

Chapter 1

What are Partial Differential Equations?

Let us begin by specifying our object of study. A *differential equation* is an equation that relates the derivatives of a (scalar) function depending on one or more variables. For example,

$$\frac{d^4u}{dx^4} + u^2 = x \quad (1.1)$$

is a differential equation for the function $u(x)$ depending on a single variable x , while

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + u = 0 \quad (1.2)$$

is a differential equation involving a function $u(t, x, y)$ of three variables.

There are two common notations for partial derivatives, and we shall employ them interchangeably. The first, used in (1.1, 2), is the familiar Leibniz notation that employs an ordinary d to denote ordinary derivatives of functions of a single variable, and the ∂ symbol (usually also pronounced “dee”) for partial derivatives of functions of more than one variable. An alternative, more compact notation is to use subscripts to indicate partial derivatives. For example, u_t represents $\partial u / \partial t$, while u_{xx} represents $\partial^2 u / \partial x^2$, and $\partial^3 u / \partial x^2 \partial y$ becomes u_{xxy} . Thus, in subscript notation, the partial differential equation (1.2) would become

$$u_t - u_{xx} - u_{yy} + u = 0. \quad (1.3)$$

We will similarly abbreviate partial differential operators, sometimes writing $\partial / \partial x$ as ∂_x , while $\partial^2 / \partial x^2$ can be written as either ∂_x^2 or ∂_{xx} , and $\partial^2 / \partial x^2 \partial y$ becomes $\partial_{xxy} = \partial_x^2 \partial_y$.

A differential equation is called *ordinary* if the function only depends on a single variable, and *partial* if it depends on more than one variable. Usually (but not quite always) the dependence of the function u can be inferred from the derivatives that appear in the differential equation. The *order* of a differential equation is the highest order derivative occurring in the equation. Thus, (1.1) is a fourth order ordinary differential equation, while (1.2) is a second order partial differential equation.

Remark: The differential equation has order 0 if no derivatives of the function actually appear. Such equations are more properly treated as *algebraic equations*, and not true differential equations. While algebraic equations are of great interest in their own right, they are not the subject of this text. A *differential equation* necessarily has order ≥ 1 .

It is worth pointing out that the preponderance of differential equations arising in applications, in science, in engineering, and within mathematics itself, are of either first

or second order, with the latter being by far the most prevalent. Third order equations arise when modeling waves in dispersive media, e.g., water waves or plasma waves. Fourth order equations appear in elasticity, particularly plate and beam mechanics. Equations of order ≥ 5 are exceedingly rare, and need not concern us.

A basic prerequisite for studying this text is the ability to solve simple ordinary differential equations: first order equations, linear constant coefficient equations, both homogeneous and inhomogeneous, and linear systems. In addition, we shall assume some familiarity with the basic theorems concerning the existence and uniqueness of solutions to initial value problems, as reviewed, for instance, in [17, 23, 40].

Partial differential equations are considerably more difficult, and challenge the analytical skills of even the most accomplished mathematician. Many of the most effective solution strategies rely on reducing the partial differential equation to one or more ordinary differential equations. And, in the course of our study of partial differential equations, we will need to develop some of the more advanced aspects of the theory of ordinary differential equations, including boundary value problems, eigenvalue problems, series solutions, singular points, and special functions.

Following the introductory remarks in the present chapter, the exposition begins in earnest with simple first order equations, concentrating on those that arise as models of wave phenomena. Most of the remainder of the text will be devoted to understanding and solving the three essential linear, second order partial differential equations in one, two, and three space dimensions[†]: the *heat equation*, modeling thermodynamics in a continuous medium, as well as diffusion of pollutants and populations; the *wave equation*, modeling vibrations of bars, strings, plates and solid bodies, as well as acoustic, fluid, and electromagnetic vibrations; and the *Laplace equation* and its inhomogeneous counterpart, the *Poisson equation*, governing the mechanical and thermal equilibria of bodies, as well as fluid mechanical and electromagnetic potentials.

Each increase in dimension requires an increase in mathematical sophistication, as well as the development of additional analytical tools — although the key ideas will have all appeared once we reach our physical, three-dimensional universe. The three starring examples — heat, wave and Laplace/Poisson— are not only essential to a wide range of applications, but also serve as instructive paradigms for the three principal classes of linear partial differential equations. Some interesting nonlinear partial differential equations — first order transport equations modeling shock waves, the second order Burgers' equation modeling nonlinear diffusion, and the third order Korteweg–deVries equation modeling dispersive waves, will also be discussed. But, in such an introductory text, the rest of the vast realm of nonlinear partial differential equations must remain unexplored, and a destination for future, more challenging mathematical excursions.

More generally, a *system of differential equations* is a collection of one or more equations relating the derivatives of one or more functions. It is essential that all the functions

[†] By *dimension*, we will always mean the number of space dimensions. Time, although theoretically also a dimension, plays a very different physical role, and therefore (at least in non-relativistic systems) is to be regarded on a separate footing.

occurring in the system *depend on the same set of variables*. The symbols for these functions are called the *dependent variables*, while the variables that they depend on are referred to as the *independent variables*. Systems of differential equations are called *ordinary* or *partial* according to the number of independent variables. The *order* of the system is the highest order derivative occurring in any of its equations.

For example, the three-dimensional *Navier–Stokes equations*

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} &= \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} &= \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right), \\ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} &= \frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right), \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0, \end{aligned} \tag{1.4}$$

is a second order system of differential equations that involves four functions, $u(t, x, y, z)$, $v(t, x, y, z)$, $w(t, x, y, z)$, $p(t, x, y, z)$, each depending on four variables. (The function p necessarily depends on t even though no time derivative of it appears in the system.) On the other hand, $\nu \geq 0$ is a constant parameter. The independent variables are t , representing time, and x, y, z , representing space coordinates. The dependent variables are u, v, w, p . The Navier–Stokes equations are the most fundamental equations of fluid mechanics, in which $\mathbf{v} = (u, v, w)$ represents the velocity vector field of an incompressible fluid flow, while p represents the accompanying pressure. The parameter ν represents the viscosity of the fluid. The Navier–Stokes equations are notoriously difficult to solve. Indeed, establishing the existence or non-existence of global solutions is a major unsolved problem in mathematics, whose resolution will earn you a \$1,000,000 prize; see <http://www.claymath.org> for details.

We shall be employing a few basic notational conventions regarding the variables that appear in our differential equations. We always use t to denote time, while x, y, z will represent (Cartesian) space coordinates. Polar coordinates r, θ , cylindrical coordinates r, θ, z , and spherical coordinates r, θ, φ , will also be used when needed, and our notational conventions appear at the appropriate places in the exposition.

An *equilibrium equation* models an unchanging physical system, and so only involves the space variables. The time variable t appears when modeling *dynamical*, meaning time-varying, processes. Both time and space coordinates are independent variables. The functions or dependent variables appearing in the differential equation or system will mostly be denoted by u, v, w , although occasionally — particularly when representing particular physical quantities — other letters may be employed, e.g., the pressure p in (1.4).

In this introductory text, we will have our work cut out for us analyzing a select few of the important partial differential equations, and so have almost no occasion to discuss the yet more challenging situation of systems of partial differential equations, whose analysis must be deferred to a more advanced text, e.g., [45, 52, 73]. Indeed, many fundamental issues remain unresolved and/or poorly understood. One of the goals of these notes is

to inspire and equip you for venturing much further into this fascinating, important, and highly active area of contemporary mathematical research, one that has a truly phenomenal range of applications throughout science, engineering, economics, and elsewhere.

Classical Solutions

For specificity, let us focus our attention on a single differential equation involving a single, scalar-valued function u that depends on one or more independent variables. The function u is usually real-valued, although complex-valued functions will play a role in the analysis. Everything that we say in general will, when suitably formulated, also apply to systems of differential equations.

By a *solution* we mean a sufficiently smooth function u of the independent variables that satisfies the differential equation at every point of its domain of definition. We do not necessarily require that the solution be defined for all possible values of the independent variables. Indeed, usually the differential equation is imposed on some domain D contained in the space of independent variables, and we seek a solution defined only on D . In general, the *domain* D will be an open subset, usually connected and, often, bounded. The boundary of the domain must not be too bizarre.

By *smooth*, we mean that the function can be differentiated sufficiently often, in order that all of the derivatives appearing in the equation be well defined on its domain of definition D . More specifically, if the differential equation has order n , then we require that the solution u be of *class* C^n , which means that it and all its derivatives of order $\leq n$ are continuous functions in D , and such that the differential equation that relates the derivatives of u holds throughout D .

Later on, we will have occasion to introduce more general types of solutions that fail to satisfy the full differentiability properties postulated above. The most important such class consists of the so-called “weak solutions” to be introduced in Section 2.3. To emphasize the distinction, the smooth solutions described above are often referred to as *classical solutions* or, occasionally, *strong solutions*. For us, the term “solution” without extra qualification will inevitably mean “classical solution”.

Example 1.1. By a (classical) *solution* to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{1.5}$$

we mean a function $u(t, x)$, defined on a domain $D \subset \mathbb{R}^2$, such that all of the functions

$$u(t, x), \quad \frac{\partial u}{\partial t}(t, x), \quad \frac{\partial u}{\partial x}(t, x), \quad \frac{\partial^2 u}{\partial t^2}(t, x), \quad \frac{\partial^2 u}{\partial t \partial x}(t, x) = \frac{\partial^2 u}{\partial x \partial t}(t, x), \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2}(t, x),$$

are well-defined and continuous[†] at every point $(t, x) \in D$, which is required in order that $u \in C^2(D)$, and, moreover, that (1.5) holds at every $(t, x) \in D$. For example,

$$u(t, x) = t + \frac{1}{2}x^2 \tag{1.6}$$

[†] The equality of the mixed partial derivatives follows from a general theorem in multivariable calculus, [8]. Classical solutions automatically enjoy equality of all their relevant mixed partial derivatives.

is a solution to the heat equation that is defined on the full domain $D = \mathbb{R}^2$ because it is C^2 — in fact, C^∞ , meaning infinitely differentiable — and, moreover,

$$\frac{\partial u}{\partial t} = 1 = \frac{\partial^2 u}{\partial x^2}.$$

Another, more complicated — but extremely important — solution is

$$u(t, x) = \frac{e^{-x^2/(4t)}}{2\sqrt{\pi t}}. \quad (1.7)$$

This function turns out to solve the heat equation on the domain $D = \{t > 0\} \subset \mathbb{R}^2$. The reader is invited to verify this by computing $\partial u/\partial t$ and $\partial^2 u/\partial x^2$, and then checking that they are equal. Finally, we note that

$$u(t, x) = e^{-t+ix} = e^{-t} \cos x + i e^{-t} \sin x, \quad (1.8)$$

where, as always, $i = \sqrt{-1}$, defines a complex-valued solution to the heat equation. This can be verified directly, since the rules for differentiating complex exponentials are identical to those for their real counterparts:

$$\frac{\partial u}{\partial t} = -e^{-t+ix}, \quad \frac{\partial u}{\partial x}(t, x) = i e^{-t+ix}, \quad \text{and so} \quad \frac{\partial^2 u}{\partial x^2}(t, x) = -e^{-t+ix} = \frac{\partial u}{\partial t}.$$

It is worth pointing out that both the real part, $e^{-t} \cos x$, and also the imaginary part, $e^{-t} \sin x$, of the complex solution (1.8) are individual real solutions, indicative of a general fact. The analogous observation is used of when solving real ordinary differential equations.

Incidentally, most partial differential equations arising in physical applications are real and, although complex solutions often facilitate their analysis, at the end of the day we require real, physically meaningful solutions. A notable exception is quantum mechanics, which is an inherently complex-valued physical theory. For example, the one-dimensional *Schrödinger equation*

$$i \hbar \frac{\partial u}{\partial t} = - \frac{\partial^2 u}{\partial x^2} + V(x) u \quad (1.9)$$

with \hbar denoting Planck's (real) constant, governs the dynamical evolution of the complex-valued wave function $u(t, x)$ describing the probabilistic distribution of a quantum particle, e.g., an electron, subject to the (real) potential function $V(x)$. While the solution u is complex-valued, the independent variables t, x remain real.

Initial Conditions and Boundary Conditions

How many solutions does a partial differential equation have? In general, lots. Even ordinary differential equations have infinitely many solutions. Indeed, the general solution to a single n^{th} order ordinary differential equation depends on n arbitrary constants. The solutions to partial differential equations are yet more numerous, in that they depend on *arbitrary functions*. Very roughly speaking, we expect the solution to an n^{th} order partial differential equation involving m independent variables to depend on n arbitrary functions of $m - 1$ variables. But this is only a rough guide, and must be taken with a large grain of

salt. Only in a few special instances will we actually be able to write the solution in terms of arbitrary functions.

The solutions to dynamical ordinary differential equations are singled out by the imposition of initial conditions, resulting in an *initial value problem*. On the other hand, equations modeling equilibrium phenomena require boundary conditions to uniquely specify their solutions, resulting in a *boundary value problem*. We assume that the reader is already familiar with the basics of initial value problems for ordinary differential equations. But we will take time to develop the perhaps less familiar case of boundary value problems in Chapter 5.

A similar specification of auxiliary conditions applies to partial differential equations. Equations modeling equilibrium phenomena are supplemented by boundary conditions imposed on the boundary of the domain of interest. In favorable circumstances, the boundary conditions serve to single out a unique solution. For example, the equilibrium temperature of a body is uniquely specified by its boundary values. If the domain is unbounded, one must also restrict the behavior of the solution at large distances, e.g., by asking that it remain bounded. The combination of a partial differential equation along with suitable boundary conditions is referred to as a *boundary value problem*.

There are three principal types of boundary value problems that arise in most applications. Specifying the value of the solution along the boundary of the domain is called a *Dirichlet boundary condition*, to honor the nineteenth century French analyst Lejeune Dirichlet. Specifying the normal derivative of the solution along the boundary results in a *Neumann boundary condition*, named after his German contemporary Carl Gottfried Neumann. Prescribing the function along part of the boundary and the normal derivative along the remainder results in a *mixed boundary value problem*. For example, in thermal equilibrium, the Dirichlet boundary value problem specifies the temperature on its boundary, and our task is to find the interior temperature distribution by solving an appropriate partial differential equation. Similarly, the Neumann boundary value problem prescribes the heat flux through the boundary. In particular, an insulated boundary has no heat flux, and hence the normal derivative of the temperature is zero on the boundary. The mixed boundary value problem prescribes the temperature along part of the boundary and the heat flux along the remainder. Again, our task is to determine the interior temperature of the body.

For partial differential equations modeling dynamical processes, in which time is one of the independent variables, the solution is to be specified by one or more initial conditions. The number of initial conditions required depends on the highest order time derivative that appears in the equation. For example, in thermodynamics, which only involves the first order time derivative, u_t , of the temperature, the initial condition requires specifying the temperature of the body at the initial time. Newtonian mechanics involves the acceleration or second order time derivative of the motion, and so requires two initial conditions: the initial position and initial velocity of the system. On bounded domains, one must also impose suitable boundary conditions in order to uniquely characterize the solution, and hence the subsequent dynamical behavior of the physical system. The combination of the partial differential equation, the initial conditions, and the boundary conditions leads to an *initial-boundary value problem*. We will encounter many important examples of such

problems during the course of these lectures.

Linear and Nonlinear Equations

As with algebraic equations, [104], and ordinary differential equations, [23], there is a crucial distinction between linear equations, which are relatively easy to solve, and nonlinear equations, which are considerably more challenging. Of course, “easy” depends on the context. While linear algebraic equations are (modulo numerical difficulties, [24]) eminently solvable by a variety of methods, starting with basic Gaussian Elimination, linear ordinary differential equations are already a challenge, and, once the order is two or more, only the constant coefficient case can be readily solved by elementary techniques. Indeed, when solving partial differential equations in higher dimensions, we will encounter several important non-constant coefficient linear ordinary differential equations, and so will need to spend time developing the proper analytical tools for analyzing their (non-elementary) solutions. Linear partial differential equations are of a yet higher level of difficulty, and only a small handful of specific equations can be truly considered as “solved”. For the vast majority of partial differential equations, the only feasible way to get a handle on their solutions is through numerical approximation. We will present and analyze some of the most basic numerical schemes during the course of this text. But it is not possible to develop the numerics before we have acquired a basic understanding of the analytical properties of partial differential equations.

The distinguishing feature of linearity is that it enables one to combine solutions to form new solutions through a general Superposition Principle. Linear superposition is universally applicable to all linear equations and systems, including linear algebraic systems, linear ordinary differential equations, linear partial differential equations, linear initial and boundary value problems, as well as linear integral equations, linear control systems, and so on.

A differential equation is called *homogeneous linear* if every term involves *only* the dependent variable u or one of its derivatives to the first power; however, the coefficients of these individual terms are allowed to be an arbitrary functions of the independent variables. Thus,

$$\frac{d^2u}{dx^2} + \frac{u}{1+x^2} = 0$$

is a homogeneous linear, second order ordinary differential equation. The nonlinear dependence of this equation on the independent variable x is irrelevant — it is only the manner in which u and its derivatives appear that distinguishes between linear and nonlinear equations. The heat equation (1.5), the partial differential equation (1.2), as well as

$$\frac{\partial u}{\partial t} = e^x \frac{\partial^2 u}{\partial x^2} + \cos(x-t)u,$$

are all homogeneous linear partial differential equations. On the other hand, Burgers’ equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \tag{1.10}$$

is not linear since the second term involves the product of u and its derivative u_x . A similar terminology is used for systems of partial differential equations. For example, the Navier–Stokes system (1.4) is nonlinear owing to the terms uu_x, vu_y , etc. — even though its final constituent equation is linear.

A more precise definition of a linear differential equation begins with the concept of a *linear differential operator* L . The operator L is assembled by summing the basic partial derivative operators, with coefficients depending on the independent variables. The operator acts on sufficiently smooth functions depending on the relevant independent variables. *Linearity* imposes two key requirements:

$$L[u + v] = L[u] + L[v], \quad L[cu] = cL[u], \quad (1.11)$$

for any two (sufficiently smooth) functions u, v and any constant c . A precise definition of linear differential equation can now be stated.

Definition 1.2. A *homogeneous linear differential equation* has the form

$$L[u] = 0, \quad (1.12)$$

where L is a linear differential operator.

As a simple example, consider the second order differential operator

$$L = \frac{\partial^2}{\partial x^2}, \quad \text{whereby} \quad L[u] = \frac{\partial^2 u}{\partial x^2}$$

for any C^2 function u . Linearity follows straightforwardly from the basic rules of differentiation:

$$\frac{\partial^2}{\partial x^2}(u + v) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2}, \quad \frac{\partial^2}{\partial x^2}(cu) = c \frac{\partial^2 u}{\partial x^2},$$

valid for any C^2 functions u, v and any constant c . The corresponding homogeneous linear differential equation is

$$\frac{\partial^2 u}{\partial x^2} = 0.$$

The heat equation (1.5) is based on the linear partial differential operator

$$L = \partial_t - \partial_x^2, \quad \text{with} \quad L[u] = \partial_t u - \partial_x^2 u = u_t - u_{xx} = 0. \quad (1.13)$$

Linearity follows as above:

$$\begin{aligned} L[u + v] &= \partial_t(u + v) - \partial_x^2(u + v) = (\partial_t u - \partial_x^2 u) + (\partial_t v - \partial_x^2 v) = L[u] + L[v], \\ L[cu] &= \partial_t(cu) - \partial_x^2(cu) = c(\partial_t u - \partial_x^2 u) = cL[u]. \end{aligned}$$

Similarly, the linear differential operator

$$L = \partial_t^2 - \partial_x \kappa(x) \partial_x,$$

where $\kappa(x)$ is a prescribed C^1 function, defines the linear partial differential equation

$$L[u] = \partial_t^2 u - \partial_x(\kappa(x)\partial_x u) = u_{tt} - \partial_x(\kappa(x)u_x) = u_{tt} - \kappa(x)u_{xx} - \kappa'(x)u_x = 0,$$

which is used to model vibrations in a non-uniform one-dimensional medium.

The defining properties of linear functions (1.11) immediately imply the key facts concerning homogeneous linear (differential) equations.

Proposition 1.3. *The sum of two solutions to a homogeneous linear differential equation is again a solution, as is the product of a solution by any constant.*

Proof: Let u_1, u_2 be solutions, meaning that $L[u_1] = 0$ and $L[u_2] = 0$. Then, thanks to linearity,

$$L[u_1 + u_2] = L[u_1] + L[u_2] = 0,$$

and hence their sum $u_1 + u_2$ is a solution. Similarly, if c is any constant, and u any solution, then

$$L[cu] = cL[u] = c0 = 0,$$

and so the scalar multiple $c u$ is also a solution. *Q.E.D.*

As a result, starting with a handful of solutions to a homogeneous linear differential equation, by repeating these operations of adding solutions and multiplying by constants, we are able to build up large families of solutions. In the case of the heat equation (1.5), we are already in possession of two solutions, namely (1.6) and (1.7). Multiplying each by a constant produces two infinite families of solutions

$$u(t, x) = c_1(t + \frac{1}{2}x^2), \quad \text{and} \quad u(t, x) = \frac{c_2 e^{-x^2/(4t)}}{2\sqrt{\pi t}},$$

where c_1, c_2 are arbitrary constants. Moreover, one can add the latter solutions together, producing a two-parameter family of solutions

$$u(t, x) = c_1(t + \frac{1}{2}x^2) + \frac{c_2 e^{-x^2/(4t)}}{2\sqrt{\pi t}}$$

valid for any choice of the constants c_1, c_2 . This construction is a special case of the general *Superposition Principle* for homogeneous linear equations.

Theorem 1.4. *If u_1, \dots, u_n are solutions to a common homogeneous linear partial differential equation $L[u] = 0$, then the linear combination $u = c_1 u_1 + \dots + c_n u_n$ is a solution for any choice of constants c_1, \dots, c_n .*

Proof: The key fact is that, thanks to the linearity requirements (1.11), for any sufficiently smooth functions u_1, \dots, u_n and any constants c_1, \dots, c_n ,

$$\begin{aligned} L[u] &= L[c_1 u_1 + \dots + c_n u_n] = L[c_1 u_1 + \dots + c_{n-1} u_{n-1}] + L[c_n u_n] \\ &= \dots = L[c_1 u_1] + \dots + L[c_n u_n] = c_1 L[u_1] + \dots + c_n L[u_n]. \end{aligned} \tag{1.14}$$

In particular, if the functions are solutions, so $L[u_1] = 0, \dots, L[u_n] = 0$, then the right hand side of (1.14) vanishes, proving that u is also a solution to the homogeneous equation $L[u] = 0$. *Q.E.D.*

In linear algebraic language, Theorem 1.4 says that the solutions to a linear partial differential equation form a vector space. The same holds true for linear algebraic equations, [104], and linear ordinary differential equations, [23]. In these two situations, once one finds a sufficient number of independent solutions, the general solution is obtained as a linear combination thereof. However, most partial differential equations admit an infinite number of independent solutions, and, as a consequence, one cannot hope to build the general solution by taking *finite* linear combinations. Instead, one requires the far more delicate operation of forming infinite series involving the basic solutions. Such considerations will soon lead us into the heart of Fourier analysis, and require spending an entire chapter developing the required analytical tools. Indeed, now would be a good time for you to brush up on the basics of infinite series that you learned in calculus, [8].

In physical applications, homogeneous linear equations model unforced systems that are subject to their own internal constraints. External forcing is represented by an additional term that does not involve the dependent variable.

Definition 1.5. An *inhomogeneous linear differential equation* has the form

$$L[v] = f, \tag{1.15}$$

where L is a linear partial differential operator, v is the dependent variable, and f is a given non-zero function of the independent variables alone.

The right hand side f of the inhomogeneous differential equation (1.15) does not depend on u or any of its derivatives. For example, the inhomogeneous form of the heat equation (1.13) is

$$L[v] = \partial_t v - \partial_x^2 v = v_t - v_{xx} = f(t, x), \tag{1.16}$$

and serves to model the thermodynamics of a one-dimensional medium subject to a, possibly time dependent, external heat source.

You already learned the basic philosophy for solving of inhomogeneous linear equations in your study of elementary ordinary differential equations, [17, 23, 40]. Step one is to determine the general solution to the homogeneous equation. Step two is to find a particular solution to the inhomogeneous version. The general solution to the inhomogeneous equation is then obtained by adding the two together. Here is the general version of this procedure:

Theorem 1.6. Let v_\star be a particular solution to the inhomogeneous linear equation $L[v_\star] = f$. Then the general solution to $L[v] = f$ is given by $v = v_\star + u$, where u is an arbitrary solution to the corresponding homogeneous equation $L[u] = 0$.

Proof: Let us first show that $v = v_\star + u$ is also a solution whenever $L[u] = 0$. By linearity,

$$L[v] = L[v_\star + u] = L[v_\star] + L[u] = f + 0 = f.$$

To show that every solution to the inhomogeneous equation can be expressed in this manner, suppose v satisfies $L[v] = f$. Set $u = v - v_\star$. Then, by linearity,

$$L[u] = L[v - v_\star] = L[v] - L[v_\star] = 0,$$

and hence u is a solution to the homogeneous differential equation. Thus, $v = v_\star + u$ has the required form. *Q.E.D.*

In physical applications, one can interpret the particular solution v_\star as a response of the system to the external forcing function, while the solution u to the homogeneous equation represents the system's internal, unforced motion. The general solution to a linear inhomogeneous equation is thus a combination of the external and internal responses.

Finally, the *Superposition Principle* for inhomogeneous linear equations allows one to combine the responses of the system to different external forcing functions. The proof of this result is left to the reader as Exercise ■.

Theorem 1.7. *Let v_1, \dots, v_k be solutions to the inhomogeneous linear systems: $L[v_1] = f_1, \dots, L[v_k] = f_k$ for the same linear operator L . Then, for constants c_1, \dots, c_k , the linear combination $v = c_1v_1 + \dots + c_kv_k$ solves the inhomogeneous system $L[v] = f$ for the combined forcing function $f = c_1f_1 + \dots + c_kf_k$.*

In contrast, nonlinear equations are much tougher nuts to crack, and, typically, knowledge of several solutions is of scant help constructing others. Indeed, even finding one solution to a nonlinear partial differential equation can be quite a challenge. The most general effective methods are based on symmetry considerations; see Chapter 8 and [30, 69, 103]. In this introductory course, we will primarily — but not exclusively — concentrate on analyzing the most basic linear partial differential equations. But we will have occasion to briefly foray into the nonlinear realm, to appreciate some recent developments in this fascinating area of contemporary research and applications.