Chapter 7

Fourier Transforms

Fourier series and their ilk are designed to solve boundary value problems on bounded intervals. The extension of Fourier methods to the entire real line leads naturally to the *Fourier transform*, an extremely powerful mathematical tool for the analysis of non-periodic functions. The Fourier transform is of fundamental importance in a broad range of applications, including both ordinary and partial differential equations, quantum mechanics, signal processing, control theory, and probability, to name but a few.

In this chapter, we begin by motivating the construction by investigating how Fourier series behave as the length of the interval goes to infinity. The resulting Fourier transform maps a function defined on physical space to a function defined on the space of frequencies, whose values quantify the amount of each periodic frequency residing in the function. The inverse Fourier transform reconstructs the original function from its transform. The integrals defining the Fourier transform and its inverse are remarkably alike, and this symmetry is exploited when assembling tables of Fourier transforms.

One of the most important properties of the Fourier transform is that it converts calculus: differentiation and integration — into algebra: multiplication and division. This underlies its application to boundary value problems for linear ordinary differential equations and, in the following chapter, boundary value and initial-boundary value problems for partial differential equations. In engineering applications, the Fourier transform is sometimes overshadowed by the Laplace transform, which is a particular subcase. The Laplace transform is better suited to solving initial value problems, [**23**, **40**], but will not be developed in this text.

The Fourier transform is, like Fourier series, completely compatible with the calculus of generalized functions. The final section contains a brief introduction to the analytical foundations of the subject, including the basics of Hilbert space. However, a full, rigorous development requires advanced tools from mathematical analysis, and the interested reader is therefore referred to more advanced texts, e.g., [**44**, **75**, **127**].

7.1. The Fourier Transform.

We begin by motivating the Fourier transform as a limiting case of Fourier series. Although the rigorous details are rather exacting, the underlying idea is not so difficult. Let $f(x)$ be a reasonably nice function defined for all $-\infty < x < \infty$. The goal is to construct a Fourier expansion for $f(x)$ in terms of basic trigonometric functions. One evident approach is to construct its Fourier series on progressively longer and longer intervals, and then take the limit as the lengths go to infinity. This limiting process converts the Fourier sums into integrals, and the resulting representation of a function is renamed the Fourier transform. Since we are dealing with an infinite interval, there are no longer any periodicity requirements on the function $f(x)$. Moreover, the frequencies represented in the Fourier transform are no longer constrained by the length of the interval, and so we are effectively decomposing a quite general, non-periodic function into a continuous superposition of trigonometric functions of all possible frequencies.

Let us present the details of this construction in a more concrete form. The computations will be significantly simpler if we work with the complex version of the Fourier series from the outset. Our starting point is the rescaled Fourier series (3.84) on a symmetric interval $[-\ell, \ell]$ of length 2ℓ, which we rewrite in the adapted form

$$f(x) \sim \sum_{\nu=-\infty}^{\infty} \sqrt{\frac{\pi}{2}}\, \frac{\widehat{f}_\ell(k_\nu)}{\ell}\, e^{\,\mathrm{i}\, k_\nu x}. \tag{7.1}$$

The sum is over the discrete collection of frequencies

$$k_\nu = \frac{\pi\nu}{\ell}, \qquad\qquad \nu = 0, \pm 1, \pm 2, \ldots\,, \tag{7.2}$$

corresponding to those trigonometric functions that have period 2ℓ. For reasons that will soon become apparent, the Fourier coefficients of f are now denoted as

$$c_\nu = \langle\, f\,; e^{\,\mathrm{i}\, k_\nu x}\,\rangle = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\, e^{-\,\mathrm{i}\, k_\nu x}\, dx = \sqrt{\frac{\pi}{2}}\, \frac{\widehat{f}_\ell(k_\nu)}{\ell}, \tag{7.3}$$

so that

$$\widehat{f}_\ell(k_\nu) = \frac{1}{\sqrt{2\pi}}\int_{-\ell}^{\ell} f(x)\, e^{-\,\mathrm{i}\, k_\nu x}\, dx. \tag{7.4}$$

This reformulation of the basic Fourier series formula allows us to smoothly pass to the limit when the interval's length $\ell \to \infty$.

On an interval of length 2ℓ, the frequencies (7.2) required to represent a function in Fourier series form are equally distributed, with interfrequency spacing

$$\Delta k = k_{\nu+1} - k_\nu = \frac{\pi}{\ell}\,. \tag{7.5}$$

As $\ell \to \infty$, the spacing $\Delta k \to 0$, and so the relevant frequencies become more and more densely packed in the space of all possible frequencies: $-\infty < k < \infty$. In the limit, we anticipate that *all* possible frequencies will be represented. Indeed, letting $k_\nu = k$ be arbitrary in (7.4), and sending $\ell \to \infty$, results in the infinite integral

$$\widehat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-\,\mathrm{i}\, k x}\, dx \tag{7.6}$$

known as the *Fourier transform* of the function $f(x)$. If $f(x)$ is a reasonably nice function, e.g., piecewise continuous and decaying to 0 reasonably quickly as $|\,x\,| \to \infty$, its Fourier transform $\widehat{f}(k)$ is defined for all possible frequencies $-\infty < k < \infty$. This formula will sometimes conveniently be abbreviated as

$$\widehat{f}(k) = \mathcal{F}[\,f(x)\,], \tag{7.7}$$

where $\mathcal{F}$ is the *Fourier transform operator*.

To reconstruct the function from its Fourier transform, we employ a similar limiting procedure on the Fourier series (7.1), which we first rewrite in a more suggestive form,

$$f(x) \sim \frac{1}{\sqrt{2\pi}} \sum_{\nu=-\infty}^{\infty} \widehat{f}_\ell(k_\nu)\, e^{\,i\,k_\nu x}\, \Delta k, \tag{7.8}$$

using (7.5). For each fixed value of x, the right hand side has the form of a Riemann sum, [**8**, **112**], approximating the integral

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}_\ell(k)\, e^{\,i\,k\,x}\, dk.$$

As $\ell \to \infty$, the functions (7.4) converge to the Fourier transform: $\widehat{f}_\ell(k) \to \widehat{f}(k)$; moreover, the interfrequency spacing $\Delta k = \pi/\ell \to 0$, and so one expects the Riemann sums to converge to the limiting integral

$$f(x) \sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(k)\, e^{\,i\,k\,x}\, dk. \tag{7.9}$$

The resulting formula serves to define the *inverse Fourier transform*, which is used to recover the original signal from its Fourier transform. In this manner, the Fourier series has become a Fourier integral that reconstructs the function $f(x)$ as a (continuous) superposition of complex exponentials $e^{\,i\,k\,x}$ of *all* possible frequencies, with $\widehat{f}(k)/\sqrt{2\pi}$ prescribing the amount of contribution of the complex exponential of frequency k. In abbreviated form, formula (7.9) can be written

$$f(x) = \mathcal{F}^{-1}[\,\widehat{f}(k)\,], \tag{7.10}$$

thus defining the inverse of the Fourier transform operator (7.7).

It is worth pointing out that both the Fourier transform (7.7) and its inverse (7.10) define linear maps on function space. This means that the Fourier transform of the sum of two functions is the sum of their individual transforms, while multiplying a function by a constant multiplies its Fourier transform by the same factor:

$$\begin{aligned} \mathcal{F}[\,f(x)+g(x)\,] &= \mathcal{F}[\,f(x)\,] + \mathcal{F}[\,g(x)\,] = \widehat{f}(k) + \widehat{g}(k), \\ \mathcal{F}[\,c\,f(x)\,] &= c\,\mathcal{F}[\,f(x)\,] = c\,\widehat{f}(k). \end{aligned} \tag{7.11}$$

A similar statement holds for the inverse Fourier transform $\mathcal{F}^{-1}$.

Recapitulating, by letting the length of the interval go to ∞, the discrete Fourier series has become a continuous Fourier integral, while the Fourier coefficients, which were defined only at a discrete collection of possible frequencies, have become a complete function $\widehat{f}(k)$ defined on all of frequency space $k \in \mathbb{R}$. The reconstruction of $f(x)$ from its Fourier transform $\widehat{f}(k)$ via (7.9) can be rigorously justified under suitable hypotheses. For example, if $f(x)$ is piecewise C^1 on all of $\mathbb{R}$ and decays reasonably rapidly, $f(x) \to 0$ as $|\,x\,| \to \infty$, in order that its Fourier integral (7.6) converges absolutely, then it can be proved, [**44**], that the inverse Fourier integral (7.9) will converge to $f(x)$ at all points of continuity, and

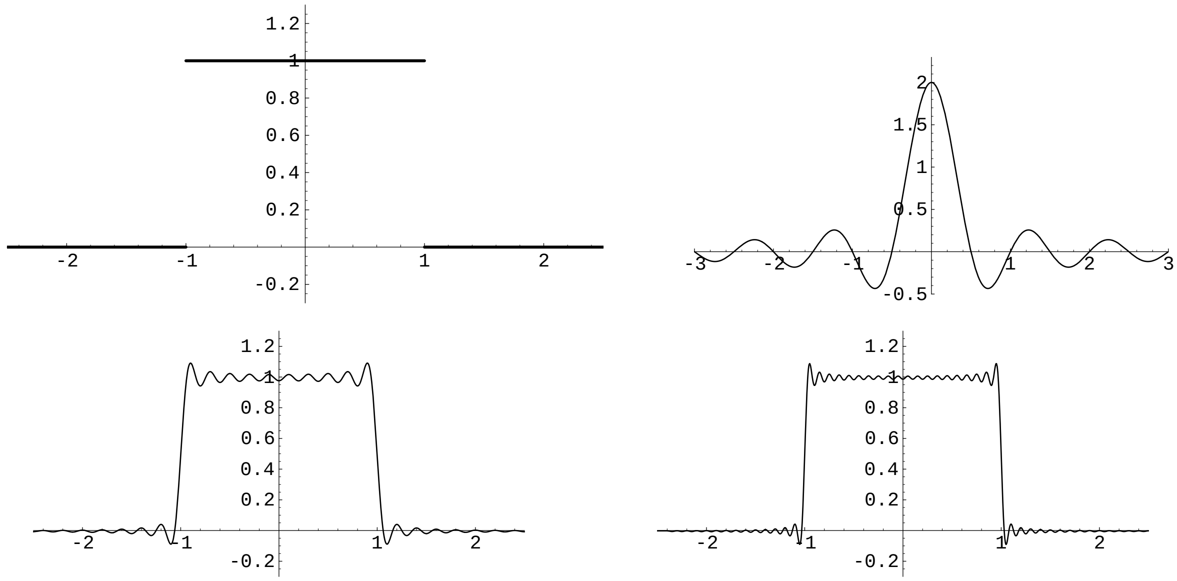


Figure 7.1. Fourier Transform of a Rectangular Pulse.

to the midpoint $\frac{1}{2}(f(x^-) + f(x^+))$ at jump discontinuities — just like a Fourier series. In particular, its Fourier transform $\widehat{f}(k) \to 0$ must also decay as $|\,k\,| \to 0$, implying that (as with Fourier series) the very high frequency modes make negligible contributions to the reconstruction of the signal. A more precise result will be formulated in Theorem 7.13 below.

Example 7.1. The Fourier transform of the rectangular pulse (or box function[†])

$$f(x) = \sigma(x+a) - \sigma(x-a) = \begin{cases} 1, & -a < x < a, \\ 0, & |\,x\,| > a. \end{cases} \tag{7.12}$$

of width $2a$ is easily computed:

$$\widehat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-\,i\,k\,x}\, dx = \frac{e^{\,i\,k a} - e^{-\,i\,k a}}{\sqrt{2\pi}\ i\,k} = \sqrt{\frac{2}{\pi}}\ \frac{\sin a\,k}{k}\ . \tag{7.13}$$

On the other hand, the reconstruction of the pulse via the inverse transform (7.9) tells us that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{\,i\,k\,x} \sin a\,k}{k}\, dk = f(x) = \begin{cases} 1, & -a < x < a, \\ \frac{1}{2}, & x = \pm\, a, \\ 0, & |\,x\,| > a. \end{cases} \tag{7.14}$$

Note the convergence to the middle of the jump discontinuities at $x = \pm\, a$. Splitting this complex integral into its real and imaginary parts, we deduce a pair of striking trigono-

[†] $\sigma(x)$ is the step function (3.46).

metric integral identities

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos kx\ \sin ak}{k}\,dk=\begin{cases}1, & -a<x<a,\\ \frac{1}{2}, & x=\pm a,\\ 0, & |\,x\,|>a,\end{cases}\qquad\qquad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin kx\ \sin ak}{k}\,dk=0. \tag{7.15}$$

Just as many Fourier series yield nontrivial summation formulae, the reconstruction of a function from its Fourier transform often leads to nontrivial integration formulas. One *cannot* compute the integral (7.14) by the Fundamental Theorem of Calculus, since there is no elementary function whose derivative equals the integrand†. Moreover, it is not even clear that the integral converges; indeed, the amplitude of the oscillatory integrand decays like $1/|\,k\,|$, but the latter function does not have a convergent integral, and so the usual comparison test for infinite integrals, [**8**], fails to apply. The convergence of the integral is marginal at best: the trigonometric oscillations somehow overcome the slow rate of decay of $1/k$ and thereby induce the (conditional) convergence of the integral. In Figure 7.1 we display the box function with $a=1$, its Fourier transform, along with a reconstruction obtained by numerically integrating (7.15). Since we are dealing with an infinite integral, we must break off the numerical integrator by restricting it to a finite interval. The first graph is obtained by integrating from $-5\le k\le 5$ while the second is from $-10\le k\le 10$. The non-uniform convergence of the integral leads to the appearance of a Gibbs phenomenon at the two discontinuities, similar to what we observed in the non-uniform convergence of a Fourier series.

Example 7.2. Consider an exponentially decaying right-handed pulse‡

$$f_r(x)=\begin{cases}e^{-ax}, & x>0,\\ 0, & x<0,\end{cases} \tag{7.16}$$

where $a>0$. We compute its Fourier transform directly from the definition:

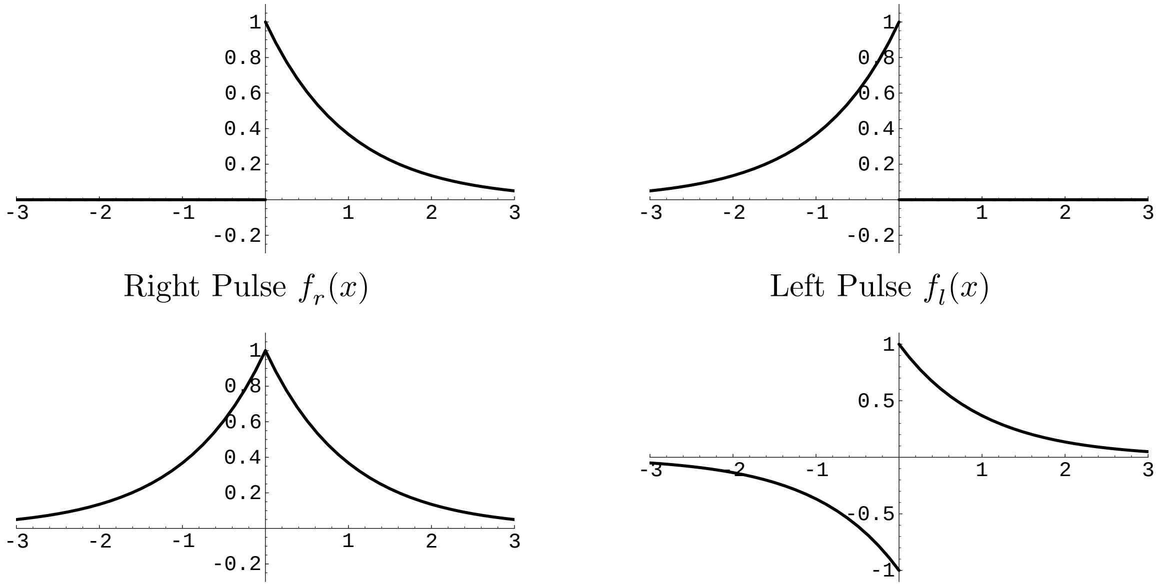
$$\widehat{f}(k)=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}e^{-ax}\,e^{-\mathrm{i}kx}\,dx=-\,\frac{1}{\sqrt{2\pi}}\,\frac{e^{-(a+\mathrm{i}k)x}}{a+\mathrm{i}k}\,\bigg|_{x=0}^{\infty}=\frac{1}{\sqrt{2\pi}\,(a+\mathrm{i}k)}\,.$$

As in the preceding example, the inverse Fourier transform produces a nontrivial complex integral identity:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\mathrm{i}kx}}{a+\mathrm{i}k}\,dk=\begin{cases}e^{-ax}, & x>0,\\ \frac{1}{2}, & x=0,\\ 0, & x<0.\end{cases} \tag{7.17}$$

† One can use Euler's formula (3.59) to reduce (7.14) to a version of the *exponential integral* $\int (e^{\alpha k}/k)\,dk$, but it can be proved, [**25**], that neither integral can be written in terms of elementary functions.

‡ Note that we can't Fourier transform the entire exponential function e^{-ax} because it does not go to zero at both $\pm\infty$, which is required for the integral (7.6) to converge.



Right Pulse $f_r(x)$ Left Pulse $f_l(x)$

Even Pulse $f_e(x)$ Odd Pulse $f_o(x)$

Figure 7.2. Exponential Pulses.

Similarly, a pulse that decays to the left,

$$f_l(x) = \begin{cases} e^{a\,x}, & x < 0, \\ 0, & x > 0, \end{cases} \tag{7.18}$$

where $a > 0$ is still positive, has Fourier transform

$$\widehat{f}(k) = \frac{1}{\sqrt{2\pi}\,(a - \mathrm{i}\,k)}\,. \tag{7.19}$$

This also follows from the general fact that the Fourier transform of $f(-x)$ is $\widehat{f}(-k)$; see Exercise ■. The even exponentially decaying pulse

$$f_e(x) = e^{-a\,|\,x\,|} \tag{7.20}$$

is merely the sum of left and right pulses: $f_e = f_r + f_l$. Thus, by linearity,

$$\widehat{f}_e(k) = \frac{1}{\sqrt{2\pi}\,(a + \mathrm{i}\,k)} \;+\; \frac{1}{\sqrt{2\pi}\,(a - \mathrm{i}\,k)} = \sqrt{\frac{2}{\pi}}\;\frac{a}{k^2 + a^2}\,, \tag{7.21}$$

The resulting Fourier transform is real and even because $f_e(x)$ is a real even function; see Exercise ■. The inverse Fourier transform (7.9) produces another nontrivial integral identity:

$$e^{-a\,|\,x\,|} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a\,e^{\,\mathrm{i}\,k\,x}}{k^2 + a^2}\,dk = \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{\cos k\,x}{k^2 + a^2}\;dk. \tag{7.22}$$

(The imaginary part of the integral vanishes because the integrand is odd.) On the other hand, the odd exponentially decaying pulse,

$$f_o(x) = (\operatorname{sign} x)\,e^{-a\,|\,x\,|} = \begin{cases} e^{-a\,x}, & x > 0, \\ -\,e^{a\,x}, & x < 0, \end{cases} \tag{7.23}$$

is the difference of the right and left pulses, $f_o = f_r - f_l$, and has purely imaginary and odd Fourier transform

$$\widehat{f}_o(k) = \frac{1}{\sqrt{2\pi}\,(a + \mathrm{i}\,k)} - \frac{1}{\sqrt{2\pi}\,(a - \mathrm{i}\,k)} = -\,\mathrm{i}\,\sqrt{\frac{2}{\pi}}\,\frac{k}{k^2 + a^2}\,. \tag{7.24}$$

The inverse transform is

$$(\operatorname{sign} x)\,e^{-a\,|\,x\,|} = -\,\frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty} \frac{k\,e^{\,\mathrm{i}\,k\,x}}{k^2 + a^2}\,dk = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{k\sin k\,x}{k^2 + a^2}\,dk. \tag{7.25}$$

As a final example, consider the rational function

$$f(x) = \frac{1}{x^2 + c^2}\,, \qquad \text{where} \qquad c > 0. \tag{7.26}$$

Its Fourier transform requires integrating

$$\widehat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{-\,\mathrm{i}\,k\,x}}{x^2 + a^2}\,dx. \tag{7.27}$$

The indefinite integral (anti-derivative) does not appear in basic integration tables, and, in fact, cannot be done in terms of elementary functions. However, we have just managed to evaluate this particular integral! Look at (7.22). If we change x to k and k to $-x$, then we exactly recover the integral (7.27) up to a factor of $a\sqrt{2/\pi}$. We conclude that the Fourier transform of (7.26) is

$$\widehat{f}(k) = \sqrt{\frac{\pi}{2}}\,\frac{e^{-a\,|\,k\,|}}{a}\,. \tag{7.28}$$

This last example is indicative of an important general fact. The reader has no doubt already noted the remarkable similarity between the Fourier transform (7.6) and its inverse (7.9). Indeed, the only difference is that the former has a minus sign in the exponential. This implies the following *Symmetry Principle* relating the direct and inverse Fourier transforms.

Theorem 7.3. *If the Fourier transform of the function $f(x)$ is $\widehat{f}(k)$, then the Fourier transform of $\widehat{f}(x)$ is $f(-k)$.*

The Symmetry Principle allows us to reduce the tabulation of Fourier transforms by half. For instance, referring back to Example 7.1, we deduce that the Fourier transform of the function

$$f(x) = \sqrt{\frac{2}{\pi}}\,\frac{\sin a\,x}{x}$$

is

$$\widehat{f}(k) = \sigma(-\,k + a) - \sigma(-\,k - a) = \sigma(k + a) - \sigma(k - a) = \begin{cases} 1, & -\,a < k < a, \\ \frac{1}{2}, & k = \pm\,a, \\ 0, & |\,k\,| > a. \end{cases} \tag{7.29}$$

Note that, by linearity, we can divide both $f(x)$ and $\widehat{f}(k)$ by $\sqrt{2/\pi}$ to deduce the Fourier transform of $\dfrac{\sin a\,x}{x}$.

Warning: Some authors omit the $\sqrt{2\pi}$ factor in the definition (7.6) of the Fourier transform $\widehat{f}(k)$. This alternative convention does have a slight advantage of eliminating many $\sqrt{2\pi}$ factors in the Fourier transforms of elementary functions. However, this necessitates an extra such factor in the reconstruction formula (7.9), which is achieved by replacing $\sqrt{2\pi}$ by 2π. A significant disadvantage is that the resulting formulae for the Fourier transform and its inverse are not as close, and so the Symmetry Principle of Theorem 7.3 requires some modification. (On the other hand, convolution — to be discussed below — is a little easier without the extra factor.) Yet another, more recent convention can be found in Exercise ■. When consulting any particular reference, the reader *always* needs to check which Fourier transform convention is being used.

All of the functions in Example 7.2 required $a > 0$ for the Fourier integrals to converge. The functions that emerge in the limit as a goes to 0 are of fundamental importance. Let us start with the odd exponential pulse (7.23). When $a \to 0$, the function $f_o(x)$ converges to the *sign function*

$$f(x) = \operatorname{sign} x = \sigma(x) - \sigma(-x) = \begin{cases} +1, & x > 0, \\ -1, & x < 0. \end{cases} \tag{7.30}$$

Taking the limit of the Fourier transform (7.24) leads to

$$\widehat{f}(k) = -\,\mathrm{i}\,\sqrt{\frac{2}{\pi}}\,\frac{1}{k}\,. \tag{7.31}$$

The nonintegrable singularity of $\widehat{f}(k)$ at $k = 0$ is indicative of the fact that the sign function does *not* decay as $|\,x\,| \to \infty$. In this case, neither the Fourier transform integral nor its inverse are well-defined as standard (Riemann, or even Lebesgue, [**112**]) integrals. Nevertheless, it is possible to rigorously justify these results within the framework of generalized functions.

More interesting are the even pulse functions $f_e(x)$, which, in the limit $a \to 0$, become the constant function

$$f(x) \equiv 1. \tag{7.32}$$

The limit of the Fourier transform (7.21) is

$$\lim_{a \to 0} \sqrt{\frac{2}{\pi}}\,\frac{2a}{k^2 + a^2} = \begin{cases} 0, & k \neq 0, \\ \infty, & k = 0. \end{cases} \tag{7.33}$$

This limiting behavior should remind the reader of our construction (5.10) of the delta function as the limit of the functions

$$\delta(x) = \lim_{n \to \infty} \frac{n}{\pi\,(1 + n^2 x^2)} = \lim_{a \to 0} \frac{a}{\pi\,(a^2 + x^2)}\,.$$

Comparing with (7.33), we conclude that the Fourier transform of the constant function (7.32) is a multiple of the delta function in the frequency variable:

$$\widehat{f}(k) = \sqrt{2\pi}\;\delta(k). \tag{7.34}$$

The direct transform integral

$$\delta(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\,\mathrm{i}\,k\,x}\,dx$$

is, strictly speaking, not defined because the infinite integrals of the oscillatory sine and cosine functions don't converge! However, this identity can be validly interpreted within the framework of weak convergence and generalized functions. On the other hand, the inverse transform formula (7.9) yields

$$\int_{-\infty}^{\infty} \delta(k)\,e^{\,\mathrm{i}\,k\,x}\,dk = e^{\,\mathrm{i}\,k\,0} = 1,$$

which is in accord with the basic definition (5.16) of the delta function. As in the preceding case, the delta function singularity at $k = 0$ manifests the lack of decay of the constant function.

Conversely, the delta function $\delta(x)$ has constant Fourier transform

$$\widehat{\delta}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x)\,e^{-\,\mathrm{i}\,k\,x}\,dx = \frac{e^{-\,\mathrm{i}\,k\,0}}{\sqrt{2\pi}} \equiv \frac{1}{\sqrt{2\pi}}\,, \tag{7.35}$$

a result that also follows from the Symmerty Principle of Theorem 7.3. To determine the Fourier transform of a delta spike $\delta_\xi(x) = \delta(x-\xi)$ concentrated at position $x = \xi$, we compute

$$\widehat{\delta_\xi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x-\xi)\,e^{-\,\mathrm{i}\,k\,x}\,dx = \frac{e^{-\,\mathrm{i}\,k\,\xi}}{\sqrt{2\pi}}\,. \tag{7.36}$$

The result is a pure exponential in frequency space. Applying the inverse Fourier transform (7.9) leads, formally, to the remarkable identity

$$\delta_\xi(x) = \delta(x-\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\,\mathrm{i}\,k(x-\xi)}\,dk = \frac{1}{2\pi}\,\langle\, e^{\,\mathrm{i}\,k\,\xi}\,; e^{\,\mathrm{i}\,k\,x}\,\rangle\,, \tag{7.37}$$

where $\langle\,\cdot\,;\cdot\,\rangle$ denotes the L^2 Hermitian inner product on $\mathbb{R}$. Since the delta function vanishes for $x \neq \xi$, this identity is telling us that complex exponentials of differing frequencies are mutually orthogonal. However, this statement must be taken with a grain of salt, since the integral does not converge in any standard sense (either Riemann or Lebesgue). But it is possible to make sense of this identity within the language of generalized functions. Indeed, multiplying both sides by $f(x)$, and then integrating with respect to x, we find

$$f(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\,e^{-\,\mathrm{i}\,k(x-\xi)}\,dx\,dk. \tag{7.38}$$

This *is* a perfectly valid formula, being a restatement (or, rather, combination) of the basic formulae (7.6) and (7.9) connecting the direct and inverse Fourier transforms of the function $f(x)$.

Vice versa, the Symmetry Principle tells us that the Fourier transform of a pure exponential $e^{\,\mathrm{i}\,\kappa\,x}$ will be a shifted delta spike $\sqrt{2\pi}\,\delta(k-\kappa)$, concentrated at frequency $k = \kappa$. Both results are particular cases of the general Shift Theorem, whose proof is left as an exercise for the reader.

Short Table of Fourier Transforms

$f(x)$	$\widehat{f}(k)$
1	$\sqrt{2\pi}\,\delta(k)$
$\delta(x)$	$\dfrac{1}{\sqrt{2\pi}}$
$\sigma(x)$	$\sqrt{\dfrac{\pi}{2}}\,\delta(k) - \dfrac{\mathrm{i}}{\sqrt{2\pi}\,k}$
$\operatorname{sign} x$	$-\,\mathrm{i}\,\sqrt{\dfrac{2}{\pi}}\,\dfrac{1}{k}$
$\sigma(x+a) - \sigma(x-a)$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin a\,k}{k}$
$e^{-a\,x}\,\sigma(x)$	$\dfrac{1}{\sqrt{2\pi}\,(a + \mathrm{i}\,k)}$
$e^{a\,x}\,(1 - \sigma(x))$	$\dfrac{1}{\sqrt{2\pi}\,(a - \mathrm{i}\,k)}$
$e^{-a\,\lvert x\rvert}$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{a}{k^2 + a^2}$
$e^{-a\,x^2}$	$\dfrac{e^{-k^2/4a}}{\sqrt{2a}}$
$\tan^{-1} x$	$-\,\mathrm{i}\sqrt{\dfrac{\pi}{2}}\,\dfrac{e^{-\lvert k\rvert}}{k} + \dfrac{\pi^{3/2}}{\sqrt{2}}\,\delta(k)$
$f(c\,x + d)$	$\dfrac{e^{\mathrm{i}\,k\,d/c}}{\lvert c\rvert}\,\widehat{f}\left(\dfrac{k}{c}\right)$
$\overline{f(x)}$	$\overline{\widehat{f}(-k)}$
$\widehat{f}(x)$	$f(-k)$
$f'(x)$	$\mathrm{i}\,k\,\widehat{f}(k)$
$x\,f(x)$	$\mathrm{i}\,\widehat{f}\,'(k)$
$f * g(x)$	$\sqrt{2\pi}\,\widehat{f}(k)\,\widehat{g}(k)$

Note: The parameters a, c, d are real, with $a > 0$ and $c \neq 0$.

Theorem 7.4. *If $f(x)$ has Fourier transform $\widehat{f}(k)$, then the Fourier transform of the shifted function $f(x-\xi)$ is $e^{-\,i\,k\,\xi}\,\widehat{f}(k)$. Similarly, the transform of the product function $e^{\,i\,\kappa\,x}\,f(x)$, for real κ, is the shifted transform $\widehat{f}(k-\kappa)$.*

Since the Fourier transform uniquely associates a function $\widehat{f}(k)$ on frequency space with each (reasonable) function $f(x)$ on physical space, one can characterize functions by their transforms. Practical applications rely on tables (or, even better, computer algebra systems such as MATHEMATICA or MAPLE) that recognize a wide variety of transforms of basic functions of importance in applications. The accompanying table lists some of the most important examples of functions and their Fourier transforms. Note that, by applying the Symmetry Principle of Theorem 7.3, each tabular entry can be used to deduce two different Fourier transforms. A more extensive collection of Fourier transforms can be found in [**100**].

7.2. Derivatives and Integrals.

One of the most remarkable and important properties of the Fourier transform is that it converts calculus into algebra! More specifically, the two basic operations in calculus — differentiation and integration of functions — are realized as algebraic operations on their Fourier transforms. (The downside is that algebraic operations on functions become more complicated in the transform domain.)

Differentiation

Let us begin with derivatives. If we differentiate[†] the basic inverse Fourier transform formula

$$f(x) \;\sim\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(k)\, e^{\,i\,k\,x}\, dk.$$

with respect to x, we obtain

$$f'(x) \;\sim\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} i\,k\,\widehat{f}(k)\, e^{\,i\,k\,x}\, dk. \tag{7.39}$$

The resulting integral is itself in the form of an inverse Fourier transform, namely of $i\,k\,\widehat{f}(k)$ which immediately implies the following key result.

Proposition 7.5. *The Fourier transform of the derivative $f'(x)$ of a function is obtained by multiplication of its Fourier transform by $i\,k$:*

$$\mathcal{F}[\,f'(x)\,] = i\,k\,\widehat{f}(k). \tag{7.40}$$

Similarly, the Fourier transform of $x\,f(x)$ is obtained by differentiating the Fourier transform of $f(x)$:

$$\mathcal{F}[\,x\,f(x)\,] = i\,\frac{d\widehat{f}}{dk}\,. \tag{7.41}$$

[†] We are assuming the integrand is sufficiently nice so that we can bring the derivative under the integral sign; see [**44**, **140**] for a fully rigorous justification.

The second statement follows from the first by use of the Symmetry Principle of Theorem 7.3.

Example 7.6. The derivative of the even exponential pulse $f_e(x) = e^{-a|x|}$ is a multiple of the odd exponential pulse $f_o(x) = (\operatorname{sign} x)\, e^{-a|x|}$:

$$f_e'(x) = -a\,(\operatorname{sign} x)\, e^{-a|x|} = -a f_o(x).$$

Proposition 7.5 says that their Fourier transforms are related by

$$\mathrm{i}\, k\, \widehat{f}_e(k) = \mathrm{i} \sqrt{\frac{2}{\pi}}\, \frac{k\, a}{k^2 + a^2} = -a\, \widehat{f}_o(k),$$

as previously noted in (7.21, 24). On the other hand, the odd exponential pulse has a jump discontinuity of magnitude 2 at $x = 0$, and so its derivative contains a delta function:

$$f_o'(x) = -a\, e^{-a|x|} + 2\,\delta(x) = -a\, f_e(x) + 2\,\delta(x).$$

This is reflected in the relation between their Fourier transforms. If we multiply (7.24) by $\mathrm{i}\, k$ we obtain

$$\mathrm{i}\, k\, \widehat{f}_o(k) = \sqrt{\frac{2}{\pi}}\, \frac{k^2}{k^2 + a^2} = \sqrt{\frac{2}{\pi}} - \sqrt{\frac{2}{\pi}}\, \frac{a^2}{k^2 + a^2} = 2\, \widehat{\delta}(k) - a\, \widehat{f}_e(k).$$

The Fourier transform, just like Fourier series, is completely compatible with the calculus of generalized functions.

Higher order derivatives are handled by iterating the first order formula (7.40).

Corollary 7.7. *The Fourier transform of $f^{(n)}(x)$ is $(\mathrm{i}\, k)^n\, \widehat{f}(k)$.*

This result has an important consequence: the smoothness of $f(x)$ is manifested in the rate of decay of its Fourier transform $\widehat{f}(k)$. We already noted that the Fourier transform of a (nice) function must decay to zero at large frequencies: $\widehat{f}(k) \to 0$ as $|k| \to \infty$. (This result can be viewed as the Fourier transform version of the Riemann–Lebesgue Lemma 3.41.) If the n^{th} derivative $f^{(n)}(x)$ is also a reasonable function, then its Fourier transform $\widehat{f^{(n)}}(k) = (\mathrm{i}\, k)^n\, \widehat{f}(k)$ must go to zero as $|k| \to \infty$. This requires that $\widehat{f}(k)$ go to zero more rapidly than $|k|^{-n}$. Thus, the smoother $f(x)$, the more rapid the decay of its Fourier transform. As a general rule of thumb, local features of $f(x)$, such as smoothness, are manifested by global features of $\widehat{f}(k)$, such as decay for large $|k|$. The Symmetry Principle implies that reverse is also true: global features of $f(x)$ correspond to local features of $\widehat{f}(k)$. For instance, smoothness of $\widehat{f}(k)$ implies decay of $f(x)$ as $x \to \pm\infty$. This local-global duality is one of the major themes of Fourier theory.

Integration

Integration is the inverse operation to differentiation, and so should correspond to division by $\mathrm{i}\, k$ in frequency space. As with Fourier series, this is not completely correct; there is an extra constant involved, which contributes an additional delta function in frequency space.

Proposition 7.8. *If $f(x)$ has Fourier transform $\widehat{f}(k)$, then the Fourier transform of its integral* $g(x) = \int_{-\infty}^{x} f(y)\,dy$ *is*

$$\widehat{g}(k) = -\,\frac{\mathrm{i}}{k}\,\widehat{f}(k) + \pi\,\widehat{f}(0)\,\delta(k). \tag{7.42}$$

Proof: First notice that

$$\lim_{x\to-\infty} g(x) = 0, \qquad\qquad \lim_{x\to+\infty} g(x) = \int_{-\infty}^{\infty} f(x)\,dx = \sqrt{2\pi}\,\widehat{f}(0).$$

Therefore, by subtracting a suitable multiple of the step function from the integral, the resulting function

$$h(x) = g(x) - \sqrt{2\pi}\,\widehat{f}(0)\,\sigma(x)$$

decays to 0 at both $\pm\infty$. Consulting our table of Fourier transforms, we find

$$\widehat{h}(k) = \widehat{g}(k) - \pi\,\widehat{f}(0)\,\delta(k) + \frac{\mathrm{i}}{k}\,\widehat{f}(0)\,. \tag{7.43}$$

On the other hand,

$$h'(x) = f(x) - \sqrt{2\pi}\,\widehat{f}(0)\,\delta(x).$$

Since $h(x) \to 0$ as $|\,x\,| \to \infty$, we can apply our differentiation rule (7.40), and conclude that

$$\mathrm{i}\,k\,\widehat{h}(k) = \widehat{f}(k) - \widehat{f}(0). \tag{7.44}$$

Combining (7.43) and (7.44) establishes the desired formula (7.42). *Q.E.D.*

Example 7.9. The Fourier transform of the inverse tangent function

$$f(x) = \tan^{-1} x = \int_0^x \frac{dy}{1+y^2} = \int_{-\infty}^{x} \frac{dy}{1+y^2} \; - \; \frac{\pi}{2}$$

can be computed by combining Proposition 7.8 with (7.28):

$$\widehat{f}(k) = -\,\mathrm{i}\,\sqrt{\frac{\pi}{2}}\;\frac{e^{-\,|\,k\,|}}{k} \;+\; \frac{\pi^{3/2}}{\sqrt{2}}\,\delta(k).$$

7.3. Green's Functions and Convolution.

The fact that the Fourier transform converts differentiation in the physical domain into multiplication in the frequency domain is one of its most compelling features. A particularly important consequence is that it effectively transforms differential equations into algebraic equations, and thereby opens the door to their solution by elementary algebra! One begins by applying the Fourier transform to both sides of the differential equation under consideration. Solving the resulting algebraic equation will produce a formula for the Fourier transform of the desired solution, which can then be immediately reconstructed via the inverse Fourier transform. In the following chapter, we will use these techniques to solve partial differential equations.

Solution of Boundary Value Problems

The Fourier transform is particularly well adapted to boundary value problems on the entire real line. In place of the boundary conditions used on finite intervals, we look for solutions that decay to zero sufficiently rapidly as $|x| \to \infty$ — in order that their Fourier transform be well-defined (in the context of ordinary functions). In quantum mechanics, [**83**, **89**], these solutions are known as the *bound states* of the system, and correspond to subatomic particles that are trapped or localized in a region of space by some sort of force field. For example, the electrons in an atom are bound states localized by the electrostatic attraction of the nucleus.

As a specific example, consider the boundary value problem

$$-\frac{d^2u}{dx^2} + \omega^2 u = h(x), \qquad -\infty < x < \infty, \tag{7.45}$$

where $\omega > 0$ is a positive constant. In lieu of boundary conditions, we require that the solution $u(x) \to 0$ as $|x| \to \infty$.

We will solve this problem by applying the Fourier transform to both sides of the differential equation. Taking Corollary 7.7 into account, the result is a linear algebraic equation

$$k^2\,\widehat{u}(k) + \omega^2\,\widehat{u}(k) = \widehat{h}(k)$$

relating the Fourier transforms of u and h. Unlike the differential equation, the transformed equation can be immediately solved for

$$\widehat{u}(k) = \frac{\widehat{h}(k)}{k^2 + \omega^2}\,. \tag{7.46}$$

Therefore, we can reconstruct the solution by applying the inverse Fourier transform formula (7.9):

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\widehat{h}(k)\, e^{\,i\,k\,x}}{k^2 + \omega^2}\, dk. \tag{7.47}$$

For example, if the forcing function is an even exponential pulse,

$$h(x) = e^{-|x|} \qquad \text{with} \qquad \widehat{h}(k) = \sqrt{\frac{2}{\pi}}\,\frac{1}{k^2+1}\,,$$

then (7.47) writes the solution as a Fourier integral:

$$u(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{\,i\,k\,x}}{(k^2+\omega^2)(k^2+1)}\, dk = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos k\,x}{(k^2+\omega^2)(k^2+1)}\, dk\,,$$

noting that the imaginary part of the complex integral vanishes because the integrand is an odd function. (Indeed, if the forcing function is real, the solution must also be real.) The Fourier integral can be explicitly evaluated by using partial fractions to rewrite

$$\widehat{u}(k) = \sqrt{\frac{2}{\pi}}\,\frac{1}{(k^2+\omega^2)(k^2+1)} = \sqrt{\frac{2}{\pi}}\,\frac{1}{\omega^2-1}\left(\frac{1}{k^2+1} - \frac{1}{k^2+\omega^2}\right), \qquad \omega^2 \neq 1.$$

Thus, according to our Fourier Transform Table, the solution to this boundary value problem is

$$u(x) = \frac{e^{-\,|\,x\,|} - \dfrac{1}{\omega}\, e^{-\,\omega\,|\,x\,|}}{\omega^2 - 1} \qquad \text{when} \qquad \omega^2 \neq 1. \tag{7.48}$$

The reader may wish to verify that this function is indeed a solution, meaning that it is twice continuously differentiable (which is not so immediately apparent from the formula), decays to 0 as $|\,x\,| \to \infty$, and satisfies the differential equation everywhere. The "resonant" case $\omega^2 = 1$ is left as an exercise.

Remark: The method of partial fractions that you learned in first year calculus is often an effective tool for evaluating (inverse) Fourier transforms of rational functions.

A particularly important case is when the forcing function

$$h(x) = \delta_\xi(x) = \delta(x - \xi)$$

represents a unit impulse concentrated at $x = \xi$. The resulting square-integrable solution is the Green's function $G(x;\xi)$ for the boundary value problem. According to (7.46), its Fourier transform with respect to x is

$$\widehat{G}(k;\xi) = \frac{1}{\sqrt{2\pi}}\,\frac{e^{-\,\mathrm{i}\,k\,\xi}}{k^2 + \omega^2}\,,$$

which is the product of an exponential factor $e^{-\,\mathrm{i}\,k\,\xi}$, representing the Fourier transform of $\delta_\xi(x)$, times a multiple of the Fourier transform of the even exponential pulse $e^{-\,\omega\,|\,x\,|}$. We apply Theorem 7.4, and conclude that the Green's function for this boundary value problem is an exponential pulse centered at ξ, namely

$$G(x;\xi) = \frac{1}{2\,\omega}\, e^{-\,\omega\,|\,x - \xi\,|}\,. \tag{7.49}$$

Observe that, as with other self-adjoint boundary value problems, the Green's function is symmetric under interchange of x and ξ. As a function of x, it satisfies the homogeneous differential equation $-\,u'' + \omega^2\,u = 0$, except at the point $x = \xi$ when its derivative has a jump discontinuity of unit magnitude. It also decays as $|\,x\,| \to \infty$, as required by the boundary conditions. The Green's function superposition principle tells us that the solution to the inhomogeneous boundary value problem (7.45) under a general forcing can be represented in the integral form

$$u(x) = \int_{-\infty}^{\infty} G(x;\xi)\,h(\xi)\,d\xi = \frac{1}{2\,\omega}\int_{-\infty}^{\infty} e^{-\,\omega|\,x-\xi\,|}\,h(\xi)\,d\xi. \tag{7.50}$$

The reader may enjoy recovering the particular exponential solution (7.48) from this integral formula.

Convolution

In our solution to the boundary value problem (7.45), we ended up deriving a formula for its Fourier transform (7.46) as the product of two known Fourier transforms. The final

Green's function formula (7.50), obtained by applying the inverse Fourier transform, is indicative of a general property. Its right hand side has the form of a *convolution product* between functions.

Definition 7.10. The *convolution* of scalar functions $f(x)$ and $g(x)$ is the scalar function $h = f * g$ defined by the formula

$$h(x) = f(x) * g(x) = \int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, d\xi. \tag{7.51}$$

We record the basic properties of the convolution product, leaving their verification as exercises for the reader. All of these assume that the implied convolution integrals converge.

(a) *Symmetry*: $f * g = g * f$,

(b) *Bilinearity*: $\begin{cases} f * (a\,g + b\,h) = a\,(f * g) + b\,(f * h), \\ (a\,f + b\,g) * h = a\,(f * h) + b\,(g * h), \end{cases} \qquad a, b \in \mathbb{C},$

(c) *Associativity*: $f * (g * h) = (f * g) * h$,

(d) *Zero function*: $f * 0 = 0$,

(e) *Delta function*: $f * \delta = f$.

One tricky feature is that the constant function 1 is *not* a unit for the convolution product; indeed,

$$f * 1 = \int_{-\infty}^{\infty} f(\xi)\, d\xi$$

is a constant function — the total integral of f — not the original function $f(x)$. In fact, according to the final property, the delta function plays the role of the "convolution unit":

$$f(x) * \delta(x) = \int_{-\infty}^{\infty} f(x-\xi)\, \delta(\xi)\, d\xi = f(x).$$

In particular, our solution (7.49) has the form of a convolution product between an even exponential pulse $g(x) = (2\,\omega)^{-1}\, e^{-\omega\,|\,x\,|}$ and the forcing function:

$$u(x) = g(x) * h(x).$$

On the other hand, its Fourier transform (7.46) is, up to a factor, the ordinary multiplicative product

$$\widehat{u}(k) = \sqrt{2\pi}\; \widehat{g}(k)\, \widehat{h}(k)$$

of the Fourier transforms of g and h. In fact, this is a general property of the Fourier transform: convolution in the physical domain corresponds to multiplication in the frequency domain, and conversely.

Theorem 7.11. *The Fourier transform of the convolution $h(x) = f(x) * g(x)$ of two functions is a multiple of the product of their Fourier transforms*:

$$\widehat{h}(k) = \sqrt{2\pi}\; \widehat{f}(k)\, \widehat{g}(k). \tag{7.52}$$

Vice versa, the Fourier transform of their product $h(x) = f(x)\, g(x)$ is, up to multiple, the convolution of their Fourier transforms:

$$\widehat{h}(k) = \frac{1}{\sqrt{2\pi}}\, \widehat{f}(k) * \widehat{g}(k). \tag{7.53}$$

Proof: Combining the definition of the Fourier transform with the convolution formula (7.51), we find

$$\widehat{h}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x)\, e^{-\,\mathrm{i}\,k\,x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, e^{-\,\mathrm{i}\,k\,x}\, dx\, d\xi,$$

where we are assuming that the integrands are sufficiently nice to allow us to interchange the order of integration, [**8**]. Applying the change of variables $\eta = x - \xi$ in the inner integral produces

$$\begin{aligned}\widehat{h}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\eta)\, g(\xi)\, e^{-\,\mathrm{i}\,k(\xi+\eta)}\, d\xi\, d\eta \\ &= \sqrt{2\pi}\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\eta)\, e^{-\,\mathrm{i}\,k\,\eta}\, d\eta \right)\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\xi)\, e^{-\,\mathrm{i}\,k\,\xi}\, d\xi \right) = \sqrt{2\pi}\; \widehat{f}(k)\, \widehat{g}(k),\end{aligned}$$

proving (7.52). The second formula can be proved in a similar fashion, or by simply noting that it follows directly from the Symmetry Principle of Theorem 7.3. *Q.E.D.*

Example 7.12. We already know, (7.29), that the Fourier transform of

$$f(x) = \frac{\sin x}{x}$$

is the box function

$$\widehat{f}(k) = \sqrt{\frac{\pi}{2}}\,\big[\,\sigma(k+1) - \sigma(k-1)\,\big] = \begin{cases} \sqrt{\dfrac{\pi}{2}}\,, & -1 < k < 1, \\ 0, & |\,k\,| > 1. \end{cases}$$

We also know that the Fourier transform of

$$g(x) = \frac{1}{x} \qquad \text{is} \qquad \widehat{g}(k) = -\,\mathrm{i}\,\sqrt{\frac{\pi}{2}}\ \operatorname{sign} k.$$

Therefore, the Fourier transform of their product

$$h(x) = f(x)\, g(x) = \frac{\sin x}{x^2}$$

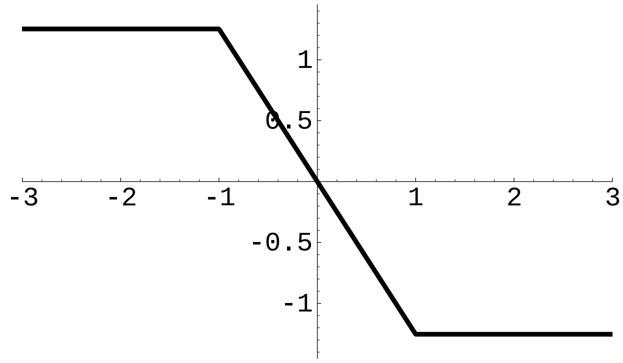


Figure 7.3. The Fourier transform of $\dfrac{\sin x}{x^2}$.

can be obtained by convolution:

$$\widehat{h}(k) = \frac{1}{\sqrt{2\pi}}\, \widehat{f}(k) * \widehat{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(\kappa)\, \widehat{g}(k-\kappa)\, d\kappa$$

$$= -\,\mathrm{i}\,\sqrt{\frac{\pi}{8}} \int_{-1}^{1} \operatorname{sign}(k-\kappa)\, d\kappa = \begin{cases} \mathrm{i}\sqrt{\dfrac{\pi}{2}} & k < -1, \\ -\,\mathrm{i}\sqrt{\dfrac{\pi}{2}}\, k, & -1 < k < 1, \\ -\,\mathrm{i}\sqrt{\dfrac{\pi}{2}} & k > 1. \end{cases}$$

A graph of $\widehat{h}(k)$ appears in Figure 7.3.

7.4. The Fourier Transform on Hilbert Space.

While we do not have the analytical tools to embark on a fully rigorous treatment of the mathematical theory underlying the Fourier transform, it is worth outlining a few of the more important features. We have already noted that the Fourier transform, when defined, is a linear map, taking functions $f(x)$ on physical space to functions $\widehat{f}(k)$ on frequency space. A critical question is precisely which function space should the theory be applied to. Not every function admits a Fourier transform in the classical sense[†] — the Fourier integral (7.6) is required to converge, and this places restrictions on the function and its asymptotics at large distances.

It turns out the proper setting for the rigorous theory is the *Hilbert space* of complex-valued square-integrable functions — the same infinite-dimensional vector space that lies at the heart of modern quantum mechanics. In Section 3.5, we already introduced the Hilbert space $\mathrm{L}^2[a,b]$ on a finite interval; here we adapt Definition 3.32 to the entire real line. Thus, the Hilbert space $\mathrm{L}^2 = \mathrm{L}^2(\mathbb{R})$ is the infinite-dimensional vector space consisting

[†] We leave aside the more advanced issues involving generalized functions in this subsection.

of all complex-valued functions $f(x)$ which are defined for all $x \in \mathbb{R}$ and have finite L^2 norm:

$$\| f \|^2 = \int_{-\infty}^{\infty} |\, f(x)\,|^2 \, dx < \infty. \tag{7.54}$$

For example, any piecewise continuous function that satisfies the decay criterion

$$|\, f(x)\,| \leq \frac{M}{|\, x\,|^{1/2+\delta}}, \qquad \text{for all sufficiently large} \qquad |\, x\,| \gg 0, \tag{7.55}$$

for some $M > 0$ and $\delta > 0$, belongs to L^2. However, as in Section 3.5, Hilbert space contains many more functions, and the precise definitions and identification of its elements is quite subtle. The Hermitian inner product on the complex Hilbert space L^2 is prescribed in the usual manner,

$$\langle\, f\,;g\,\rangle = \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\, dx, \tag{7.56}$$

so that $\| f \|^2 = \langle\, f\,; f\,\rangle$. The Cauchy–Schwarz inequality

$$|\,\langle\, f\,;g\,\rangle\,| \leq \| f \| \, \| g \| \tag{7.57}$$

ensures that the inner product integral is finite whenever $f, g \in \mathrm{L}^2$.

Let us state the fundamental result governing the effect of the Fourier transform on functions in Hilbert space. It can be regarded as a direct analog of the Pointwise Convergence Theorem 3.8 for Fourier series. A rigorous proof of this result can be found in [**44**, **127**].

Theorem 7.13. *If $f(x) \in \mathrm{L}^2$ is square-integrable, then its Fourier transform $\widehat{f}(k) \in \mathrm{L}^2$ is a well-defined, square-integrable function of the frequency variable k. If $f(x)$ is continuously differentiable at a point x, then its inverse Fourier transform* (7.9) *equals its value $f(x)$. More generally, if the right and left hand limits $f(x^-)$, $f(x^+)$, and also $f'(x^-)$, $f'(x^+)$ exist, then the inverse Fourier transform integral converges to the average value $\frac{1}{2}\left[\, f(x^-) + f(x^+)\,\right]$.*

Thus, the Fourier transform $\widehat{f} = \mathcal{F}[\,f\,]$ defines a linear transformation from L^2 functions of x to L^2 functions of k. In fact, the Fourier transform preserves inner products. This important result is known as *Parseval's formula*, whose Fourier series counterpart appeared in (3.119).

Theorem 7.14. *If $\widehat{f}(k) = \mathcal{F}[\,f(x)\,]$ and $\widehat{g}(k) = \mathcal{F}[\,g(x)\,]$, then $\langle\, f\,;g\,\rangle = \langle\, \widehat{f}\,;\widehat{g}\,\rangle$, i.e.,*

$$\int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\, dx = \int_{-\infty}^{\infty} \widehat{f}(k)\,\overline{\widehat{g}(k)}\, dk. \tag{7.58}$$

Proof: Let us sketch a formal proof that serves to motivate why this result is valid. We use the definition (7.6) of the Fourier transform to evaluate

$$\begin{aligned}\int_{-\infty}^{\infty} \widehat{f}(k)\,\overline{\widehat{g}(k)}\, dk &= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-\,i\,k\,x}\, dx \right) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{g(y)}\, e^{+\,i\,k\,y}\, dy \right) dk \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\,\overline{g(y)} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\,i\,k(x-y)}\, dk \right) dx\, dy.\end{aligned}$$

Now according to (7.37), the inner k integral can be replaced by a delta function $\delta(x-y)$, and hence

$$\int_{-\infty}^{\infty} \widehat{f}(k)\,\overline{\widehat{g}(k)}\,dk = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\,\overline{g(y)}\,\delta(x-y)\,dx\,dy = \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\,dx.$$

This completes our "proof"; see [**44**, **127**] for a rigorous version. *Q.E.D.*

In particular, orthogonal functions, satisfying $\langle\, f\,;g\,\rangle = 0$, will have orthogonal Fourier transforms, $\langle\, \widehat{f}\,;\widehat{g}\,\rangle = 0$. Choosing $f = g$ in Parseval's formula (7.58) results in the *Plancherel formula*

$$\|\, f\,\|^2 = \|\, \widehat{f}\,\|^2, \qquad \text{or, explicitly,} \qquad \int_{-\infty}^{\infty} |\, f(x)\,|^2\,dx = \int_{-\infty}^{\infty} |\, \widehat{f}(k)\,|^2\,dk. \tag{7.59}$$

Thus, the Fourier transform $\mathcal{F}\colon \mathrm{L}^2 \to \mathrm{L}^2$ defines a *unitary* or norm-preserving linear transformation on Hilbert space, mapping L^2 functions of the physical variable x to L^2 functions of the frequency variable k.

Quantum Mechanics and the Uncertainty Principle

In its popularized form, the Heisenberg Uncertainty Principle is a by now familiar philosophical concept. It was first formulated in the 1920's by the German physicist Werner Heisenberg, one of the founders of modern quantum mechanics, and states that, in a physical system, certain quantities cannot be simultaneously measured with complete accuracy. For instance, the more precisely one measures the position of a particle, the less accuracy there will be in the measurement of its momentum; vice versa, the greater the accuracy in the momentum, the less certainty in its position. A similar uncertainty couples energy and time. Experimental verification of the uncertainty principle can be found even in fairly simple situations. Consider a light beam passing through a small hole. The position of the photons is constrained by the hole; the effect of their momenta is in the pattern of light diffused on a screen placed beyond the hole. The smaller the hole, the more constrained the position, and the wider the image on the screen, meaning the less certainty there is in the observed momentum.

This is not the place to discuss the philosophical and experimental consequences of Heisenberg's principle. What we will show is that the Uncertainty Principle is, in fact, a mathematical property of the Fourier transform! In quantum theory, each of the paired quantities, e.g., position and momentum, are interrelated by the Fourier transform. Indeed, Proposition 7.5 says that the Fourier transform of the differentiation operator representing momentum is a multiplication operator representing position and vice versa. This Fourier transform-based duality between position and momentum, or multiplication and differentiation, lies at the heart of the Uncertainty Principle.

In quantum mechanics, the wave functions of a quantum system are characterized as the elements of unit norm, $\|\,\varphi\,\| = 1$, belonging to the underlying state space, which, in a one-dimensional model of a single particle, is the Hilbert space $\mathrm{L}^2 = \mathrm{L}^2(\mathbb{R})$ consisting of square integrable, complex valued functions of x. As we already noted in Section 3.5, the squared modulus of the wave function, $|\,\varphi(x)\,|^2$, represents the probability density of the particle being found at position x. Consequently, the *mean value* of any function $f(x)$ of the position variable is given by its integral against the system's probability density, and denoted by

$$\langle\, f(x)\,\rangle = \int_{-\infty}^{\infty} f(x)\,|\,\varphi(x)\,|^2\,dx. \tag{7.60}$$

In particular,

$$\langle\, x\,\rangle = \int_{-\infty}^{\infty} x\,|\,\varphi(x)\,|^2\,dx \tag{7.61}$$

is the mean or average measured position of the particle, while Δx, defined by

$$(\Delta x)^2 = \langle\, \big(\, x - \langle\, x\,\rangle\,\big)^2\,\rangle = \langle\, x^2\,\rangle - \langle\, x\,\rangle^2 \tag{7.62}$$

is the *variance*, that is, the statistical deviation of the particle's measured position from the mean. We note that

$$\langle\, x^2\,\rangle = \int_{-\infty}^{\infty} x^2\,|\,\varphi(x)\,|^2\,dx = \|\, x\,\varphi(x)\,\|^2. \tag{7.63}$$

On the other hand, the momentum variable p is related to the Fourier transform frequency via the de Broglie relation $p = \hbar\, k$, where

$$\hbar = \frac{h}{2\,\pi} \approx 1.055 \times 10^{-34} \quad \text{Joule seconds} \tag{7.64}$$

is *Planck's constant*, whose value governs the quantization of physical quantities. Therefore, the *mean value* of any function of momentum $g(p)$ is given by its integral against the squared modulus of the Fourier transformed wave function:

$$\langle\, g(p)\,\rangle = \int_{-\infty}^{\infty} g(\hbar\, k)\,|\,\widehat{\varphi}(k)\,|^2\,dk. \tag{7.65}$$

In particular, the mean of the momentum measurements of the particle is given by

$$\langle\, p\,\rangle = \hbar \int_{-\infty}^{\infty} k\,|\,\widehat{\varphi}(k)\,|^2\,dk = -\,\mathrm{i}\,\hbar \int_{-\infty}^{\infty} \varphi'(x)\,\overline{\varphi(x)}\,dx = -\,\mathrm{i}\,\hbar\,\langle\, \varphi'\,;\varphi\,\rangle, \tag{7.66}$$

where we used Parseval's formula formula (7.58) to convert to an integral over position, and (7.40) to infer that $k\,\widehat{\varphi}(k)$ is the Fourier transform of $-\,\mathrm{i}\,\varphi'(x)$. Similarly,

$$(\Delta p)^2 = \langle\, \big(\, p - \langle\, p\,\rangle\,\big)^2\,\rangle = \langle\, p^2\,\rangle - \langle\, p\,\rangle^2 \tag{7.67}$$

is the squared *variance* in the momentum, where, by a similar computation,

$$\begin{aligned} \langle\, p^2\,\rangle &= \hbar^2 \int_{-\infty}^{\infty} k^2\,|\,\widehat{\varphi}(k)\,|^2\,dk = -\,\hbar^2 \int_{-\infty}^{\infty} \varphi(x)\,\overline{\varphi''(x)}\,dx \\ &= \hbar^2 \int_{-\infty}^{\infty} |\,\varphi'(x)\,|^2\,dx = \hbar^2\,\|\,\varphi'(x)\,\|^2. \end{aligned} \tag{7.68}$$

With this interpretation, the Uncertainty Principle can be stated as follows.

Theorem 7.15. *If $\varphi(x)$ is a wave function, so $\|\varphi\| = 1$, then the variances in position and momentum satisfy the inequality*

$$\Delta x \, \Delta p \geq \tfrac{1}{2}\,\hbar. \tag{7.69}$$

The smaller the variance of a quantity such as position or momentum, the more accurate will be its measurement. Thus, the Heisenberg inequality (7.69) quantifies the statement that the more accurately we are able to measure the momentum p, the less accurate will be any measurement of its position x, and vice versa. For more details and physical consequences, you should consult an introductory text on mathematical quantum mechanics, e.g., [**83**, **89**].

Proof: For any value of the real parameter t,

$$\begin{aligned} 0 &\leq \|\, t\, x\, \varphi(x) + \varphi'(x)\,\|^2 \\ &= t^2\, \|\, x\,\varphi(x)\,\|^2 + t\,\big[\,\langle\, \varphi'(x)\,;x\,\varphi(x)\,\rangle + \langle\, x\,\varphi(x)\,;\varphi'(x)\,\rangle\,\big] + \|\,\varphi'(x)\,\|^2. \end{aligned} \tag{7.70}$$

The middle term can be evaluated as follows:

$$\begin{aligned} \langle\, \varphi'(x)\,;x\,\varphi(x)\,\rangle + \langle\, x\,\varphi(x)\,;\varphi'(x)\,\rangle &= \int_{-\infty}^{\infty} \big[\, x\,\varphi'(x)\,\overline{\varphi(x)} + x\,\varphi(x)\,\overline{\varphi'(x)}\,\big]\,dx \\ = \int_{-\infty}^{\infty} x\,\frac{d}{dx}\,|\,\varphi(x)\,|^2\,dx &= -\int_{-\infty}^{\infty} |\,\varphi(x)\,|^2\,dx = -1, \end{aligned}$$

where we employed integration by parts, noting that the boundary terms vanish since $\varphi(x) \to 0$ as $|\,x\,| \to \infty$. Thus, in view of (7.63, 68), (7.70) reads

$$\langle\, x^2\,\rangle\, t^2 - t + \frac{\langle\, p^2\,\rangle}{\hbar^2} \geq 0$$

for all real t. The minimum value of the left hand side of this inequality occurs at $t_\star = 1/(2\,\langle\, x^2\,\rangle)$, where its value is

$$\frac{\langle\, p^2\,\rangle}{\hbar^2} - \frac{1}{4\,\langle\, x^2\,\rangle} \geq 0 \qquad \text{and hence} \qquad \langle\, x^2\,\rangle\,\langle\, p^2\,\rangle \geq \tfrac{1}{4}\,\hbar^2.$$

To obtain the Uncertainty Relation, one performs the selfsame calculation, but with $x - \langle\, x\,\rangle$ replacing x and $p - \langle\, p\,\rangle$ replacing p. The result is

$$\big\langle\, \big(\, x - \langle\, x\,\rangle\,\big)^2\,\big\rangle\, t^2 - t + \frac{\big\langle\, \big(\, p - \langle\, p\,\rangle\,\big)^2\,\big\rangle}{\hbar^2} = (\Delta x)^2\, t^2 - t + \frac{(\Delta p)^2}{\hbar^2} \geq 0. \tag{7.71}$$

Substituting $t = 1/(2\,(\Delta x)^2)$ produces the Heisenberg inequality (7.69). *Q.E.D.*