} \left[\frac{x \sin kx}{k} + \frac{\cos kx}{k^2} \right]_{x=0}^\pi = \begin{cases} 0, & 0 \neq k \text{ even,} \\ -\frac{4}{k^2\pi}, & k \text{ odd.} \end{cases}$$

Therefore

$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi} \left(\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \frac{\cos 7x}{49} + \dots \right). \quad (3.55)$$

According to Theorem 3.8, this Fourier cosine series converges to the 2π periodic extension of $|x|$, the “sawtooth function” graphed in Figure 3.8.

In particular, if we substitute $x = 0$, we obtain another interesting series

$$\frac{\pi^2}{8} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \dots = \sum_{j=0}^{\infty} \frac{1}{(2j+1)^2}. \quad (3.56)$$

It converges faster than Gregory’s series (3.43), and, while far from optimal in this regards, can be used to compute reasonable approximations to π . One can further manipulate this result to compute the sum of the series

$$S = \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \dots$$

We note that

$$\frac{S}{4} = \sum_{k=1}^{\infty} \frac{1}{4k^2} = \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \frac{1}{64} + \dots$$

Therefore, by (3.56),

$$\frac{3}{4}S = S - \frac{S}{4} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots = \frac{\pi^2}{8},$$

from which we conclude that

$$S = \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}. \quad (3.57)$$

Remark: The most famous function in number theory — and the source of the most outstanding problem in mathematics, the *Riemann hypothesis* — is the *Riemann zeta function*

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}. \quad (3.58)$$

Formula (3.57) shows that $\zeta(2) = \frac{1}{6}\pi^2$. In fact, the value of the zeta function at any *even* positive integer $s = 2j$ is a rational polynomial in π , [8]. Because of its importance to the study of prime numbers, locating all the complex zeros of the zeta function will earn you \$1,000,000 — see <http://www.claymath.org> for details.

Any function $f(x)$ defined on $[0, \pi]$ has a unique even extension to $[-\pi, \pi]$, obtained by setting $f(-x) = f(x)$ for $-\pi \leq x < 0$, and also a unique odd extension, where now $f(-x) = -f(x)$ and $f(0) = 0$. These in turn can be periodically extended to the entire real line. The *Fourier cosine series* of $f(x)$ is defined by the formulas (3.50–51), and represents the even, 2π periodic extension. Similarly, the formulas (3.52–53) define the *Fourier sine series* of $f(x)$, representing its odd, 2π periodic extension.

Example 3.16. Suppose $f(x) = \sin x$. Its Fourier cosine series has coefficients

$$a_k = \frac{2}{\pi} \int_0^{\pi} \sin x \cos kx \, dx = \begin{cases} \frac{2}{\pi}, & k = 0, \\ 0, & k \text{ odd}, \\ -\frac{4}{(k^2 - 1)\pi}, & 0 < k \text{ even}. \end{cases}$$

The resulting cosine series represents the even, 2π periodic extension of $\sin x$, namely

$$|\sin x| \sim \frac{2}{\pi} - \frac{4}{\pi} \sum_{j=1}^{\infty} \frac{\cos 2jx}{4j^2 - 1}.$$

On the other hand, $f(x) = \sin x$ is already odd, and so its Fourier sine series coincides with its ordinary Fourier series, namely $\sin x$, all the other Fourier sine coefficients being zero: $b_1 = 1$, while $b_k = 0$ for $k > 1$.

Complex Fourier Series

An alternative, and often more convenient, approach to Fourier series is to use complex exponentials instead of sines and cosines. Indeed, *Euler's formula*

$$e^{ikx} = \cos kx + i \sin kx, \quad e^{-ikx} = \cos kx - i \sin kx, \quad (3.59)$$

shows how to write the trigonometric functions

$$\cos kx = \frac{e^{ikx} + e^{-ikx}}{2}, \quad \sin kx = \frac{e^{ikx} - e^{-ikx}}{2i}, \quad (3.60)$$

in terms of complex exponentials and so we can easily go back and forth between the two representations.

Like their trigonometric antecedents, complex exponentials are also endowed with an underlying orthogonality. But here, since we are dealing with the vector space of complex-valued functions on the interval $[-\pi, \pi]$, we need to use the rescaled L^2 *Hermitian inner product*

$$\langle f; g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx, \quad (3.61)$$

in which the second function acquires a complex conjugate, as indicated by the overbar. This is needed to ensure that the associated L^2 *Hermitian norm*

$$\|f\| = \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx} \quad (3.62)$$

is real and positive for all nonzero complex functions: $\|f\| > 0$ when $f \neq 0$. Orthonormality of the complex exponentials is proved by direct computation:

$$\begin{aligned} \langle e^{ikx}; e^{ilx} \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-l)x} dx = \begin{cases} 1, & k = l, \\ 0, & k \neq l, \end{cases} \\ \|e^{ikx}\|^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |e^{ikx}|^2 dx = 1. \end{aligned} \quad (3.63)$$

The *complex Fourier series* for a (piecewise continuous) real or complex function f is

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx} = \dots + c_{-2} e^{-2ix} + c_{-1} e^{-ix} + c_0 + c_1 e^{ix} + c_2 e^{2ix} + \dots \quad (3.64)$$

The orthonormality formulae (3.61) imply that the *complex Fourier coefficients* are obtained by taking the inner products

$$c_k = \langle f; e^{ikx} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx. \quad (3.65)$$

Pay particular attention to the minus sign appearing in the integrated exponential, which is because the second argument in the Hermitian inner product (3.61) requires a complex conjugate.

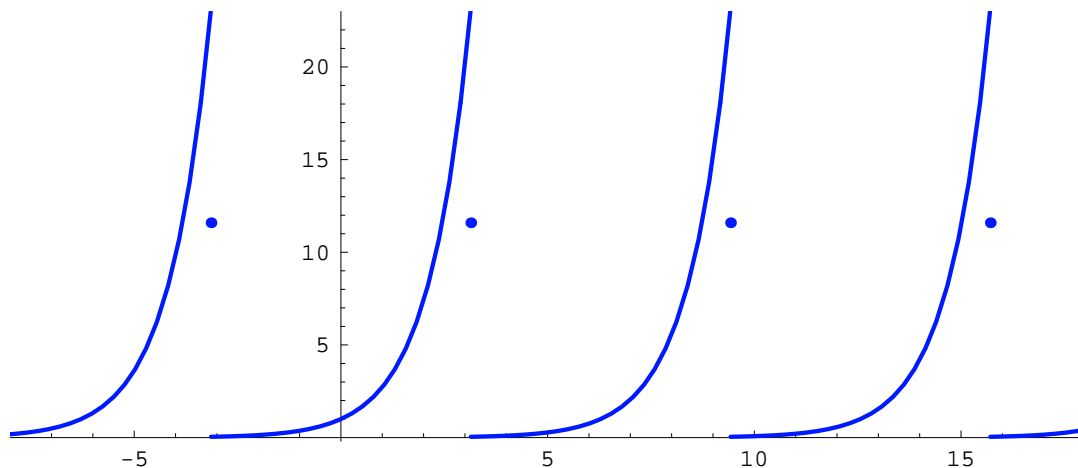


Figure 3.9. Periodic Extension of e^x .

It must be emphasized that the real (3.34) and complex (3.64) Fourier formulae are just two different ways of writing the *same* series! Indeed, if we substitute Euler's formula (3.59) into (3.65) and compare the result with the real Fourier formulae (3.35), we find that the real and complex Fourier coefficients are related by

$$\begin{aligned} a_k &= c_k + c_{-k}, & c_k &= \frac{1}{2}(a_k - i b_k), \\ b_k &= i(c_k - c_{-k}), & c_{-k} &= \frac{1}{2}(a_k + i b_k), \end{aligned} \quad k = 0, 1, 2, \dots \quad (3.66)$$

Remark: We already see one advantage of the complex version. The constant function $1 = e^{0ix}$ no longer plays an anomalous role — the annoying factor of $\frac{1}{2}$ in the real Fourier series (3.34) has mysteriously disappeared!

Example 3.17. For the step function $\sigma(x)$ considered in Example 3.9, the complex Fourier coefficients are

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sigma(x) e^{-ikx} dx = \frac{1}{2\pi} \int_0^{\pi} e^{-ikx} dx = \begin{cases} \frac{1}{2}, & k = 0, \\ 0, & 0 \neq k \text{ even}, \\ \frac{1}{ik\pi}, & k \text{ odd}. \end{cases}$$

Therefore, the step function has the complex Fourier series

$$\sigma(x) \sim \frac{1}{2} - \frac{i}{\pi} \sum_{l=-\infty}^{\infty} \frac{e^{(2l+1)ix}}{2l+1}.$$

You should convince yourself that this is *exactly the same series* as the real Fourier series (3.49). We are merely rewriting it using complex exponentials instead of real sines and cosines.

Example 3.18. Let us find the Fourier series for the exponential function e^{ax} . It is much easier to evaluate the integrals for the complex Fourier coefficients, and so

$$\begin{aligned} c_k &= \langle e^{ax}; e^{ikx} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(a-ik)x} dx = \frac{e^{(a-ik)x}}{2\pi(a-ik)} \Big|_{x=-\pi}^{\pi} \\ &= \frac{e^{(a-ik)\pi} - e^{-(a-ik)\pi}}{2\pi(a-ik)} = (-1)^k \frac{e^{a\pi} - e^{-a\pi}}{2\pi(a-ik)} = \frac{(-1)^k (a+ik) \sinh a\pi}{\pi(a^2+k^2)}. \end{aligned}$$

Therefore, the desired Fourier series is

$$e^{ax} \sim \frac{\sinh a\pi}{\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^k (a+ik)}{a^2+k^2} e^{ikx}. \quad (3.67)$$

As an exercise, the reader should try writing this as a real Fourier series, either by breaking up the complex series into its real and imaginary parts, or by direct evaluation of the real coefficients via their integral formulae (3.35). According to Theorem 3.8 (which is equally valid for complex Fourier series) the Fourier series converges to the 2π periodic extension of the exponential function, as graphed in Figure 3.9. In particular, its values at odd multiples of π is the average of the limiting values there, namely $\cosh a\pi = \frac{1}{2}(e^{a\pi} + e^{-a\pi})$.

3.3. Differentiation and Integration.

If a series of functions converges “nicely”, then one expects to be able to integrate and differentiate it term by term. The resulting series should converge to the integral and derivative of the original sum. For example, integration and differentiation of power series is always valid within the range of convergence, and is used extensively in the construction of series solutions of differential equations, series for integrals of non-elementary functions, and so on. (See Section 12.3 for further details.) The convergence of Fourier series is considerably more delicate, and so one must exercise due care when differentiating or integrating. Nevertheless, in favorable situations, both operations lead to valid results, and are quite useful for constructing Fourier series of more intricate functions.

Integration of Fourier Series

Integration is a smoothing operation — the integrated function is always nicer than the original. Therefore, we should anticipate being able to integrate Fourier series without difficulty. There is, however, one complication: the integral of a periodic function is not necessarily periodic. The simplest example is the constant function 1, which is certainly periodic, but its integral, namely x , is not. On the other hand, integrals of all the other periodic sine and cosine functions appearing in the Fourier series are periodic. Thus, only the constant term

$$\frac{a_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \quad (3.68)$$

might cause us difficulty when we try to integrate a Fourier series (3.34). Note that (3.68) is the *mean* or *average* of the function $f(x)$ over the interval $[-\pi, \pi]$, and so a function has no constant term in its Fourier series if and only if it has *mean zero*. It is easily shown,

cf. Exercise ■, that the mean zero functions are precisely those that remain periodic upon integration. In particular, Lemma 3.13 implies that all odd functions automatically have mean zero, and hence periodic integrals.

Lemma 3.19. *If $f(x)$ is 2π periodic, then its integral $g(x) = \int_0^x f(y) dy$ is 2π periodic if and only if $\int_{-\pi}^{\pi} f(x) dx = 0$, so that f has mean zero on the interval $[-\pi, \pi]$.*

In view of the elementary integration formulae

$$\int \cos kx dx = \frac{\sin kx}{k}, \quad \int \sin kx dx = -\frac{\cos kx}{k}, \quad (3.69)$$

termwise integration of a Fourier series without constant term is straightforward.

Theorem 3.20. *If f is piecewise continuous and has mean zero on the interval $[-\pi, \pi]$, then its Fourier series*

$$f(x) \sim \sum_{k=1}^{\infty} [a_k \cos kx + b_k \sin kx],$$

can be integrated term by term, to produce the Fourier series

$$g(x) = \int_0^x f(y) dy \sim m + \sum_{k=1}^{\infty} \left[-\frac{b_k}{k} \cos kx + \frac{a_k}{k} \sin kx \right]. \quad (3.70)$$

The constant term

$$m = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) dx \quad (3.71)$$

is the mean of the integrated function.

Example 3.21. The function $f(x) = x$ is odd, and so has mean zero: $\int_{-\pi}^{\pi} x dx = 0$. Let us integrate its Fourier series

$$x \sim 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sin kx \quad (3.72)$$

that we found in Example 3.3. The result is the Fourier series

$$\begin{aligned} \frac{1}{2}x^2 &\sim \frac{\pi^2}{6} - 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} \cos kx \\ &= \frac{\pi^2}{6} - 2 \left(\cos x - \frac{\cos 2x}{4} + \frac{\cos 3x}{9} - \frac{\cos 4x}{16} + \dots \right), \end{aligned} \quad (3.73)$$

whose constant term is the mean of the left hand side:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{x^2}{2} dx = \frac{\pi^2}{6}.$$

Let us revisit the derivation of the integrated Fourier series from a slightly different standpoint. If we were to integrate each trigonometric summand in a Fourier series (3.34) from 0 to x , we would obtain

$$\int_0^x \cos ky \, dy = \frac{\sin kx}{k}, \quad \text{whereas} \quad \int_0^x \sin ky \, dy = \frac{1}{k} - \frac{\cos kx}{k}.$$

The extra $1/k$ terms coming from the definite sine integrals did not appear explicitly in our previous expression for the integrated Fourier series, (3.70), and so must be hidden in the constant term m . We deduce that the mean value of the integrated function can be computed using the Fourier sine coefficients of f via the formula

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) \, dx = m = \sum_{k=1}^{\infty} \frac{b_k}{k}. \quad (3.74)$$

For example, the result of integrating both sides of the Fourier series (3.72) for $f(x) = x$ from 0 to x is

$$\frac{x^2}{2} \sim 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} (1 - \cos kx).$$

The constant terms sum up to yield the mean value of the integrated function:

$$2 \left(1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots \right) = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{x^2}{2} \, dx = \frac{\pi^2}{6}, \quad (3.75)$$

which reproduces a formula established in Exercise ■.

More generally, if $f(x)$ does not have mean zero, its Fourier series contains a nonzero constant term,

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos kx + b_k \sin kx].$$

In this case, the result of integration will be

$$g(x) = \int_0^x f(y) \, dy \sim \frac{a_0}{2} x + m + \sum_{k=1}^{\infty} \left[-\frac{b_k}{k} \cos kx + \frac{a_k}{k} \sin kx \right], \quad (3.76)$$

where m is given in (3.74). The right hand side is not, strictly speaking, a Fourier series. There are two ways to interpret this formula within the Fourier framework. Either we can write (3.76) as the Fourier series for the difference

$$g(x) - \frac{a_0}{2} x \sim m + \sum_{k=1}^{\infty} \left[-\frac{b_k}{k} \cos kx + \frac{a_k}{k} \sin kx \right], \quad (3.77)$$

which, by Exercise ■(d), is a 2π periodic function. Alternatively, one can replace x by its Fourier series (3.37), and the result will be the Fourier series for the 2π periodic extension of the integral $g(x) = \int_0^x f(y) \, dy$.

Differentiation of Fourier Series

Differentiation has the opposite effect — it makes a function worse. Therefore, to justify taking the derivative of a Fourier series, we need to know that the differentiated function remains reasonably nice. Since we need the derivative $f'(x)$ to be piecewise C^1 for the convergence Theorem 3.8 to be applicable, we will require that $f(x)$ itself be continuous and piecewise C^2 .

Theorem 3.22. *If $f(x)$ has a piecewise C^2 and continuous 2π periodic extension, then its Fourier series can be differentiated term by term, to produce the Fourier series for its derivative*

$$f'(x) \sim \sum_{k=1}^{\infty} [k b_k \cos kx - k a_k \sin kx] = \sum_{k=-\infty}^{\infty} i k c_k e^{i k x}. \quad (3.78)$$

Example 3.23. The derivative (6.31) of the absolute value function $f(x) = |x|$ is the sign function

$$\frac{d}{dx} |x| = \text{sign } x = \begin{cases} +1, & x > 0 \\ -1, & x < 0. \end{cases}$$

Therefore, if we differentiate its Fourier series (3.55), we obtain the Fourier series

$$\text{sign } x \sim \frac{4}{\pi} \left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \dots \right). \quad (3.79)$$

Note that $\text{sign } x = \sigma(x) - \sigma(-x)$ is the difference of two step functions. Indeed, subtracting the step function Fourier series (3.49) at x from the same series at $-x$ reproduces (3.79).

3.4. Change of Scale.

So far, we have only dealt with Fourier series on the standard interval of length 2π . (We chose $[-\pi, \pi]$ for convenience, but all of the results and formulas are easily adapted to any other interval of the same length, e.g., $[0, 2\pi]$.) Since physical objects like bars and strings do not all come in this particular length, we need to understand how to adapt the formulas to more general intervals.

Any symmetric interval $[-\ell, \ell]$ of length 2ℓ can be rescaled (stretched) to the standard interval $[-\pi, \pi]$ by using the linear change of variables

$$x = \frac{\ell}{\pi} y, \quad \text{so that} \quad -\pi \leq y \leq \pi \quad \text{whenever} \quad -\ell \leq x \leq \ell. \quad (3.80)$$

Given a function $f(x)$ defined on $[-\ell, \ell]$, the *rescaled function* $F(y) = f\left(\frac{\ell}{\pi} y\right)$ lives on $[-\pi, \pi]$. Let

$$F(y) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos ky + b_k \sin ky],$$

be the standard Fourier series for $F(y)$, so that

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F(y) \cos ky \, dy, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F(y) \sin ky \, dy. \quad (3.81)$$

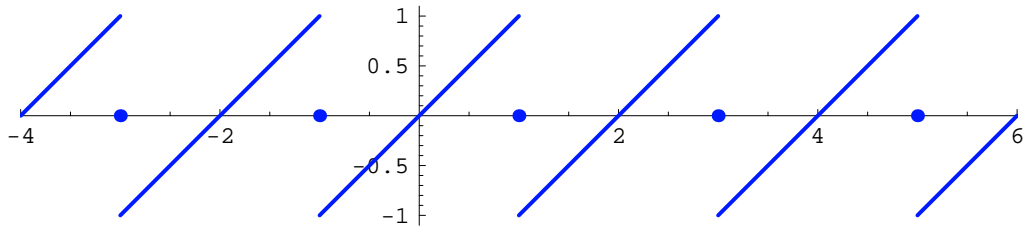


Figure 3.10. 2 Periodic Extension of x .

Then, reverting to the unscaled variable x , we deduce that

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos \frac{k\pi x}{\ell} + b_k \sin \frac{k\pi x}{\ell} \right] \quad (3.82)$$

is the Fourier series of $f(x)$ on the interval $[-\ell, \ell]$. The Fourier coefficients a_k, b_k can, in fact, be computed directly without appealing to the rescaling. Indeed, replacing the integration variable in (3.81) by $y = \pi x/\ell$, and noting that $dy = (\pi/\ell) dx$, we deduce the rescaled formulae

$$a_k = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{k\pi x}{\ell} dx, \quad b_k = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{k\pi x}{\ell} dx, \quad (3.83)$$

for the Fourier coefficients of $f(x)$ on the interval $[-\ell, \ell]$.

All of the convergence results, integration and differentiation formulae, etc., that are valid for the interval $[-\pi, \pi]$ carry over, essentially unchanged, to Fourier series on nonstandard intervals. In particular, adapting our basic convergence Theorem 3.8, we conclude that if $f(x)$ is piecewise C^1 , then its rescaled Fourier series (3.82) converges to its 2ℓ periodic extension $\tilde{f}(x)$, subject to the proviso that $\tilde{f}(x)$ takes on the midpoint values at all jump discontinuities.

Example 3.24. Let us compute the Fourier series for the function $f(x) = x$ on the interval $-1 \leq x \leq 1$. Since f is odd, only the sine coefficients will be nonzero. We have

$$b_k = \int_{-1}^1 x \sin k\pi x dx = \left[-\frac{x \cos k\pi x}{k\pi} + \frac{\sin k\pi x}{(k\pi)^2} \right]_{x=-1}^1 = \frac{2(-1)^{k+1}}{k\pi}.$$

The resulting Fourier series is

$$x \sim \frac{2}{\pi} \left(\sin \pi x - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \dots \right).$$

The series converges to the 2 periodic extension of the function x , namely

$$\tilde{f}(x) = \begin{cases} x - 2m, & 2m - 1 < x < 2m + 1, \\ 0, & x = m, \end{cases} \quad \text{where } m \text{ is an arbitrary integer,}$$

which is plotted in Figure 3.10.

We can similarly reformulate complex Fourier series on the nonstandard interval $[-\ell, \ell]$. Using (3.80) to rescale the variables in (3.64), we find

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ik\pi x/\ell}, \quad \text{where} \quad c_k = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) e^{-ik\pi x/\ell} dx. \quad (3.84)$$

Again, this is merely an alternative way of writing the real Fourier series (3.82).

When dealing with a more general interval $[a, b]$, there are two possible options. The first is to take a function $f(x)$ defined for $a \leq x \leq b$ and periodically extend it to a function $\tilde{f}(x)$ that agrees with $f(x)$ on $[a, b]$ and has period $b-a$. One can then compute the Fourier series (3.82) for its periodic extension $\tilde{f}(x)$ on the symmetric interval $[-\ell, \ell]$ of width $2\ell = b-a$; the resulting Fourier series will (under the appropriate hypotheses) converge to $\tilde{f}(x)$ and hence agree with $f(x)$ on the original interval. An alternative approach is to translate the interval by an amount $\frac{1}{2}(a+b)$ so as to make it symmetric around the origin; this is accomplished by the change of variables $\hat{x} = x - \frac{1}{2}(a+b)$, followed by an additional rescaling to convert the interval into $[-\pi, \pi]$. The two methods are essentially equivalent, and details are left to the reader.

3.5. Convergence of the Fourier Series.

The purpose of this final section is to establish the most basic convergence results for Fourier series. This is not a purely theoretical enterprise, since convergence considerations impinge directly upon applications. One particularly important consequence is the connection between the degree of smoothness of a function and the decay rate of its high order Fourier coefficients — a result that is exploited in signal and image denoising and in the analytical properties of solutions to partial differential equations.

The material in this section is more theoretical than elsewhere, and those who are applications-oriented may consider omitting it on a first reading. However, a full understanding of the scope of Fourier analysis as well as its limitations does require some familiarity with the underlying theory. Moreover, the required techniques and proofs serve as an excellent introduction to some of the most important tools of modern mathematical analysis. Any effort expended to assimilate this material will be more than amply rewarded.

Unlike power series, which converge to analytic functions on the interval of convergence, and diverge elsewhere (the only tricky point being whether or not the series converges at the endpoints), the convergence of a Fourier series is a much more subtle matter, and still not understood in complete generality. A large part of the difficulty stems from the intricacies of convergence in infinite-dimensional function spaces. Let us therefore begin with a brief outline of the key issues.

We assume that you are familiar with the usual calculus definition of the limit of a sequence of real numbers: $\lim_{n \rightarrow \infty} a_n = a^*$. In any finite-dimensional vector space, e.g., \mathbb{R}^m , there is essentially only one way for a sequence of vectors $\mathbf{v}^{(0)}, \mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \dots \in \mathbb{R}^m$ to converge, as guaranteed by any one of the following equivalent criteria:

- The vectors converge: $\mathbf{v}^{(n)} \rightarrow \mathbf{v}^* \in \mathbb{R}^m$ as $n \rightarrow \infty$.

- The individual components of $\mathbf{v}^{(n)} = (v_1^{(n)}, \dots, v_m^{(n)})$ converge, so $\lim_{n \rightarrow \infty} v_i^{(n)} = v_i^*$ for all $i = 1, \dots, m$.
- The difference in norms goes to zero: $\|\mathbf{v}^{(n)} - \mathbf{v}^*\| \rightarrow 0$ as $n \rightarrow \infty$.

The last requirement, known as *convergence in norm*, does not, in fact, depend on which norm is chosen. Indeed, on a finite-dimensional vector space, all norms are essentially equivalent, and if one norm goes to zero, so does any other norm, [108].

On the other hand, the analogous convergence criteria are *not the same* in infinite-dimensional function spaces. There are, in fact, a bewildering variety of convergence mechanisms in function space, that include pointwise convergence, uniform convergence, convergence in norm, weak convergence, and so on. Each plays a significant role in advanced mathematical analysis, and hence all are deserving of study. Here, though, we shall cover just the most basic aspects of convergence of the Fourier series and their applications to partial differential equations, leaving further developments to more specialized texts, e.g., [44, 117, 153].

Pointwise and Uniform Convergence

The most familiar convergence mechanism for a sequence of functions $v_n(x)$ is *pointwise convergence*. This requires that the functions' values at each individual point converge in the usual sense:

$$\lim_{n \rightarrow \infty} v_n(x) = v_*(x) \quad \text{for all } x \in I, \quad (3.85)$$

where I denotes an interval belonging to the common domain of all the functions. Even more explicitly, pointwise convergence requires that, for every $\varepsilon > 0$ and every $x \in I$, there exists an integer N , depending on ε and x , such that

$$|v_n(x) - v_*(x)| < \varepsilon \quad \text{for all } n \geq N. \quad (3.86)$$

Pointwise convergence can be viewed as the function space version of the convergence of the components of a vector. We have already stated the Fundamental Theorem 3.8 regarding pointwise convergence of Fourier series. The proof will be deferred until the end of this section.

On the other hand, proving uniform convergence of a Fourier series is not so difficult, and so we will begin there. The basic definition of uniform convergence looks very similar to that of pointwise convergence, but with a subtle, but important difference.

Definition 3.25. A sequence of functions $v_n(x)$ is said to converge *uniformly* to a function $v_*(x)$ on a subset $I \subset \mathbb{R}$ if, for every $\varepsilon > 0$, there exists an integer N , depending on ε , such that

$$|v_n(x) - v_*(x)| < \varepsilon \quad \text{for all } x \in I \text{ and all } n \geq N. \quad (3.87)$$

Clearly, a uniformly convergent sequence of functions converges pointwise, but the converse does not hold. The key difference — and the reason for the term “uniform convergence” — is that the integer N depends only upon ε and, unlike pointwise convergence, *not* on the point $x \in I$. According to (3.87), the sequence converges uniformly if and only

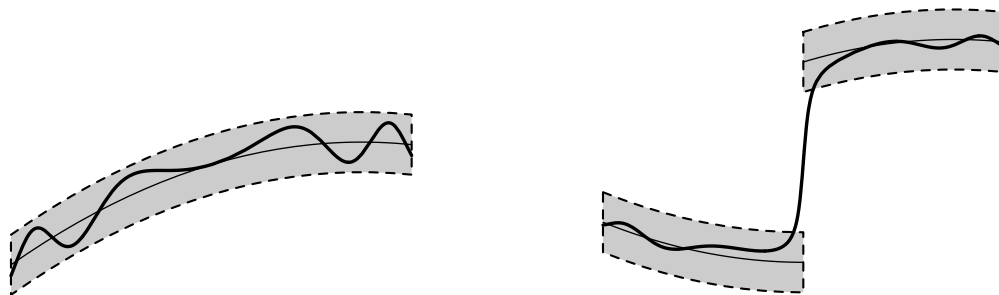


Figure 3.11. Uniform and Non-Uniform Convergence of Functions.

if for any small ε , the graphs of the functions eventually lie inside a band of width 2ε centered around the graph of the limiting function, as in the first plot in Figure 3.11.

A key feature of uniform convergence is that it preserves continuity.

Theorem 3.26. *If each $v_n(x)$ is continuous and $v_n(x) \rightarrow v_*(x)$ converges uniformly, then $v_*(x)$ is also a continuous function.*

The proof is by contradiction. Intuitively, if $v_*(x)$ were to have a discontinuity, then, as sketched in the second plot in Figure 3.11, a sufficiently small band around its graph would not connect together, and this prevents the graph of any continuous function, such as $v_n(x)$, from remaining entirely within the band. A detailed discussion of these issues, including the proofs of the basic theorems, can be found in any basic real analysis text, e.g., [7, 116, 117].

The Gibbs phenomenon shown in Figure 3.7 is a prototypical example of a nonuniform convergence: For a given $\varepsilon > 0$, the closer x is to the discontinuity, the larger n must be chosen so that the inequality in (3.87) holds. Hence, there is *no* uniform choice of N that makes (3.87) valid for *all* x and *all* $n \geq N$.

Warning: A sequence of continuous functions can converge *non-uniformly* to a continuous function. An example is the sequence

$$v_n(x) = \frac{2nx}{1+n^2x^2},$$

which converges pointwise to $v_*(x) \equiv 0$ (why?) but not uniformly since

$$\max |v_n(x)| = v_n\left(\frac{1}{n}\right) = 1,$$

which implies that (3.87) cannot hold when $\varepsilon < 1$.

The convergence (pointwise, uniform, etc.) of a series $\sum_{k=1}^n u_k(x)$ is, by definition, governed by the convergence of its sequence of *partial sums*

$$v_n(x) = \sum_{k=1}^n u_k(x). \tag{3.88}$$

The most useful test for uniform convergence of series of functions is known as the *Weierstrass M-test*, in honor of the nineteenth century German mathematician Karl Weierstrass, known as the “father of modern analysis”.

Theorem 3.27. Let $I \subset \mathbb{R}$. Suppose the functions $u_k(x)$ are bounded:

$$|u_k(x)| \leq m_k \quad \text{for all } x \in I, \quad (3.89)$$

where $m_k \geq 0$ are fixed positive constants. If the constant series

$$\sum_{k=1}^{\infty} m_k < \infty \quad (3.90)$$

converges, then the series

$$\sum_{k=1}^{\infty} u_k(x) = f(x) \quad (3.91)$$

converges uniformly and absolutely[†] to a function $f(x)$ for all $x \in I$. In particular, if the summands $u_k(x)$ are continuous, so is the sum $f(x)$.

With some care, we can manipulate uniformly convergent series just like finite sums. Thus, if (3.91) is a uniformly convergent series, so is the term-wise product

$$\sum_{k=1}^{\infty} g(x) u_k(x) = g(x) f(x) \quad (3.92)$$

with any bounded function: $|g(x)| \leq C$ for all $x \in I$. We can integrate a uniformly convergent series term by term[‡], and the resulting integrated series

$$\int_a^x \left(\sum_{k=1}^{\infty} u_k(y) \right) dy = \sum_{k=1}^{\infty} \int_a^x u_k(y) dy = \int_a^x f(y) dy \quad (3.93)$$

is uniformly convergent. Differentiation is also allowed — but only when the differentiated series converges uniformly.

Proposition 3.28. If $\sum_{k=1}^{\infty} u'_k(x) = g(x)$ is a uniformly convergent series, then $\sum_{k=1}^{\infty} u_k(x) = f(x)$ is also uniformly convergent, and, moreover, $f'(x) = g(x)$.

We are particularly interested in the convergence of a Fourier series, which, for convenience, we take in its complex form

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}. \quad (3.94)$$

[†] Recall that a series $\sum_{n=1}^{\infty} a_n = a^*$ is said to converge *absolutely* if and only if $\sum_{n=1}^{\infty} |a_n|$ converges, [7].

[‡] Assuming that the individual functions are all integrable.

Since x is real, $|e^{ikx}| \leq 1$, and hence the individual summands are bounded by

$$|c_k e^{ikx}| \leq |c_k| \quad \text{for all } x.$$

Applying the Weierstrass M -test, we immediately deduce the basic result on uniform convergence of Fourier series.

Theorem 3.29. *If the Fourier coefficients c_k satisfy*

$$\sum_{k=-\infty}^{\infty} |c_k| < \infty, \tag{3.95}$$

then the Fourier series (3.94) converges uniformly to a continuous function $\tilde{f}(x)$ that has the same Fourier coefficients: $c_k = \langle f; e^{ikx} \rangle = \langle \tilde{f}; e^{ikx} \rangle$.

Proof: Uniform convergence and continuity of the limiting function follow from Theorem 3.27. To show that the c_k actually are the Fourier coefficients of the sum, we multiply the Fourier series by e^{-ikx} and integrate term by term from $-\pi$ to π . As in (3.92, 93), both operations are valid thanks to the uniform convergence of the series. *Q.E.D.*

The one thing that the theorem does not guarantee is that the original function $f(x)$ used to compute the Fourier coefficients c_k is the *same* as the function $\tilde{f}(x)$ obtained by summing the resulting Fourier series! Indeed, this may very well not be the case. As we know, the function that the series converges to is necessarily 2π periodic. Thus, at the very least, $\tilde{f}(x)$ will be the 2π periodic extension of $f(x)$. But even this may not suffice. Indeed, two functions $f(x)$ and $\hat{f}(x)$ that have the same values except at a finite set of points x_1, \dots, x_m have the same Fourier coefficients. (Why?) For example, the discontinuous function $f(x) = \begin{cases} 1, & x = 0, \\ 0, & \text{otherwise,} \end{cases}$ has all zero Fourier coefficients, and hence its Fourier series converges to the continuous zero function. More generally, two functions which agree everywhere outside a set of “measure zero” will have identical Fourier coefficients. In this way, a convergent Fourier series singles out a distinguished representative from a collection of essentially equivalent 2π periodic functions.

Remark: The term “measure” refers to a rigorous generalization of the notion of the length of an interval to more general subsets $S \subset \mathbb{R}$. In particular, S has *measure zero* if it can be covered by a collection of intervals of arbitrarily small total length. For example, any collection of finitely many points, or even countably many points, e.g., the rational numbers, has measure zero. The proper development of the notion of measure, and the consequential Lebesgue theory of integration, is properly studied in a course in real analysis, [116, 117].

As a consequence of Theorem 3.27, Fourier series cannot converge uniformly when discontinuities are present. However, it can be proved, [26, 44, 153], that even when the function fails to be everywhere continuous, its Fourier series is uniformly convergent on any closed subset of continuity.

Theorem 3.30. *Let $f(x)$ be 2π periodic and piecewise C^1 . If f is continuous for $a < x < b$, then its Fourier series converges uniformly to $f(x)$ on any closed subinterval $a + \delta \leq x \leq b - \delta$, with $\delta > 0$.*

For example, the Fourier series (3.49) for the step function does converge uniformly if we stay away from the discontinuities; for instance, by restriction to a subinterval of the form $[\delta, \pi - \delta]$ or $[-\pi + \delta, -\delta]$ for any $0 < \delta < \frac{1}{2}\pi$. This reconfirms our observation that the nonuniform Gibbs behavior becomes progressively more and more localized at the discontinuities.

Smoothness and Decay

The criterion (3.95), that guarantees uniform convergence, requires, at the very least, that the Fourier coefficients go to zero: $c_k \rightarrow 0$ as $k \rightarrow \pm\infty$. And they cannot decay too slowly. For example, the individual summands of the infinite series

$$\sum_{k=-\infty}^{\infty} \frac{1}{|k|^\alpha} \quad (3.96)$$

go to 0 as $k \rightarrow \infty$ whenever $\alpha > 0$, but the series only converges when $\alpha > 1$. (This follows from the standard integral convergence test, [7].) Thus, if we can bound the Fourier coefficients by

$$|c_k| \leq \frac{M}{|k|^\alpha} \quad \text{for all } |k| \gg 0, \quad (3.97)$$

for some exponent $\alpha > 1$ and some positive constant $M > 0$, then the Weierstrass M test will guarantee that the Fourier series converges uniformly to a continuous function.

An important consequence of the differentiation formulae (3.78) for Fourier series is the fact that the faster the Fourier coefficients of a function tend to zero as $k \rightarrow \infty$, the smoother the function is. Thus, one can detect the degree of smoothness of a function by seeing how rapidly its Fourier coefficients decay to zero. More rigorously:

Theorem 3.31. *If the Fourier coefficients satisfy*

$$\sum_{k=-\infty}^{\infty} |k|^n |c_k| < \infty, \quad (3.98)$$

then the Fourier series (3.64) converges to an n times continuously differentiable 2π periodic function $f(x) \in C^n$. Moreover, for any $m \leq n$, the m times differentiated Fourier series converges uniformly to the corresponding derivative $f^{(m)}(x)$.

Proof: Iterating (3.78), the Fourier series for the n^{th} derivative of a function is

$$f^{(n)}(x) \sim \sum_{k=-\infty}^{\infty} i^n k^n c_k e^{ikx}. \quad (3.99)$$

If (3.98) holds, the Weierstrass M test implies the uniform convergence of the series to a continuous function. Proposition 3.28 guarantees that the limit is the n^{th} derivative of the original Fourier series. *Q.E.D.*

As a direct consequence, we can quantify the statement that the smaller the high frequency Fourier coefficients, the smoother the function.

Corollary 3.32. *If the Fourier coefficients satisfy (3.97) for some $\alpha > n + 1$, then the Fourier series converges to an n times continuously differentiable 2π periodic function $f \in C^n$.*

If the Fourier coefficients go to zero faster than any power of k , e.g., exponentially fast, then the function is infinitely differentiable. Analyticity is a little more delicate, and we refer the reader to [44, 153] for details.

Example 3.33. The 2π periodic extension of the function $|x|$ is continuous with piecewise continuous first derivative. Its Fourier coefficients (3.54) satisfy the estimate (3.97) for $\alpha = 2$ — not quite fast enough to ensure a continuous second derivative. On the other hand, the Fourier coefficients (3.36) of the step function $\sigma(x)$ only tend to zero as $1/k$, so $\alpha = 1$, reflecting the fact that its periodic extension is only piecewise continuous, not continuous.

Hilbert Space

In order to make further progress, we must take a little detour. The proper setting for the rigorous theory of Fourier series turns out to be the most important function space in modern analysis and modern physics, known as *Hilbert space* in honor of the great turn-of-the-twentieth-century German mathematician David Hilbert. The precise definition of this infinite-dimensional inner product space is rather technical, but a rough version goes as follows:

Definition 3.34. A complex-valued function $f(x)$ is called *square-integrable* on the interval $[-\pi, \pi]$ if it has finite L^2 norm:

$$\|f\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx < \infty. \quad (3.100)$$

The *Hilbert space* $L^2 = L^2[-\pi, \pi]$ is the vector space consisting of all complex-valued *square-integrable* functions.

The triangle inequality

$$\|f + g\| \leq \|f\| + \|g\|,$$

implies that if $f, g \in L^2$, so $\|f\|, \|g\| < \infty$, then $\|f + g\| < \infty$, and so $f + g \in L^2$. Moreover, for any complex constant c

$$\|cf\| = |c| \|f\|,$$

and so $cf \in L^2$ also. Thus, as claimed, Hilbert space is a complex vector space. The Cauchy–Schwarz inequality

$$|\langle f; g \rangle| \leq \|f\| \|g\|$$

implies that the L^2 inner product of two square-integrable functions is well-defined and finite. In particular, the Fourier coefficients of a function $f \in L^2$ are specified by its inner products

$$c_k = \langle f; e^{ikx} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

with the complex exponentials (which, by (3.63), are in L^2), and hence are all well-defined and finite.

There are some interesting analytical subtleties that arise when one tries to prescribe precisely which functions are in the Hilbert space. Every piecewise continuous function belongs to L^2 . But some functions with singularities are also members. For example, the power function $|x|^{-\alpha}$ belongs to L^2 for any $\alpha < \frac{1}{2}$, but not if $\alpha \geq \frac{1}{2}$.

Analysis relies on limiting procedures, and Hilbert space must be “complete” in the sense that appropriately convergent[†] sequences of functions have a limit. The completeness requirement is not elementary, and relies on the development of the more sophisticated Lebesgue theory of integration, which was formalized in the early part of the twentieth century by the French mathematician Henri Lebesgue. Any function which is square-integrable in the Lebesgue sense is admitted into L^2 . This includes such non-piecewise continuous functions as $\sin \frac{1}{x}$ and $x^{-1/3}$, as well as the strange function

$$r(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases} \quad (3.101)$$

Thus, while well-behaved in some respects, square-integrable functions can be quite wild in others.

A second complication is that (3.100) does not, strictly speaking, define a norm once we allow discontinuous functions into the fold. For example, the piecewise continuous function

$$f_0(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0, \end{cases} \quad (3.102)$$

has norm zero, $\|f_0\| = 0$, even though it is not zero everywhere. Indeed, any function which is zero except on a set of measure zero also has norm zero, including the function (3.101). Therefore, in order to make (3.100) into a legitimate norm, we must agree to identify any two functions which have the same values except on a set of measure zero. Thus, the zero function 0 along with the preceding examples $f_0(x)$ and $r(x)$ are all viewed as defining the *same* element of Hilbert space. Thus, an element of Hilbert space is not, in fact, a function, but, rather, an equivalence class of functions all differing on a set of measure zero. All this may strike the applications-oriented reader as becoming much too abstract and arcane. In practice, you will not lose much by working with the elements of L^2 as if they were ordinary functions, and, even better, assuming that said “functions” are always piecewise continuous and square-integrable. Nevertheless, the full analytical power

[†] The precise technical requirement is that every *Cauchy sequence* of functions $v_k \in L^2$ converges to a function $v_* \in L^2$; see [38, 116, 117], and also Exercise ■ for details.

of Hilbert space theory is only unleashed by allowing completely general square integrable functions into the fold.

After its invention by pure mathematicians around the turn of the twentieth century, physicists in the 1920's suddenly realized that Hilbert space was the ideal setting for the modern theory of quantum mechanics, [41, 85, 91, 136]. A quantum mechanical *wave function* is an element[†] $\varphi \in L^2$ that has unit norm: $\|\varphi\| = 1$. Thus, the set of wave functions is merely the “unit sphere” in Hilbert space. Quantum mechanics endows each physical wave function with a probabilistic interpretation. Suppose the wave function represents a single subatomic particle — photon, electron, etc. Then, according to the Copenhagen interpretation of quantum mechanics, the squared modulus of the wave function, $|\varphi(x)|^2$, represents the probability density that quantifies the chances of the particle being located at position x . More precisely, the probability that the particle resides in a

prescribed interval $[a, b]$ is equal to $\sqrt{\frac{1}{2\pi} \int_a^b |\varphi(x)|^2 dx}$. In particular, the wave function has unit norm,

$$\|\varphi\| = \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} |\varphi(x)|^2 dx} = 1,$$

because the particle must certainly, i.e., with probability 1, be *somewhere*!

Convergence in Norm

We are now in a position to discuss convergence in norm of the Fourier series. We begin with the basic definition, which makes sense on any normed vector space.

Definition 3.35. Let V be a normed vector space. A sequence $\mathbf{v}^{(n)} \in V$ is said to *converge in norm* to $\mathbf{v}^* \in V$ if $\|\mathbf{v}^{(n)} - \mathbf{v}^*\| \rightarrow 0$ as $n \rightarrow \infty$.

As we noted earlier, on finite-dimensional vector spaces such as \mathbb{R}^m , convergence in norm is equivalent to ordinary convergence. On the other hand, on infinite-dimensional function spaces, convergence in norm is very different from pointwise convergence. For instance, it is possible, cf. Exercise ■, to construct a sequence of functions that converges in norm to 0, but does not converge pointwise *anywhere*!

We are particularly interested in the convergence in norm of the Fourier series of a square integrable function $f \in L^2$. Let

$$s_n(x) = \sum_{k=-n}^n c_k e^{ikx} \tag{3.103}$$

be the n^{th} partial sum of its Fourier series (3.64). The partial sum (3.103) belongs to the finite-dimensional subspace $\mathcal{T}^{(n)} \subset L^2$ consisting of all *trigonometric polynomials* (finite

[†] Here we are acting as if the physical universe were represented by the one-dimensional interval $[-\pi, \pi]$. The more apt context of three-dimensional physical space is developed analogously, replacing the single integral by a triple integral over all of \mathbb{R}^3 . See also Section 8.4.

Fourier sums)

$$p_n(x) = \sum_{k=-n}^n r_k e^{ikx} = \frac{u_0}{2} + \sum_{k=1}^n [u_k \cos kx + v_k \sin kx] \quad (3.104)$$

of degree at most n . An important fact is that the Fourier partial sum $s_n(x)$ is distinguished as the trigonometric polynomial that *best approximates* the function $f(x)$ in the *least squares* sense, [108], or, equivalently, is the *closest point* to f on the subspace $\mathcal{T}^{(n)}$, where the *distance* between functions $f, g \in L^2$ is measured by the L^2 norm of their difference: $\text{dist}(f, g) = \|f - g\|$.

Theorem 3.36. *The n^{th} order Fourier partial sum $s_n(x)$ is the best least squares approximation to $f(x)$ among all trigonometric polynomials $p_n \in \mathcal{T}^{(n)}$ of degree $\leq n$, meaning that it minimizes the L^2 norm of the difference*

$$\|f - p_n\| = \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - p_n(x)|^2 dx}. \quad (3.105)$$

Proof: We first note that, thanks to the orthonormality of the basis exponentials e^{ikx} , cf. (3.63), we can compute the squared norm of a trigonometric polynomial (3.104) by summing the squared moduli of its Fourier coefficients:

$$\begin{aligned} \|p_n\|^2 &= \langle p_n; p_n \rangle = \left\langle \sum_{k=-n}^n r_k e^{ikx}; \sum_{l=-n}^n r_l e^{ilx} \right\rangle \\ &= \sum_{k,l=-n}^n r_k \bar{r}_l \langle e^{ikx}; e^{ilx} \rangle = \sum_{k=-n}^n |r_k|^2, \end{aligned} \quad (3.106)$$

which reproduces the general formula (B.26) for the norm with respect to an orthonormal basis. Therefore, employing the identity in Exercise ■(a),

$$\begin{aligned} \|f - p_n\|^2 &= \|f\|^2 - 2 \operatorname{Re} \langle f; p_n \rangle + \|p_n\|^2 = \|f\|^2 - 2 \operatorname{Re} \sum_{k=-n}^n \bar{r}_k \langle f; e^{ikx} \rangle + \|p_n\|^2 \\ &= \|f\|^2 - 2 \sum_{k=-n}^n \operatorname{Re} (c_k \bar{r}_k) + \sum_{k=-n}^n |r_k|^2 = \|f\|^2 - \sum_{k=-n}^n |c_k|^2 + \sum_{k=-n}^n |r_k - c_k|^2. \end{aligned}$$

The final equality results from adding and subtracting the squared norm

$$\|s_n\|^2 = \sum_{k=-n}^n |c_k|^2 \quad (3.107)$$

of the Fourier partial sum, which is a particular case of (3.106). We conclude that

$$\|f - p_n\|^2 = \|f\|^2 - \|s_n\|^2 + \sum_{k=-n}^n |r_k - c_k|^2. \quad (3.108)$$

The first and second terms on the right hand side of (3.108) are uniquely determined by $f(x)$ and hence cannot be altered by the choice of trigonometric polynomial $p_n(x)$, which only affects the final summation. Since the latter is a sum of nonnegative quantities, it is minimized by setting all its summands to zero, i.e., setting $r_k = c_k$. We conclude that $\|f - p_n\|$ achieves its minimum value among all $p_n \in \mathcal{T}^{(n)}$ if and only if $r_k = c_k$, and hence $p_n(x) = s_n(x)$ is the Fourier partial sum. *Q.E.D.*

Setting $p_n = s_n$, so $r_k = c_k$, in (3.108), we conclude that the minimizing least squares error for the Fourier partial sum is

$$0 \leq \|f - s_n\|^2 = \|f\|^2 - \|s_n\|^2 = \|f\|^2 - \sum_{k=-n}^n |c_k|^2.$$

Therefore, the Fourier coefficients of the function f must satisfy the inequality

$$\sum_{k=-n}^n |c_k|^2 \leq \|f\|^2. \tag{3.109}$$

Consider what happens in the limit as $n \rightarrow \infty$. Since we are summing a sequence of nonnegative numbers, with uniformly bounded partial sums, the limiting summation must exist, and be subject to the same bound. We have thus established *Bessel's inequality*:

$$\sum_{k=-\infty}^{\infty} |c_k|^2 \leq \|f\|^2, \tag{3.110}$$

valid for any $f \in L^2$ — an important waystation on the road to the general theory. Now, as noted earlier, if a series is to converge, the individual summands must go to zero: $|c_k|^2 \rightarrow 0$. Therefore, Bessel's inequality immediately implies the following simplified form of the *Riemann–Lebesgue Lemma*.

Lemma 3.37. *If $f \in L^2$ is square integrable, then its Fourier coefficients satisfy*

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx \quad \longrightarrow \quad 0 \quad \text{as} \quad |k| \rightarrow \infty. \tag{3.111}$$

Remark: This result is equivalent to the decay of the real Fourier coefficients

$$\left. \begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx \\ b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx \end{aligned} \right\} \longrightarrow 0 \quad \text{as} \quad k \rightarrow \infty. \tag{3.112}$$

As before, the convergence of the sum (3.110) requires that the coefficients c_k not tend to zero too slowly. For instance, requiring the power bound $|c_k| \leq M|k|^{-\alpha}$ for some $\alpha > \frac{1}{2}$ suffices to ensure that $\sum_{k=-\infty}^{\infty} |c_k|^2 < \infty$. Thus, as we should have expected, convergence in norm of the Fourier series imposes less restrictive requirements on the decay

of the Fourier coefficients than uniform convergence — which needed $\alpha > 1$. Indeed, a Fourier series may very well converge in norm to a discontinuous function, which is not possible under uniform convergence. In fact, there are pathological continuous functions whose Fourier series do not converge uniformly, even failing to converge at some points. A deep result states that the Fourier series of a continuous function converges everywhere except possibly on a set of measure zero, [153]. Again, the subtle details of the convergence of Fourier series are rather delicate, and lack of space and analytical savvy precludes us from delving any further into these topics.

Completeness

Calculations in vector spaces rely on the specification of a basis, meaning a set of linearly independent elements that span the space. The choice of basis serves to introduce a system of local coordinates on the space, namely, the coefficients in the expression of an element as a linear combination of basis elements. Orthogonal and orthonormal bases are particularly handy, since the coordinates are immediately calculated by taking inner products, while general bases require solving linear systems. In finite-dimensional vector spaces, all bases contain the same number of elements, which, by definition, is the dimension of the space. A vector space is, therefore, infinite-dimensional if it contains an infinite number of linearly independent elements. However, the question of when such a collection forms a basis for the space is considerably more subtle, and mere counting will no longer suffice. Indeed, omitting a finite number of elements from an infinite collection would still leave an infinite number, but the latter will certainly not span the space. Moreover, we cannot, in general, expect to write a general element of an infinite-dimensional space as a finite linear combination of basis elements, and so subtle questions of convergence of infinite series must also be addressed if we are to properly define the concept.

The definition of a basis of an infinite-dimensional vector space rests on the idea of completeness. We shall discuss completeness in a general, abstract setting, but the key example is, of course, the Hilbert space $L^2[-\pi, \pi]$ and the systems of trigonometric or complex exponential functions. Additional examples, including Bessel functions, spherical harmonics, and other systems of eigenfunctions of self-adjoint boundary value problems arising from the solution of partial differential equations, will appear in later chapters.

For simplicity, we only define completeness in the case of orthonormal systems. (Similar arguments will clearly apply to orthogonal systems, but normality helps to streamline the presentation.) Let V be an infinite-dimensional complex[†] inner product space. Suppose that $\varphi_1, \varphi_2, \varphi_3, \dots \in V$ form an *orthonormal* collection of elements of V , meaning that

$$\langle \varphi_i; \varphi_j \rangle = \begin{cases} 1 & i = j, \\ 0, & i \neq j. \end{cases} \quad (3.113)$$

A straightforward argument — see Exercise ■ — proves that the φ_i are linearly indepen-

[†] The results are equally valid in real inner product spaces, with slightly simpler proofs.

dent. Given $f \in V$, we form its *generalized Fourier series*

$$f \sim \sum_{k=1}^{\infty} c_k \varphi_k, \quad \text{where} \quad c_k = \langle f; \varphi_k \rangle. \quad (3.114)$$

The coefficients are obtained by formally taking the inner product of the series with φ_k and invoking the orthonormality conditions (3.113).

Definition 3.38. An orthonormal system $\varphi_1, \varphi_2, \varphi_3, \dots \in V$ is called *complete* if, for any $f \in V$, the generalized Fourier series (3.114) converges in norm to f :

$$\|f - s_n\| \longrightarrow 0, \quad \text{as } n \rightarrow \infty, \quad \text{where } s_n = \sum_{k=1}^n c_k \varphi_k \quad (3.115)$$

is the n^{th} partial sum of the generalized Fourier series (3.114).

Thus, completeness requires that every element can be arbitrarily closely approximated (in norm) by a suitable linear combination of the basis elements. A complete orthonormal system should be viewed as the infinite-dimensional version of an orthonormal basis of a finite-dimensional vector space.

The key result for classical Fourier series is that the complex exponentials, or, equivalently, the trigonometric functions, form a complete system. An indication of its proof will appear below.

Theorem 3.39. *The complex exponentials e^{ikx} , $k = 0, \pm 1, \pm 2, \dots$, form a complete orthonormal system in $L^2 = L^2[-\pi, \pi]$. In other words, if $s_n(x)$ denotes the n^{th} partial sum of the Fourier series of the square-integrable function $f(x) \in L^2$, then $\lim_{n \rightarrow \infty} \|f - s_n\| = 0$.*

In order to understand completeness, let us describe some equivalent characterizations. *Plancherel's formula* is the infinite-dimensional counterpart of formula (B.26) for the norm of a vector in terms of its coordinates with respect to an orthonormal basis.

Theorem 3.40. *The orthonormal system $\varphi_1, \varphi_2, \varphi_3, \dots \in V$ is complete if and only if Plancherel's formula*

$$\|f\|^2 = \sum_{k=1}^{\infty} |c_k|^2 = \sum_{k=1}^{\infty} \langle f; \varphi_k \rangle^2, \quad (3.116)$$

holds for every $f \in V$.

Proof: We begin by computing[†] the squared norm

$$\|f - s_n\|^2 = \|f\|^2 - 2 \operatorname{Re} \langle f; s_n \rangle + \|s_n\|^2.$$

[†] We are, in essence, repeating the proofs of Theorem 3.36 and the subsequent trigonometric Bessel inequality (3.110) in a more abstract setting.

Substituting the formula (3.115) for the partial sums, we find, by orthonormality,

$$\|s_n\|^2 = \sum_{k=1}^n |c_k|^2, \quad \text{while} \quad \langle f; s_n \rangle = \sum_{k=1}^n \overline{c_k} \langle f; \varphi_k \rangle = \sum_{k=1}^n |c_k|^2.$$

Therefore,

$$0 \leq \|f - s_n\|^2 = \|f\|^2 - \sum_{k=1}^n |c_k|^2. \quad (3.117)$$

The fact that the left hand side of (3.117) is non-negative for all n implies the abstract form of *Bessel's inequality*

$$\sum_{k=1}^{\infty} |c_k|^2 \leq \|f\|^2, \quad (3.118)$$

which is valid for *any* orthonormal system of elements in an inner product space. The Fourier series version (3.110) is a particular case of this general result. As we noted above, Bessel's inequality implies that the generalized Fourier coefficients c_k must tend to zero reasonably rapidly in order that the sum of their squares converges.

Plancherel's Theorem 3.40, thus, states that the system of functions is complete if and only if the Bessel inequality is, in fact, an equality. Indeed, letting $n \rightarrow \infty$ in (3.117), we find

$$\lim_{n \rightarrow \infty} \|f - s_n\|^2 = \|f\|^2 - \lim_{n \rightarrow \infty} \sum_{k=1}^n |c_k|^2 = \|f\|^2 - \sum_{k=1}^{\infty} |c_k|^2.$$

Therefore, the completeness condition (3.115) holds if and only if the right hand side vanishes, which is the Plancherel identity (3.116). *Q.E.D.*

Corollary 3.41. *Let $f, g \in V$. Then their Fourier coefficients $c_k = \langle f; \varphi_k \rangle$, $d_k = \langle g; \varphi_k \rangle$, satisfy Parseval's formula*

$$\langle f; g \rangle = \sum_{k=1}^{\infty} c_k \overline{d_k}. \quad (3.119)$$

Proof: Using the identity in Exercise ■(b),

$$\langle f; g \rangle = \frac{1}{4} (\|f + g\|^2 - \|f - g\|^2 + i\|f + ig\|^2 - i\|f - ig\|^2).$$

Parseval's formula results from applying Plancherel's formula (3.116) to each term on the right hand side:

$$\langle f; g \rangle = \frac{1}{4} \sum_{k=1}^{\infty} (|c_k + d_k|^2 - |c_k - d_k|^2 + i|c_k + id_k|^2 - i|c_k - id_k|^2) = \sum_{k=1}^{\infty} c_k \overline{d_k},$$

by a straightforward algebraic manipulation. *Q.E.D.*

Note that Plancherel's formula is a special case of Parseval's formula, obtained by setting $f = g$. In the particular case of the complex exponential basis of $L^2[-\pi, \pi]$, the Plancherel and Parseval formulae tell us that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{k=-\infty}^{\infty} |c_k|^2, \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx = \sum_{k=-\infty}^{\infty} c_k \overline{d_k}, \quad (3.120)$$

where $c_k = \langle f; e^{ikx} \rangle$, $d_k = \langle g; e^{ikx} \rangle$ are the ordinary Fourier coefficients of the complex-valued functions $f(x)$ and $g(x)$. In Exercise ■, you are asked to rewrite these formulas in terms of the real Fourier coefficients.

Completeness also tells us that a function is uniquely determined by its Fourier coefficients.

Proposition 3.42. *If the orthonormal system $\varphi_1, \varphi_2, \dots \in V$ is complete, then the only element $f \in V$ with all zero Fourier coefficients, $0 = c_1 = c_2 = \dots$, is the zero element: $f = 0$. More generally, two elements $f, g \in V$ have the same Fourier coefficients if and only if they are the same: $f = g$.*

Proof: The proof is an immediate consequence of Plancherel's formula. Indeed, if $c_k = 0$, then (3.116) implies that $\|f\| = 0$ and hence $f = 0$. The second statement follows by applying the first to their difference $f - g$. *Q.E.D.*

Another way of stating this result is that the only function which is orthogonal to every element of a complete orthonormal system is the zero function[†]. In other words, a complete orthonormal system is maximal in the sense that no further orthonormal elements can be appended to it.

Let us now discuss the completeness of the Fourier trigonometric and complex exponential functions. We shall prove the completeness criterion only for continuous functions, leaving the harder general proof to the references, [44, 153]. According to Theorem 3.30, if $f(x)$ is continuous, 2π periodic, and piecewise C^1 , its Fourier series converges uniformly

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \quad \text{for all } -\pi \leq x \leq \pi.$$

The same holds for its complex conjugate $\overline{f(x)}$. Therefore,

$$|f(x)|^2 = f(x) \overline{f(x)} = f(x) \sum_{k=-\infty}^{\infty} \overline{c_k} e^{-ikx} = \sum_{k=-\infty}^{\infty} \overline{c_k} f(x) e^{-ikx},$$

which also converges uniformly by (3.92). Formula (3.93) permits us to integrate both sides from $-\pi$ to π , yielding

$$\|f\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{c_k} f(x) e^{-ikx} dx = \sum_{k=-\infty}^{\infty} c_k \overline{c_k} = \sum_{k=-\infty}^{\infty} |c_k|^2.$$

[†] Or, to be more technically accurate, any function which is zero outside a set of measure zero.

Therefore, Plancherel's formula (3.116) holds for any continuous, piecewise C^1 function.

With some additional technical work, this result is used to establish the validity of Plancherel's formula for all $f \in L^2$, the key step being to suitably approximate f by such continuous, piecewise C^1 functions. With this in hand, completeness is an immediate consequence of Theorem 3.40. *Q.E.D.*

Pointwise Convergence

Let us finally return to the Pointwise Convergence Theorem 3.8. The goal is to prove that, under the appropriate hypotheses on $f(x)$, namely 2π periodic and piecewise C^1 , the limit of its partial Fourier sums is

$$\lim_{n \rightarrow \infty} s_n(x) = \frac{1}{2} [f(x^+) + f(x^-)]. \quad (3.121)$$

We begin by substituting the formulae (3.65) for the complex Fourier coefficients into the formula (3.103) for the n^{th} partial sum:

$$\begin{aligned} s_n(x) &= \sum_{k=-n}^n c_k e^{ikx} = \sum_{k=-n}^n \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iky} dy \right) e^{ikx} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \left(\sum_{k=-n}^n e^{ik(x-y)} \right) dy. \end{aligned} \quad (3.122)$$

To proceed further, we need to calculate the final summation

$$\sum_{k=-n}^n e^{ikx} = e^{-inx} + \dots + e^{-ix} + 1 + e^{ix} + \dots + e^{inx}.$$

This, in fact, has the form of a geometric sum,

$$\sum_{k=0}^m ar^k = a + ar + ar^2 + \dots + ar^m = a \left(\frac{r^{m+1} - 1}{r - 1} \right), \quad (3.123)$$

with $m + 1 = 2n + 1$ summands, initial term $a = e^{-inx}$, and ratio $r = e^{ix}$. Therefore,

$$\begin{aligned} \sum_{k=-n}^n e^{ikx} &= e^{-inx} \left(\frac{e^{i(2n+1)x} - 1}{e^{ix} - 1} \right) = \frac{e^{i(n+1)x} - e^{-inx}}{e^{ix} - 1} \\ &= \frac{e^{i(n+\frac{1}{2})x} - e^{-i(n+\frac{1}{2})x}}{e^{ix/2} - e^{-ix/2}} = \frac{\sin(n + \frac{1}{2})x}{\sin \frac{1}{2}x}. \end{aligned} \quad (3.124)$$

In this computation, to pass from the first to the second line, we multiplied numerator and denominator by $e^{-ix/2}$, after which we used the formula (3.60) for the sine function in terms of complex exponentials. Incidentally, (3.124) is equivalent to the intriguing trigonometric summation formula

$$1 + 2(\cos x + \cos 2x + \cos 3x + \dots + \cos nx) = \frac{\sin(n + \frac{1}{2})x}{\sin \frac{1}{2}x}. \quad (3.125)$$

Therefore, substituting back into (3.122), we find

$$\begin{aligned} s_n(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \frac{\sin(n + \frac{1}{2})(x - y)}{\sin \frac{1}{2}(x - y)} dy \\ &= \frac{1}{2\pi} \int_{x-\pi}^{x+\pi} f(x + y) \frac{\sin(n + \frac{1}{2})y}{\sin \frac{1}{2}y} dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x + y) \frac{\sin(n + \frac{1}{2})y}{\sin \frac{1}{2}y} dy. \end{aligned}$$

The second equality is the result of changing the integration variable from y to $x + y$; the final equality follows since the integrand is 2π periodic, and so its integrals over *any* interval of length 2π all have the same value; see Exercise ■.

Thus, to prove (3.121), it suffices to show that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{\pi} \int_0^{\pi} f(x + y) \frac{\sin(n + \frac{1}{2})y}{\sin \frac{1}{2}y} dy &= f(x^+), \\ \lim_{n \rightarrow \infty} \frac{1}{\pi} \int_{-\pi}^0 f(x + y) \frac{\sin(n + \frac{1}{2})y}{\sin \frac{1}{2}y} dy &= f(x^-). \end{aligned} \tag{3.126}$$

The proofs of the two formulae are identical, and so we concentrate on establishing the first. Since the integrand is even,

$$\frac{1}{\pi} \int_0^{\pi} \frac{\sin(n + \frac{1}{2})y}{\sin \frac{1}{2}y} dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(n + \frac{1}{2})y}{\sin \frac{1}{2}y} dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-n}^n e^{iky} dy = 1,$$

because only the constant term has a nonzero integral. Multiplying this formula by $f(x^+)$ and then subtracting the result from the first formula in (3.126) leads to

$$\lim_{n \rightarrow \infty} \frac{1}{\pi} \int_0^{\pi} \frac{f(x + y) - f(x^+)}{\sin \frac{1}{2}y} \sin(n + \frac{1}{2})y dy = 0, \tag{3.127}$$

which we now proceed to prove.

We claim that, for each fixed value of x , the function

$$g(y) = \frac{f(x + y) - f(x^+)}{\sin \frac{1}{2}y}$$

is piecewise continuous for all $0 \leq y \leq \pi$. Owing to our hypotheses on $f(x)$, the only problematic point is when $y = 0$, but then, by l'Hôpital's rule (for one-sided limits),

$$\lim_{y \rightarrow 0^+} g(y) = \lim_{y \rightarrow 0^+} \frac{f(x + y) - f(x^+)}{\sin \frac{1}{2}y} = \lim_{y \rightarrow 0^+} \frac{f'(x + y)}{\frac{1}{2} \cos \frac{1}{2}y} = 2f'(x^+).$$

Consequently, (3.127) will be established if we can show that

$$\lim_{n \rightarrow \infty} \frac{1}{\pi} \int_0^{\pi} g(y) \sin(n + \frac{1}{2})y dy = 0 \tag{3.128}$$

whenever g is piecewise continuous. Were it not for the extra $\frac{1}{2}$, this would immediately follow from the simplified Riemann–Lebesgue Lemma 3.37. More honestly, we can invoke the addition formula for $\sin\left(n + \frac{1}{2}\right)y$ to write

$$\frac{1}{\pi} \int_0^\pi g(y) \sin\left(n + \frac{1}{2}\right)y \, dy = \frac{1}{\pi} \int_0^\pi \left(g(y) \sin \frac{1}{2}y\right) \cos ny \, dy + \frac{1}{\pi} \int_0^\pi \left(g(y) \cos \frac{1}{2}y\right) \sin ny \, dy$$

The first integral is the n^{th} Fourier cosine coefficient for the piecewise continuous function $g(y) \sin \frac{1}{2}y$, while the second integral is the n^{th} Fourier sine coefficient for the piecewise continuous function $g(y) \cos \frac{1}{2}y$. Lemma 3.37 implies that both of these converge to zero as $n \rightarrow \infty$, and hence (3.128) holds. This completes the proof, thus establishing pointwise convergence of the Fourier series. *Q.E.D.*

Remark: An alternative approach to the last part of the proof is to use the general *Riemann–Lebesgue Lemma*, whose proof can be found in [44, 153].

Lemma 3.43. *Suppose $g(x)$ is piecewise continuous on $[a, b]$. Then*

$$0 = \lim_{\omega \rightarrow \infty} \int_a^b g(x) e^{i\omega x} \, dx = \lim_{\omega \rightarrow \infty} \int_a^b g(x) \cos \omega x \, dx + i \lim_{\omega \rightarrow \infty} \int_a^b g(x) \sin \omega x \, dx. \tag{3.129}$$

Intuitively, the Riemann–Lebesgue Lemma says that, as the frequency ω gets larger and larger, the increasingly rapid oscillations of $e^{i\omega x}$ tend to cancel each other out.

Remark: While the Fourier series of a merely continuous function need not converge pointwise everywhere, a deep theorem due to Carleson in 1966 states that the set of points where it does not converge has measure zero, and hence the exceptional points comprise a very small subset; see [44, 153].