Appendix B

Linear Algebra

In this appendix, we collect basic results and definitions from linear algebra that are used in our study of partial differential equations. The reader is referred to [**104**] for the proofs and further details.

B.1. Vector Spaces and Subspaces.

Vector spaces and their ancillary structures provide the common language of linear algebra. The basic definition is modeled on the prototypical finite-dimensional example: the *Euclidean space* $\mathbb{R}^n$, which is the set of all real (column) vectors with n entries, equipped with the operations of vector addition and scalar multiplication. More generally:

Definition B.1. A (real) *vector space* is a set V equipped with two operations:

(i) *Addition*: adding any pair of elements $\mathbf{v}, \mathbf{w} \in V$ produces another vector $\mathbf{v} + \mathbf{w} \in V$;

(ii) *Scalar Multiplication*: multiplying an element $\mathbf{v} \in V$ by a scalar $c \in \mathbb{R}$ produces a vector $c\,\mathbf{v} \in V$,

which are subject to the following axioms: for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all scalars $c, d \in \mathbb{R}$,

(a) *Commutativity of Addition*: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.

(b) *Associativity of Addition*: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

(c) *Additive Identity*: There is a zero element $\mathbf{0} \in V$ satisfying $\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}$.

(d) *Additive Inverse*: For each $\mathbf{v} \in V$ there is an element $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0} = (-\mathbf{v}) + \mathbf{v}$.

(e) *Distributivity*: $(c + d)\,\mathbf{v} = (c\,\mathbf{v}) + (d\,\mathbf{v})$, and $c\,(\mathbf{v} + \mathbf{w}) = (c\,\mathbf{v}) + (c\,\mathbf{w})$.

(f) *Associativity of Scalar Multiplication*: $c\,(d\,\mathbf{v}) = (c\,d)\,\mathbf{v}$.

(g) *Unit for Scalar Multiplication*: the scalar $1 \in \mathbb{R}$ satisfies $1\,\mathbf{v} = \mathbf{v}$.

Complex vector spaces are defined in an identical manner, the only difference being that the scalars are allowed to complex numbers. In this case, the prototype is the space $\mathbb{C}^n$ consisting of column vectors with n complex entries.

While finite-dimensional vector spaces do play a role in the study of partial differential equations, particularly in the design of numerical solution schemes, for us the more important examples are infinite-dimensional vector spaces whose elements ("vectors") are functions. The main example is:

Example B.2. Let $I \subset \mathbb{R}$ be an interval. The *function space* $\mathcal{F} = \mathcal{F}(I)$, whose elements are all real-valued functions $f(x)$ defined for $x \in I$, has the structure of a vector space. Addition of functions in $\mathcal{F}$ is defined in the usual manner: $(f + g)(x) = f(x) + g(x)$

for all $x \in I$. Multiplication by scalars $c \in \mathbb{R}$ is the same as multiplication by constants, $(c\,f)(x) = c\,f(x)$. The zero element is the constant function that is identically 0 for all $x \in I$. With these operations, all the vector space axioms listed in Definition B.1 are valid, and hence $\mathcal{F}(I)$ is a real vector space.

More generally, if $\Omega \subset \mathbb{R}^n$ is any subset of n-dimensional Euclidean space, the function space $\mathcal{F}(\Omega)$ is defined as the set of all real-valued functions $f(x_1, \ldots, x_n)$ defined for all $x = (x_1, \ldots, x_n) \in \Omega$. Addition and scalar (constant) multiplication of functions are defined in the same manner.

A *subspace* of a vector space V is a subset $W \subset V$ which is a vector space in its own right. In particular, a subspace W *must* contain the zero element of V.

Proposition B.3. *A nonempty subset $W \subset V$ of a vector space is a subspace if and only if*

(*a*) *for every $\mathbf{v}, \mathbf{w} \in W$, the sum $\mathbf{v} + \mathbf{w} \in W$, and*

(*b*) *for every $\mathbf{v} \in W$ and every $c \in \mathbb{R}$, the scalar product $c\,\mathbf{v} \in W$.*

For example, a complete list of subspaces of $V = \mathbb{R}^3$ is (*i*) the origin $\{\mathbf{0}\}$; (*ii*) any line through the origin; (*iii*) any plane through the origin; (*iv*) all of $\mathbb{R}^3$.

Example B.4. Here are some examples of subspaces of the function space $\mathcal{F}(I)$.

(*a*) The space $\mathrm{C}^0(I)$ of all continuous functions.

(*b*) The space $\mathrm{C}^n(I)$ consisting of all functions $f(x)$ that have n continuous derivatives $f'(x), f''(x), \ldots, f^{(n)}(x)$ on† I.

(*c*) The space $\mathrm{C}^\infty(I) = \bigcap_{n \geq 0} \mathrm{C}^n(I)$ of infinitely differentiable or *smooth* functions is also a subspace.

(*d*) The space $\mathcal{A}(I)$ of analytic functions on the interval I. Recall that a function $f(x)$ is called *analytic* at a point a if it is smooth, and, moreover, its Taylor series

$$f(a) + f'(a)\,(x-a) + \tfrac{1}{2}\,f''(a)\,(x-a)^2 + \;\cdots\; = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}\,(x-a)^n \qquad \text{(B.1)}$$

converges to $f(x)$ for all x sufficiently close to a. (The series is not required to converge on the entire interval I.) Not every smooth function is analytic, and so $\mathcal{A}(I) \subsetneq \mathrm{C}^\infty(I)$. An explicit example is the function

$$f(x) = \begin{cases} e^{-1/x}, & x > 0, \\ 0, & x \leq 0. \end{cases} \qquad \text{(B.2)}$$

It can be shown that every derivative of this function at 0 exists and equals zero: $f^{(n)}(0) = 0$, $n = 0, 1, 2, \ldots$, and so the function is smooth. However, its Taylor series at $a = 0$ is $0 + 0\,x + 0\,x^2 + \;\cdots\; \equiv 0$, which converges to the zero function, not to $f(x)$. Therefore $f(x)$ is *not* analytic at $a = 0$.

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† We use one-sided derivatives at any endpoint belonging to the interval.

B.2. Bases and Dimension.

Definition B.5. Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ belong to a vector space V. A sum of the form

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_k \mathbf{v}_k = \sum_{i=1}^{k} c_i \mathbf{v}_i, \tag{B.3}$$

where the coefficients $c_1, c_2, \ldots, c_k$ are any scalars, is known as a *linear combination* of the elements $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Their *span* is the subspace $W = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subset V$ consisting of all possible linear combinations.

Definition B.6. The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$ are called *linearly dependent* if there exist scalars $c_1, \ldots, c_k$, *not all zero*, such that

$$c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k = \mathbf{0}. \tag{B.4}$$

Vectors that are not linearly dependent are called *linearly independent*.

In particular, a collection of functions $f_1(x), \ldots, f_n(x)$ is linearly dependent if and only if there exist constants $c_1, \ldots, c_n$, *not all zero*, such that the linear combination

$$c_1 f_1(x) + \cdots + c_n f_n(x) \equiv 0 \tag{B.5}$$

is identically zero. Vice versa, if the only choice of constants for which (B.5) holds is $c_1 = \cdots = c_n = 0$, then the functions are linearly independent.

Definition B.7. A *basis* of a vector space V is a finite collection of elements $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ that (*a*) spans V, and (*b*) is linearly independent.

Lemma B.8. *The elements $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis of V if and only if every $\mathbf{v} \in V$ can be written* uniquely *as a linear combination of the basis elements*:

$$\mathbf{v} = c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n = \sum_{i=1}^{n} c_i \mathbf{v}_i. \tag{B.6}$$

The coefficients $(c_1, \ldots, c_n)$ are called the coordinates *of the vector $\mathbf{v}$ with respect to the given basis.*

Theorem B.9. *Suppose the vector space V has a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Then every other basis of V has the same number of elements in it. This number is called the* dimension *of V, and written* $\dim V = n$.

On the other hand, if the vector space contains infinitely many linearly independent elements, then it does not have a basis in the sense of Definition B.7, and is thus *infinite dimensional*. All of the function spaces and subspaces listed above are infinite-dimensional vector spaces. An example of a finite-dimensional function space is the space $\mathcal{P}^{(n)} \subset \mathcal{F}(\mathbb{R})$ consisting of all polynomials $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ of degree $\leq n$. The monomials $1, x, x^2, \ldots, x^n$ form a basis, and hence $\mathcal{P}^{(n)}$ has dimension $n+1$. (On the other hand, the vector space containing *all* polynomials is infinite-dimensional.)

B.3. Inner Products and Norms.

The dot product on Euclidean space $\mathbb{R}^n$ plays an essential role in geometry, analysis, and mechanics. Its basic properties inspire the general definition of an inner product on a vector space.

Definition B.10. An *inner product* on the real vector space V is a pairing that takes two elements $\mathbf{v}, \mathbf{w} \in V$ and produces a real number $\langle \mathbf{v} ; \mathbf{w} \rangle \in \mathbb{R}$, subject to the following three axioms for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and scalars $c, d \in \mathbb{R}$.

(i) *Bilinearity*:

$$\begin{aligned} \langle c\,\mathbf{u} + d\,\mathbf{v} ; \mathbf{w} \rangle &= c\,\langle \mathbf{u} ; \mathbf{w} \rangle + d\,\langle \mathbf{v} ; \mathbf{w} \rangle, \\ \langle \mathbf{u} ; c\,\mathbf{v} + d\,\mathbf{w} \rangle &= c\,\langle \mathbf{u} ; \mathbf{v} \rangle + d\,\langle \mathbf{u} ; \mathbf{w} \rangle. \end{aligned} \tag{B.7}$$

(ii) *Symmetry*:

$$\langle \mathbf{v} ; \mathbf{w} \rangle = \langle \mathbf{w} ; \mathbf{v} \rangle. \tag{B.8}$$

(iii) *Positivity*:

$$\langle \mathbf{v} ; \mathbf{v} \rangle > 0 \qquad \text{whenever} \qquad \mathbf{v} \neq \mathbf{0}, \qquad \text{while} \qquad \langle \mathbf{0} ; \mathbf{0} \rangle = 0. \tag{B.9}$$

Given an inner product, the associated *norm* of an element $\mathbf{v} \in V$ is defined as the positive square root of its inner product with itself:

$$\| \mathbf{v} \| = \sqrt{\langle \mathbf{v} ; \mathbf{v} \rangle}\,. \tag{B.10}$$

Bilinearity of the inner product implies that

$$\| c\,\mathbf{v} \| = |\, c\,|\, \| v \|$$

for any scalar c. The positivity axiom implies that $\| \mathbf{v} \| \geq 0$ is real and non-negative, and equals 0 if and only if $\mathbf{v} = \mathbf{0}$ is the zero element. A vector space norm induces a notion of *distance* between elements $\mathbf{v}, \mathbf{w} \in V$, with $\operatorname{dist}(\mathbf{v}, \mathbf{w}) = \| \mathbf{v} - \mathbf{w} \|$. In particular, $\operatorname{dist}(\mathbf{v}, \mathbf{w}) = 0$ if and only if $\mathbf{v} = \mathbf{w}$.

Example B.11. The most familiar example of an inner product is the *dot product*†

$$\langle \mathbf{v} ; \mathbf{w} \rangle = \mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w} = v_1\, w_1 + v_2\, w_2 + \ \cdots \ + v_n\, w_n \tag{B.11}$$

on the Euclidean space $\mathbb{R}^n$. The associated *Euclidean norm*

$$\| \mathbf{v} \| \ = \ \sqrt{\mathbf{v} \cdot \mathbf{v}} \ = \ \sqrt{v_1^2 + v_2^2 + \ \cdots \ + v_n^2} \tag{B.12}$$

conforms to our usual notion of distance between points in Euclidean space.

To find the most general inner product on $\mathbb{R}^n$, we need to introduce the important class of positive definite matrices.

† The elements $\mathbf{v} \in \mathbb{R}^n$ are to be regarded as column vectors, while the transpose, written $\mathbf{v}^T$, is the corresponding row vector.

Definition B.12. An $n \times n$ matrix C is called *positive definite* if it satisfies the positivity condition

$$\mathbf{v}^T C\,\mathbf{v} > 0 \qquad \text{for all} \qquad \mathbf{0} \neq \mathbf{v} \in \mathbb{R}^n. \tag{B.13}$$

We will sometimes write $C > 0$ to mean that C is a positive definite matrix.

Warning: The condition $C > 0$ does *not* mean that all the entries of C are positive. For example, $\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$ is positive definite, whereas $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ is not.

Many authors, e.g., [**104**], require that a positive definite matrix also be symmetric. We will not impose this condition here a priori. However, most of the positive definite matrices we will encounter in applications will be symmetric (or, more generally, self-adjoint — as in Example 9.15). For a symmetric matrix, the most useful test for positive definiteness is to perform Gaussian Elimination on C, which is positive definite if and only if no row interchanges are needed, and all the pivots are positive, [**104**].

Proposition B.13. *Every inner product on $\mathbb{R}^n$ is given by*

$$\langle\, \mathbf{v}\,;\mathbf{w}\,\rangle = \mathbf{v}^T C\,\mathbf{w} \qquad \textit{for} \qquad \mathbf{v}, \mathbf{w} \in \mathbb{R}^n, \tag{B.14}$$

where $C > 0$ is a symmetric, positive definite matrix.

The next example is of particular significance in Fourier analysis and partial differential equations.

Example B.14. Let $[a, b] \subset \mathbb{R}$ be a bounded closed interval. The integral

$$\langle\, f\,; g\,\rangle = \int_a^b f(x)\,g(x)\,dx \tag{B.15}$$

defines an inner product on space $\mathrm{C}^0[a, b]$ of continuous functions. The associated norm

$$\|\, f\,\| = \sqrt{\int_a^b f(x)^2\,dx}\;, \tag{B.16}$$

is known as the L^2 *norm* of the function f over the interval $[a, b]$. The positivity of the norm, $\|\, f\,\| > 0$ for $f \neq 0$, follows from the fact that the only continuous, non-negative function $g(x) \geq 0$ that satisfies $\displaystyle\int_a^b g(x)\,dx = 0$ is the zero function $g(x) \equiv 0$. Extending this construction to spaces containing discontinuous functions is more tricky, since there are discontinuous functions that are not identically zero, but nevertheless have zero norm integral. An example is a function that is zero except at a single point. Further discussion can be found in Section 3.5.

We will also have occasion to use inner products on complex vector spaces. To ensure that the associated norm remain positive, the real definition must be modified. The complex conjugate of a complex scalar $c = a + \mathrm{i}\,b$, with $a, b \in \mathbb{R}$, will be indicated by an overbar: $\overline{c} = a - \mathrm{i}\,b$. When dealing with a complex inner product space, one must pay careful attention to complex conjugation.

Definition B.15. An *inner product* on the complex vector space V is a pairing that takes two vectors $\mathbf{v}, \mathbf{w} \in V$ and produces a complex number $\langle \mathbf{v} ; \mathbf{w} \rangle \in \mathbb{C}$, subject to the following requirements, for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and $c, d \in \mathbb{C}$:

(*i*) *Sesquilinearity*:

$$\begin{aligned} \langle c\,\mathbf{u} + d\,\mathbf{v} ; \mathbf{w} \rangle &= c \langle \mathbf{u} ; \mathbf{w} \rangle + d \langle \mathbf{v} ; \mathbf{w} \rangle, \\ \langle \mathbf{u} ; c\,\mathbf{v} + d\,\mathbf{w} \rangle &= \overline{c} \langle \mathbf{u} ; \mathbf{v} \rangle + \overline{d} \langle \mathbf{u} ; \mathbf{w} \rangle. \end{aligned} \tag{B.17}$$

(*ii*) *Conjugate Symmetry*:

$$\langle \mathbf{v} ; \mathbf{w} \rangle = \overline{\langle \mathbf{w} ; \mathbf{v} \rangle}. \tag{B.18}$$

(*iii*) *Positivity*:

$$\| \mathbf{v} \|^2 = \langle \mathbf{v} ; \mathbf{v} \rangle \geq 0, \qquad \text{and} \qquad \langle \mathbf{v} ; \mathbf{v} \rangle = 0 \quad \text{if and only if} \quad \mathbf{v} = \mathbf{0}. \tag{B.19}$$

Example B.16. The simplest example is the *Hermitian dot product*

$$\mathbf{z} \cdot \mathbf{w} = \mathbf{z}^T\, \overline{\mathbf{w}} = z_1 \overline{w}_1 + z_2 \overline{w}_2 + \;\cdots\; + z_n \overline{w}_n, \quad \text{for} \quad \mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \tag{B.20}$$

between complex vectors $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$.

Example B.17. Let $\mathrm{C}^0[\,-\pi, \pi\,]$ denote the complex vector space consisting of all complex valued continuous functions $f(x) = u(x) + \mathrm{i}\, v(x)$ depending upon the *real* variable $-\pi \leq x \leq \pi$. The L^2 *Hermitian inner product* on $\mathrm{C}^0[\,-\pi, \pi\,]$ is defined as

$$\langle f ; g \rangle = \int_{-\pi}^{\pi} f(x)\, \overline{g(x)}\, dx\,, \tag{B.21}$$

i.e., the integral of f times the complex conjugate of g, with corresponding norm

$$\| f \| \;=\; \sqrt{\int_{-\pi}^{\pi} |\, f(x) \,|^2 \, dx} \;=\; \sqrt{\int_{-\pi}^{\pi} \left[\, u(x)^2 + v(x)^2 \,\right] dx}\,. \tag{B.22}$$

The two most important inequalities in mathematical analysis apply to any (complex) inner product space. See [**104**] for the proof.

Theorem B.18. *Every inner product satisfies the Cauchy–Schwarz and* triangle *inequalities*

$$|\, \langle \mathbf{v} ; \mathbf{w} \rangle \,| \;\leq\; \| \mathbf{v} \| \, \| \mathbf{w} \|, \qquad\qquad \| \mathbf{v} + \mathbf{w} \| \;\leq\; \| \mathbf{v} \| + \| \mathbf{w} \|, \tag{B.23}$$

for all $\mathbf{v}, \mathbf{w} \in V$. *Equality holds if and only if* $\mathbf{v}$ *and* $\mathbf{w}$ *are parallel, i.e., scalar multiples of each other.*

B.4. Orthogonality.

Definition B.19. Two elements $\mathbf{v}, \mathbf{w} \in V$ of an inner product space V are called *orthogonal* if their inner product vanishes: $\langle \mathbf{v} ; \mathbf{w} \rangle = 0$.

For ordinary Euclidean space equipped with the dot product, two vectors are orthogonal if and only if they are perpendicular, i.e., meet at a right angle.

Definition B.20. A basis $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of an inner product space V is called *orthogonal* if $\langle \mathbf{u}_i ; \mathbf{u}_j \rangle = 0$ for all $i \neq j$. The basis is called *orthonormal* if, in addition, each vector has unit length: $\| \mathbf{u}_i \| = 1$, for all $i = 1, \ldots, n$.

Theorem B.21. *If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form an orthogonal basis, then the corresponding coordinates of a vector*

$$\mathbf{v} = a_1 \mathbf{v}_1 + \cdots + a_n \mathbf{v}_n \qquad \text{are given by} \qquad a_i = \frac{\langle \mathbf{v} ; \mathbf{v}_i \rangle}{\| \mathbf{v}_i \|^2}. \tag{B.24}$$

Moreover, the vector's norm can be computed using the formula

$$\| \mathbf{v} \|^2 = \sum_{i=1}^{n} a_i^2 \, \| \mathbf{v}_i \|^2 = \sum_{i=1}^{n} \left(\frac{\langle \mathbf{v} ; \mathbf{v}_i \rangle}{\| \mathbf{v}_i \|} \right)^2 . \tag{B.25}$$

Proof: We compute the inner product of (B.24) with one of the basis vectors. By orthogonality,

$$\langle \mathbf{v} ; \mathbf{v}_i \rangle = \left\langle \sum_{j=1}^{n} a_j \mathbf{v}_j \, ; \, \mathbf{v}_i \right\rangle = \sum_{j=1}^{n} a_j \, \langle \mathbf{u}_j ; \mathbf{u}_i \rangle = a_i \, \| \mathbf{v}_i \|^2.$$

To prove formula (B.25), we similarly expand

$$\| \mathbf{v} \|^2 = \langle \mathbf{v} ; \mathbf{v} \rangle = \sum_{i,j=1}^{n} a_i \, a_j \, \langle \mathbf{v}_i ; \mathbf{v}_j \rangle = \sum_{i=1}^{n} a_i^2 \, \| \mathbf{v}_i \|^2. \qquad \textit{Q.E.D.}$$

In the case of an orthonormal basis, the formulas (B.24–25) simplify to

$$\mathbf{v} = c_1 \mathbf{u}_1 + \cdots + c_n \mathbf{u}_n, \quad \text{where} \quad c_i = \langle \mathbf{v} ; \mathbf{u}_i \rangle, \qquad \| \mathbf{v} \| = c_1^2 + \cdots + c_n^2. \tag{B.26}$$

B.5. Eigenvalues and Eigenvectors.

The eigenvalues and eigenvectors of a matrix first appear when solving linear systems of ordinary differential equations. But their essential importance extends across all of mathematics and its manifold applications. Extensions of the eigenvalue method to linear operators on function spaces are critical to the analysis of partial differential equations.

Definition B.22. Let A be an $n \times n$ matrix. A scalar λ is called an *eigenvalue* of A if there is a *non-zero* vector $\mathbf{v} \neq \mathbf{0}$, called an associated *eigenvector*, such that

$$A\mathbf{v} = \lambda \mathbf{v}. \tag{B.27}$$

Even if A is a real matrix, we must allow the possibility of complex eigenvectors. Matrices with a "complete" set of eigenvectors are the most common, and also the easiest to deal with.

Definition B.23. An $n \times n$ real or complex matrix A is called *complete* if and only if there exists a basis of $\mathbb{C}^n$ consisting of its (complex) eigenvectors.

It is not hard to show that eigenvectors corresponding to different eigenvalues are necessarily linearly independent. This means that matrices with all distinct (and hence simple) eigenvalues are necessarily complete:

Proposition B.24. *Any $n \times n$ matrix with n distinct eigenvalues is complete.*

Unfortunately, not all matrices with repeated eigenvalues are complete. For instance, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is complete since, for instance, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ form an eigenvector basis of $\mathbb{C}^2$, whereas $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not, since it has only 1 independent eigenvector, namely $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Incomplete matrices are much more challenging to deal with, both theoretically and numerically. Fortunately, we can safely ignore the incomplete cases in this text.

The most common way for orthogonal bases to arise is as eigenvector bases of symmetric matrices. (Orthogonality is with respect to the standard dot product on $\mathbb{R}^n$.) The extension of this result to "self-adjoint" operators on function space forms the foundation of Fourier analysis and its generalizations.

Theorem B.25. *Let $A = A^T$ be a real, symmetric $n \times n$ matrix. Then*

(*a*) *All the eigenvalues of A are real.*
(*b*) *Eigenvectors corresponding to distinct eigenvalues are orthogonal.*
(*c*) *There is an orthonormal basis of $\mathbb{R}^n$ consisting of n eigenvectors of A.*

Let us demonstrate orthogonality, leaving the remaining steps in the proof to [**104**; Theorem 8.20]. If

$$A\mathbf{v} = \lambda \mathbf{v}, \qquad A\mathbf{w} = \mu \mathbf{w},$$

where $\lambda \neq \mu$ are distinct real eigenvalues, then, by symmetry of A,

$$\lambda\, \mathbf{v} \cdot \mathbf{w} = (A\mathbf{v}) \cdot \mathbf{w} = (A\mathbf{v})^T \mathbf{w} = \mathbf{v}^T A \mathbf{w} = \mathbf{v} \cdot (A\mathbf{w}) = \mathbf{v} \cdot (\mu\, \mathbf{w}) = \mu\, \mathbf{v} \cdot \mathbf{w},$$

and hence

$$(\lambda - \mu)\, \mathbf{v} \cdot \mathbf{w} = 0.$$

Since $\lambda \neq \mu$, this implies that the eigenvectors $\mathbf{v}, \mathbf{w}$ are necessarily orthogonal.

B.6. Linear Iterative Systems.

For numerical applications, we will require some basic results on linear iterative systems. Consider first a *homogeneous linear iterative system* of the form

$$\mathbf{u}^{(k+1)} = A\,\mathbf{u}^{(k)}, \qquad \mathbf{u}^{(0)} = \mathbf{u}_0, \tag{B.28}$$

in which A is an $n \times n$ matrix and $\mathbf{u}_0 \in \mathbb{R}^n$ or $\mathbb{C}^n$. The solution to such a system is evidently obtained by repeatedly multiplying the initial vector $\mathbf{u}_0$ by the matrix A, and so

$$\mathbf{u}^{(k)} = A^k \mathbf{u}_0. \tag{B.29}$$

Definition B.26. A matrix A is called *convergent* if every solution to the homogeneous linear iterative system (B.28) tends to zero in the limit: $\mathbf{u}^{(k)} \to \mathbf{0}$ as $k \to \infty$. Equivalently, A is convergent if and only if its powers converge to the zero matrix: $A^k \to \mathrm{O}$ as $k \to \infty$.

The solution formula (B.29), while elementary, is not particularly enlightening. An alternative approach is to recognize that if λ_j is an eigenvalue of A and $\mathbf{v}_j$ a corresponding eigenvector, then

$$\mathbf{u}_j^{(k)} = \lambda_j^k\,\mathbf{v}_j \tag{B.30}$$

is a solution, since

$$A\,\mathbf{u}_j^{(k)} = \lambda_j^k\,A\,\mathbf{v}_j = \lambda_j^{k+1}\,\mathbf{v}_j = \mathbf{u}_j^{(k+1)}.$$

Moreover, linear combinations of such *eigensolutions* are also solutions. In particular, if A is complete, then we can write down the general solution to (B.28) as a linear combination of the independent eigensolutions:

$$\mathbf{u}^{(k)} = c_1\,\lambda_1^k\,\mathbf{v}_1 + c_2\,\lambda_2^k\,\mathbf{v}_2 + \cdots + c_n\,\lambda_n^k\,\mathbf{v}_n, \tag{B.31}$$

where $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is the eigenvector basis. The coefficients $c_1, \ldots, c_n$ are uniquely determined by the initial conditions:

$$\mathbf{u}^{(0)} = c_1\,\mathbf{v}_1 + c_2\,\mathbf{v}_2 + \cdots + c_n\,\mathbf{v}_n = \mathbf{u}_0,$$

which relies on the fact that the eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis. Now, A is convergent if and only if all solutions $\mathbf{u}^{(k)} \to \mathbf{0}$. The individual eigensolution (B.30) goes to zero if and only if its associated eigenvalue is strictly less than 1 in modulus: $|\,\lambda_j\,| < 1$. This proves the following result for complete matrices. The proof in the incomplete case relies on the Jordan canonical form, [**104**; Chapter 10].

Theorem B.27. *The matrix A is convergent if and only if all its eigenvalues satisfy $|\,\lambda\,| < 1$.*

Definition B.28. The *spectral radius* of a matrix A is defined as the maximal modulus of all of its real and complex eigenvalues: $\rho(A) = \max\,\{\,|\,\lambda_1\,|, \ldots, |\,\lambda_k\,|\,\}$.

Corollary B.29. *The matrix A is convergent if and only if $\rho(A) < 1$.*

Indeed, the spectral radius essentially governs the rate of convergence of the iterative system — the closer it is to 0, the faster the convergence rate.

Next, consider the *inhomogeneous linear iterative system*

$$\mathbf{v}^{(k+1)} = A\,\mathbf{v}^{(k)} + \mathbf{b}, \qquad \mathbf{v}^{(0)} = \mathbf{v}_0, \tag{B.32}$$

where $\mathbf{b}$ a fixed vector. A *fixed point* is a vector $\mathbf{v}^\star$ that satisfies

$$\mathbf{v}^\star = A\,\mathbf{v}^\star + \mathbf{b}, \qquad \text{or, equivalently} \qquad (\mathrm{I} - A)\mathbf{v}^\star = \mathbf{b}. \tag{B.33}$$

Thus, if 1 is not an eigenvalue of A (which cannot happen when A is convergent), then $\mathrm{I} - A$ is nonsingular, and so the iterative system has a unique fixed point.

Theorem B.30. *Assuming* 1 *is not an eigenvalue of A, all solutions to* (B.32) *converge to the fixed point, $\mathbf{v}^{(k)} \to \mathbf{v}^\star$ as $k \to \infty$ if and only if A is a convergent matrix.*

Proof: Let $\mathbf{u}^{(k)} = \mathbf{v}^{(k)} - \mathbf{v}^\star$, so that $\mathbf{v}^{(k)} \to \mathbf{v}^\star$ if and only if $\mathbf{u}^{(k)} \to \mathbf{0}$. Now,

$$\mathbf{u}^{(k+1)} = \mathbf{v}^{(k+1)} - \mathbf{v}^\star = (A\,\mathbf{v}^{(k)} + \mathbf{b}) - (A\,\mathbf{v}^\star + \mathbf{b}) = A(\mathbf{v}^{(k)} - \mathbf{v}^\star) = A\,\mathbf{u}^{(k)},$$

and hence $\mathbf{u}^{(k)}$ solves the homogeneous version (B.28). Thus, the result is an immediate consequence of Definition B.26. *Q.E.D.*

Illustrations and applications of these results can be found in Chapter 10.

B.7. Linear Functions and Systems.

The most basic structural features of linear differential equations, both ordinary and partial, linear boundary value problems, etc., are founded on the concept of a linear function between vector spaces.

Definition B.31. Let U and V be real vector spaces. A function $L: U \to V$ is called *linear* if it obeys two basic rules:

$$L[\,\mathbf{u} + \mathbf{v}\,] = L[\,\mathbf{u}\,] + L[\,\mathbf{v}\,], \qquad L[\,c\,\mathbf{u}\,] = c\,L[\,\mathbf{u}\,], \tag{B.34}$$

for all $\mathbf{u}, \mathbf{v} \in U$ and all scalars c.

We will refer to U as the *domain space* of the function L, and V as the *target space*. The latter is to emphasize the fact that the range of L, namely

$$\operatorname{rng} L = \{\, \mathbf{v} \in V \mid \ \mathbf{v} = L[\,\mathbf{u}\,] \text{ for some } \mathbf{u} \in U \,\}, \tag{B.35}$$

may very well be a proper subspace of the target space V.

Theorem B.32. *Every linear function $L: \mathbb{R}^n \to \mathbb{R}^m$ is given by matrix multiplication, $L[\,\mathbf{v}\,] = A\,\mathbf{v}$, where A is an $m \times n$ matrix.*

Proving that matrix multiplication satisfies the linearity conditions (B.34) is easy. The converse is established by seeing what the linear function does to the basis vectors of $\mathbb{R}^n$; see [**104**; Theorem 7.5].

Corollary B.33. *Every linear function $L: \mathbb{R}^n \to \mathbb{R}$ is given by taking the dot product with a fixed vector $\mathbf{a} \in \mathbb{R}^n$:*

$$L[\mathbf{v}] = \mathbf{a} \cdot \mathbf{v}. \tag{B.36}$$

When U is a function space, a linear function is also referred to as a *linear operator* in order to avoid confusion with the elements of U. If the target space $V = \mathbb{R}$, then the term *linear functional* is also often used for $L: U \to \mathbb{R}$.

Here are some representative examples that arise in applications.

Example B.34. (*a*) Evaluation of the function at a point, namely $L[f] = f(x_0)$, defines a linear operator $L: \mathrm{C}^0[a, b] \to \mathbb{R}$.

(*b*) Integration,

$$I[f] = \int_a^b f(x)\, dx, \tag{B.37}$$

also defines a linear functional $I: \mathrm{C}^0[a, b] \to \mathbb{R}$.

(*c*) The operation $M_a[f(x)] = a(x)\, f(x)$ of multiplication by a continuous function a defines a linear operator $M_a: \mathrm{C}^0[a, b] \to \mathrm{C}^0[a, b]$.

(*d*) Differentiation of functions $D[f] = f'$ defines a linear operator $D: \mathrm{C}^1[a, b] \to \mathrm{C}^0[a, b]$.

(*e*) A general *linear ordinary differential operator* of order n

$$L = a_n(x)\, D^n + a_{n-1}(x)\, D^{n-1} + \cdots + a_1(x)\, D + a_0(x) \tag{B.38}$$

is obtained by summing such operators. If the coefficient functions $a_0(x), \ldots, a_n(x)$ are continuous, then

$$L[u] = a_n(x)\, \frac{d^x u}{dn^x} + a_{n-1}(x)\, \frac{d^x u}{dn - 1^x} + \cdots + a_1(x)\, \frac{du}{dx} + a_0(x) u \tag{B.39}$$

defines a linear operator from $\mathrm{C}^n[a, b]$ to $\mathrm{C}^0[a, b]$.

Linear partial differential equations are based on linear partial differential operators, which are discussed in Chapter 1. They are particular examples of the general concept of a linear system.

Definition B.35. A *linear system* is an equation of the form

$$L[\mathbf{u}] = \mathbf{f}, \tag{B.40}$$

in which $L: U \to V$ is a linear function, $\mathbf{f} \in V$, while the desired solution $\mathbf{u} \in U$. The system is *homogeneous* if $\mathbf{f} = \mathbf{0}$; otherwise, it is called *inhomogeneous*.

Note that, by the definition (B.35) of the range of L, the linear system (B.40) will have a solution if and only if $\mathbf{f} \in \operatorname{rng} L$. In particular, a homogeneous linear system always has a solution, namely $\mathbf{u} = \mathbf{0}$. However, it may possibly admit other, non-zero solutions.

Theorem B.36. *If* $\mathbf{z}_1, \ldots, \mathbf{z}_k$ *are all solutions to the same homogeneous linear system*

$$L[\,\mathbf{z}\,] = \mathbf{0}, \tag{B.41}$$

then any linear combination $c_1 \mathbf{z}_1 + \cdots + c_k \mathbf{z}_k$*, for any scalars* $c_1, \ldots, c_k$*, is also a solution.*

In other words, the set of solutions to a homogeneous linear system (B.41) forms a subspace of the domain space U, known as the *kernel* of the linear function L:

$$\ker L = \{\, \mathbf{z} \in U \mid L[\,\mathbf{z}\,] = \mathbf{0} \,\}. \tag{B.42}$$

Theorem B.37. *If the inhomogeneous linear system* $L[\,\mathbf{u}\,] = \mathbf{f}$ *has a solution* $\mathbf{u}^\star$*, which requires* $\mathbf{f} \in \operatorname{rng} L$*, then the general solution is* $\mathbf{u} = \mathbf{u}^\star + \mathbf{z}$*, where* $\mathbf{z} \in \ker L$ *is any solution to the corresponding homogeneous system* $L[\,\mathbf{z}\,] = \mathbf{0}$.

The *Superposition Principle* for inhomogeneous linear systems allows us to combine solutions corresponding to different right hand sides.

Theorem B.38. *Suppose that, for each* $i = 1, \ldots, k$*, we know a particular solution* $\mathbf{u}_i^\star$ *to the inhomogeneous linear system* $L[\,\mathbf{u}\,] = \mathbf{f}_i$ *for some* $\mathbf{f}_i \in \operatorname{rng} L$*. Then, given scalars* $c_1, \ldots, c_k$*, a particular solution to the combined inhomogeneous system*

$$L[\,\mathbf{u}\,] = c_1 \mathbf{f}_1 + \cdots + c_k \mathbf{f}_k \tag{B.43}$$

is the corresponding linear combination

$$\mathbf{u}^\star = c_1 \mathbf{u}_1^\star + \cdots + c_k \mathbf{u}_k^\star \tag{B.44}$$

of particular solutions. The general solution to the inhomogeneous system (B.43) *is*

$$\mathbf{u} = \mathbf{u}^\star + \mathbf{z} = c_1 \mathbf{u}_1^\star + \cdots + c_k \mathbf{u}_k^\star + \mathbf{z}, \tag{B.45}$$

where $\mathbf{z} \in \ker L$ *is an arbitrary solution to the associated homogeneous system* $L[\,\mathbf{z}\,] = \mathbf{0}$.

