

# Chapter 1

## What are Partial Differential Equations?

Let us begin by specifying our object of study. A *differential equation* is an equation that relates the derivatives of a (scalar) function depending on one or more variables. For example,

$$\frac{d^4u}{dx^4} + u^2 \frac{du}{dx} = \cos x \quad (1.1)$$

is a differential equation for the function  $u(x)$  depending on a single variable  $x$ , while

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + u = 0 \quad (1.2)$$

is a differential equation involving a function  $u(t, x, y)$  of three variables.

There are two common notations for partial derivatives, and we shall employ them interchangeably. The first, used in (1.1) and (1.2), is the familiar Leibniz notation that employs a  $d$  to denote ordinary derivatives of functions of a single variable, and the  $\partial$  symbol (usually also pronounced “dee”) for partial derivatives of functions of more than one variable. An alternative, more compact notation employs subscripts to indicate partial derivatives. For example,  $u_t$  represents  $\partial u/\partial t$ , while  $u_{xx}$  is used for  $\partial^2 u/\partial x^2$ , and  $\partial^3 u/\partial x^2 \partial y$  for  $u_{xxy}$ . Thus, in subscript notation, the partial differential equation (1.2) is written

$$u_t - u_{xx} - u_{yy} + u = 0. \quad (1.3)$$

We will similarly abbreviate partial differential operators, sometimes writing  $\partial/\partial x$  as  $\partial_x$ , while  $\partial^2/\partial x^2$  can be written as either  $\partial_x^2$  or  $\partial_{xx}$ , and  $\partial^2/\partial x^2 \partial y$  becomes  $\partial_{xxy} = \partial_x^2 \partial_y$ .

A differential equation is called *ordinary* if the function  $u$  only depends on a single variable, and *partial* if it depends on more than one variable. Usually (but not quite always) the dependence of  $u$  can be inferred from the derivatives that appear in the differential equation. The *order* of a differential equation is that of the highest order derivative that appears in the equation. Thus, (1.1) is a fourth order ordinary differential equation, while (1.2) is a second order partial differential equation. To be a true differential equation, the equation must contain at least one derivative of  $u$ , and hence its order must be  $\geq 1$ .

It is worth pointing out that the preponderance of differential equations arising in applications, in science, in engineering, and within mathematics itself, are of either first or second order, with the latter being by far the most prevalent. Third order equations arise when modeling waves in dispersive media, e.g., water waves or plasma waves. Fourth order equations show up in elasticity, particularly plate and beam mechanics. Equations of order  $\geq 5$  are rather rare.

A basic prerequisite for studying this text is the ability to solve simple ordinary differential equations: first order equations, linear constant coefficient equations, both homogeneous and inhomogeneous, and linear systems. In addition, we shall assume some familiarity with the basic theorems concerning the existence and uniqueness of solutions to initial value problems. There are many good introductory texts, including [15, 17, 23, 40]. More advanced treatises include [32, 62, 66, 72]. Partial differential equations are considerably more demanding, and can challenge the analytical skills of even the most accomplished mathematician. Many of the most effective solution strategies rely on reducing the partial differential equation to one or more ordinary differential equations. Thus, in the course of our study of partial differential equations, we will need to develop, ab initio, some of the more advanced aspects of the theory of ordinary differential equations, including boundary value problems, eigenvalue problems, series solutions, singular points, and special functions.

Following the introductory remarks in the present chapter, the exposition begins in earnest with simple first order equations, concentrating on those that arise as models of wave phenomena. Most of the remainder of the text will be devoted to understanding and solving the three essential linear, second order partial differential equations in one, two, and three space dimensions<sup>†</sup>: the *heat equation*, modeling thermodynamics in a continuous medium, as well as diffusion of populations and pollutants; the *wave equation*, modeling vibrations of bars, strings, plates, and solid bodies, as well as acoustic, fluid, and electromagnetic vibrations; and the *Laplace equation* and its inhomogeneous counterpart, the *Poisson equation*, governing the mechanical and thermal equilibria of bodies, as well as fluid mechanical and electromagnetic potentials.

Each increase in dimension requires an increase in mathematical sophistication, as well as the development of additional analytical tools — although the key ideas will have all appeared once we reach our physical, three-dimensional universe. The three starring examples — heat, wave and Laplace/Poisson — are not only essential to a wide range of applications, but also serve as instructive paradigms for the three principal classes of linear partial differential equations — parabolic, hyperbolic and elliptic. Some interesting nonlinear partial differential equations, including first order transport equations modeling shock waves, the second order Burgers’ equation governing simple nonlinear diffusion processes, and the third order Korteweg–deVries equation governing dispersive waves, will also be discussed. But, in such an introductory text, the rest of the vast realm of nonlinear partial differential equations must remain unexplored, awaiting more advanced mathematical explorations.

More generally, a *system of differential equations* is a collection of one or more equations relating the derivatives of one or more functions. It is essential that all the functions occurring in the system *depend on the same set of variables*. The symbols for these functions are known as the *dependent variables*, while the variables that they depend on are called the *independent variables*. Systems of differential equations are called *ordinary* or

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<sup>†</sup> For us *dimension* always refers to the number of space dimensions. Time, although theoretically also a dimension, plays a very different physical role, and therefore (at least in non-relativistic systems) is to be treated on a separate footing.

*partial* according to whether there are one or more independent variables. The *order* of the system is the highest order derivative occurring in any of its equations.

For example, the three-dimensional *Navier–Stokes equations*

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} &= - \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} &= - \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right), \\ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} &= - \frac{\partial p}{\partial z} + \nu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right), \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0, \end{aligned} \tag{1.4}$$

is a second order system of differential equations that involves four functions,  $u(t, x, y, z)$ ,  $v(t, x, y, z)$ ,  $w(t, x, y, z)$ ,  $p(t, x, y, z)$ , each depending on four variables. (The function  $p$  necessarily depends on  $t$  even though no  $t$  derivative of it appears in the system.) On the other hand,  $\nu \geq 0$  is a fixed constant. The independent variables are  $t$ , representing time, and  $x, y, z$ , representing space coordinates. The dependent variables are  $u, v, w, p$ , with  $\mathbf{v} = (u, v, w)$  representing the velocity vector field of an incompressible fluid flow, e.g., water, and  $p$  the accompanying pressure. The parameter  $\nu$  indicates the viscosity of the fluid. The Navier–Stokes equations are fundamental in fluid mechanics, and notoriously difficult to solve, either analytically or numerically. Indeed, establishing the existence or non-existence of solutions for all time is a major unsolved problem in mathematics, whose resolution will earn you a \$1,000,000 prize; see <http://www.claymath.org> for details.

We shall be employing a few basic notational conventions regarding the variables that appear in our differential equations. We always use  $t$  to denote time, while  $x, y, z$  will represent (Cartesian) space coordinates. Polar coordinates  $r, \theta$ , cylindrical coordinates  $r, \theta, z$ , and spherical coordinates  $r, \theta, \varphi$ , will also be used when needed. An *equilibrium equation* models an unchanging physical system, and so only involves the space variable(s). The time variable appears when modeling *dynamical*, meaning time-varying, processes. Both time and space coordinates are (usually) independent variables. The dependent variables will mostly be denoted by  $u, v, w$ , although occasionally — particularly when representing particular physical quantities — other letters may be employed, e.g., the pressure  $p$  in (1.4).

In this introductory text, we must confine our attention to the most basic analytical and numerical solution techniques for a select few of the most important partial differential equations. More advanced topics, including all systems of partial differential equations, must be deferred to graduate and research level texts, e.g., [36, 45, 52, 75]. In fact, many fundamental issues remain unresolved and/or poorly understood, making partial differential equations one of the most active fields of contemporary mathematical research. One of my goals is that, by reading this book, you will be both inspired and equipped to venture much further into this fascinating and essential area of mathematics and/or its phenomenal range of applications throughout science, engineering, economics, biology, and elsewhere.

## Classical Solutions

Let us now focus our attention on a single differential equation involving a single, scalar-valued function  $u$  that depends on one or more independent variables. The function  $u$  is usually real-valued, although complex-valued functions can, and do, play a role in the analysis. Everything that we say in this section will, when suitably adapted, apply to systems of differential equations.

By a *solution* we mean a sufficiently smooth function  $u$  of the independent variables that satisfies the differential equation at every point of its domain of definition. We do not necessarily require that the solution be defined for all possible values of the independent variables. Indeed, usually the differential equation is imposed on some domain  $D$  contained in the space of independent variables, and we seek a solution defined only on  $D$ . In general, the *domain*  $D$  will be an open subset, usually connected and, particularly in equilibrium equations, often bounded, with a reasonably nice boundary, denoted  $\partial D$ .

We will call a function *smooth* if it can be differentiated sufficiently often, at least so that all of the derivatives appearing in the equation be well-defined on the domain of interest  $D$ . More specifically, if the differential equation has order  $n$ , then we require that the solution  $u$  be of *class*  $C^n$ , which means that it and all its derivatives of order  $\leq n$  are continuous functions in  $D$ , and such that the differential equation that relates the derivatives of  $u$  holds throughout  $D$ . However, on occasion, e.g., when dealing with shock waves, we will consider more general types of solutions. The most important such class consists of the so-called “weak solutions” to be introduced in Section 2.3. To emphasize the distinction, the smooth solutions described above are often referred to as *classical solutions* or, occasionally, *strong solutions*. In this book, the term “solution” without extra qualification will inevitably mean “classical solution”.

**Example 1.1.** A *classical solution* to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{1.5}$$

is a function  $u(t, x)$ , defined on a domain  $D \subset \mathbb{R}^2$ , such that all of the functions

$$u(t, x), \quad \frac{\partial u}{\partial t}(t, x), \quad \frac{\partial u}{\partial x}(t, x), \quad \frac{\partial^2 u}{\partial t^2}(t, x), \quad \frac{\partial^2 u}{\partial t \partial x}(t, x) = \frac{\partial^2 u}{\partial x \partial t}(t, x), \quad \frac{\partial^2 u}{\partial x^2}(t, x),$$

are well-defined and continuous<sup>†</sup> at every point  $(t, x) \in D$ , so that  $u \in C^2(D)$ , and, moreover, that (1.5) holds at every  $(t, x) \in D$ . For example,

$$u(t, x) = t + \frac{1}{2}x^2 \tag{1.6}$$

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<sup>†</sup> The equality of the mixed partial derivatives follows from a general theorem in multivariable calculus, [7]. Classical solutions automatically enjoy equality of all their relevant mixed partial derivatives.

is a solution to the heat equation that is defined on the full domain  $D = \mathbb{R}^2$  because it is<sup>‡</sup>  $C^2$ , and, moreover,

$$\frac{\partial u}{\partial t} = 1 = \frac{\partial^2 u}{\partial x^2}.$$

Another, more complicated, but extremely important, solution is

$$u(t, x) = \frac{e^{-x^2/(4t)}}{2\sqrt{\pi t}}. \quad (1.7)$$

One easily checks that  $u \in C^2$  and, moreover, solves the heat equation on the domain  $D = \{t > 0\} \subset \mathbb{R}^2$ . The reader is invited to verify this by computing  $\partial u/\partial t$  and  $\partial^2 u/\partial x^2$ , and then checking that they are equal. Finally, with  $i = \sqrt{-1}$  denoting the imaginary unit, we note that

$$u(t, x) = e^{-t+ix} = e^{-t} \cos x + i e^{-t} \sin x \quad (1.8)$$

defines a complex-valued solution to the heat equation. This can be verified directly, since the rules for differentiating complex exponentials are identical to those for their real counterparts:

$$\frac{\partial u}{\partial t} = -e^{-t+ix}, \quad \frac{\partial u}{\partial x}(t, x) = i e^{-t+ix}, \quad \text{and so} \quad \frac{\partial^2 u}{\partial x^2}(t, x) = -e^{-t+ix} = \frac{\partial u}{\partial t}.$$

It is worth pointing out that both the real part,  $e^{-t} \cos x$ , and the imaginary part,  $e^{-t} \sin x$ , of the complex solution (1.8) are individual real solutions, which is indicative of a fairly general property.

Incidentally, most partial differential equations arising in physical applications are real and, although complex solutions often facilitate their analysis, at the end of the day we require real, physically meaningful solutions. A notable exception is quantum mechanics, which is an inherently complex-valued physical theory. For example, the one-dimensional *Schrödinger equation*

$$i \hbar \frac{\partial u}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 u}{\partial x^2} + V(x) u \quad (1.9)$$

with  $\hbar$  denoting Planck's (real) constant, governs the dynamical evolution of the complex-valued wave function  $u(t, x)$  describing the probabilistic distribution of a quantum particle of mass  $m$ , e.g., an electron, subject to the (real) potential function  $V(x)$ . While the solution  $u$  is complex-valued, the independent variables  $t, x$ , representing time and space, remain real.

### *Initial Conditions and Boundary Conditions*

How many solutions does a partial differential equation have? In general, lots. Even ordinary differential equations have infinitely many solutions. Indeed, the general solution to a single  $n^{\text{th}}$  order ordinary differential equation depends on  $n$  arbitrary constants. The

<sup>‡</sup> In fact, the function (1.6) is  $C^\infty$ , meaning infinitely differentiable, on all of  $\mathbb{R}^2$ .

solutions to partial differential equations are yet more numerous, in that they depend on *arbitrary functions*. Very roughly, we can expect the solution to an  $n^{\text{th}}$  order partial differential equation involving  $m$  independent variables to depend on  $n$  arbitrary functions of  $m - 1$  variables. But this is only a rough guide, and must be taken with a large grain of salt. Only in a few special instances will we actually be able to write the solution in terms of arbitrary functions.

The solutions to dynamical ordinary differential equations are singled out by the imposition of initial conditions, resulting in an *initial value problem*. On the other hand, equations modeling equilibrium phenomena require boundary conditions to uniquely specify their solutions, resulting in a *boundary value problem*. We assume that the reader is already familiar with the basics of initial value problems for ordinary differential equations. But we will take time to develop the perhaps less familiar case of boundary value problems for ordinary differential equations in Chapter 6.

A similar specification of auxiliary conditions applies to partial differential equations. Equations modeling equilibrium phenomena are supplemented by boundary conditions imposed on the boundary of the domain of interest. In favorable circumstances, the boundary conditions serve to single out a unique solution. For example, the equilibrium temperature of a body is uniquely specified by its boundary behavior. If the domain is unbounded, one must also restrict the nature of the solution at large distances, e.g., by asking that it remain bounded. The combination of a partial differential equation along with suitable boundary conditions is referred to as a *boundary value problem*.

There are three principal types of boundary value problems that arise in most applications. Specifying the value of the solution along the boundary of the domain is called a *Dirichlet boundary condition*, to honor the nineteenth century analyst Johann Peter Gustav Lejeune Dirichlet. Specifying the normal derivative of the solution along the boundary results in a *Neumann boundary condition*, named after his contemporary Carl Gottfried Neumann. Prescribing the function along part of the boundary and the normal derivative along the remainder results in a *mixed boundary value problem*. For example, in thermal equilibrium, the Dirichlet boundary value problem specifies the temperature on its boundary, and our task is to find the interior temperature distribution by solving an appropriate partial differential equation. Similarly, the Neumann boundary value problem prescribes the heat flux through the boundary. In particular, an insulated boundary has no heat flux, and hence the normal derivative of the temperature is zero on the boundary. The mixed boundary value problem prescribes the temperature along part of the boundary and the heat flux along the remainder. Again, our task is to determine the interior temperature of the body.

For partial differential equations modeling dynamical processes, in which time is one of the independent variables, the solution is to be specified by one or more initial conditions. The number of initial conditions required depends on the highest order time derivative that appears in the equation. For example, in thermodynamics, which only involves the first order time derivative of the temperature, the initial condition requires specifying the temperature of the body at the initial time. Newtonian mechanics describes the acceleration or second order time derivative of the motion, and so requires two initial conditions: the initial position and initial velocity of the system. On bounded domains, one must also

impose suitable boundary conditions in order to uniquely characterize the solution, and hence the subsequent dynamical behavior of the physical system. The combination of the partial differential equation, the initial conditions, and the boundary conditions leads to an *initial-boundary value problem*. We will encounter, and solve, many important examples of such problems during the course of this text.

### *Linear and Nonlinear Equations*

As with algebraic equations and ordinary differential equations, there is a crucial distinction between linear partial differential equations, which are comparatively easy to solve, and nonlinear partial differential equations, which are considerably more challenging. Of course, “easy” depends on the context. While linear algebraic equations are (modulo numerical difficulties) eminently solvable by a variety of methods, linear ordinary differential equations are already a challenge; once the order is two or more, only the constant coefficient case can be readily solved by elementary techniques; indeed, as we will learn in Chapter 12, solving linear ordinary differential equations requires new types of functions, known as “special functions”. Linear partial differential equations are of a yet higher level of difficulty, and only a small handful of specific equations can be truly considered as “solved”. For the vast majority of partial differential equations, the only feasible means of producing general solutions is through numerical approximation. In this book, we will study the two most basic numerical schemes, finite differences and finite elements. Keep in mind that, in order to develop and understand the numerics of partial differential equations, one must already have some understanding of their analytical properties.

The distinguishing feature of linearity is that it enables one to combine solutions so as to form new solutions through a general Superposition Principle. Linear superposition is universally applicable to all linear equations and systems, including linear algebraic systems, linear ordinary differential equations, linear partial differential equations, linear initial and boundary value problems, as well as linear integral equations, linear control systems, and so on. Let us introduce the basic idea in the context of a single differential equation.

A differential equation is called *homogeneous linear* if it is a sum of terms, each of which involves *only* the dependent variable  $u$  or one of its derivatives to the first power. There is no restriction on how the terms involve the independent variables. Thus,

$$\frac{d^2u}{dx^2} + \frac{u}{1+x^2} = 0$$

is a homogeneous linear, second order ordinary differential equation. Examples of homogeneous linear partial differential equations include the heat equation (1.5), the partial differential equation (1.2), and the equation

$$\frac{\partial u}{\partial t} = e^x \frac{\partial^2 u}{\partial x^2} + \cos(x-t)u.$$

On the other hand, Burgers’ equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \tag{1.10}$$

is not linear since the second term involves the product of  $u$  and its derivative  $u_x$ . A similar terminology is applied to systems of partial differential equations. For example, the Navier–Stokes system (1.4) is not linear because of the terms  $uu_x, vu_y$ , etc. — even though its final constituent equation is linear.

A more precise definition of a homogeneous linear differential equation begins with the concept of a *linear differential operator*  $L$ . Such operators are assembled by summing the basic partial derivative operators, with either constant coefficients, or, more generally, coefficients depending on the independent variables. The operator acts on sufficiently smooth functions depending on the relevant independent variables. *Linearity* imposes two key requirements:

$$L[u + v] = L[u] + L[v], \quad L[cu] = cL[u], \quad (1.11)$$

for any two (sufficiently smooth) functions  $u, v$ , and any constant  $c$ .

**Definition 1.2.** A *homogeneous linear differential equation* has the form

$$L[u] = 0, \quad (1.12)$$

where  $L$  is a linear differential operator.

As a simple example, consider the second order differential operator

$$L = \frac{\partial^2}{\partial x^2}, \quad \text{whereby} \quad L[u] = \frac{\partial^2 u}{\partial x^2}$$

for any  $C^2$  function  $u(x, y)$ . The linearity requirements (1.11) follow immediately from basic properties of differentiation:

$$\frac{\partial^2}{\partial x^2} (u + v) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2}, \quad \frac{\partial^2}{\partial x^2} (cu) = c \frac{\partial^2 u}{\partial x^2},$$

which are valid for any  $C^2$  functions  $u, v$  and any constant  $c$ . The corresponding homogeneous linear differential equation  $L[u] = 0$  is

$$\frac{\partial^2 u}{\partial x^2} = 0.$$

The heat equation (1.5) is based on the linear partial differential operator

$$L = \partial_t - \partial_x^2, \quad \text{with} \quad L[u] = \partial_t u - \partial_x^2 u = u_t - u_{xx} = 0. \quad (1.13)$$

Linearity follows as above:

$$\begin{aligned} L[u + v] &= \partial_t(u + v) - \partial_x^2(u + v) = (\partial_t u - \partial_x^2 u) + (\partial_t v - \partial_x^2 v) = L[u] + L[v], \\ L[cu] &= \partial_t(cu) - \partial_x^2(cu) = c(\partial_t u - \partial_x^2 u) = cL[u]. \end{aligned}$$

Similarly, the linear differential operator

$$L = \partial_t^2 - \partial_x \kappa(x) \partial_x = \partial_t^2 - \kappa(x) \partial_x^2 - \kappa'(x) \partial_x,$$

where  $\kappa(x)$  is a prescribed  $C^1$  function of  $x$  alone, defines the homogeneous linear partial differential equation

$$L[u] = \partial_t^2 u - \partial_x(\kappa(x)\partial_x u) = u_{tt} - \partial_x(\kappa(x)u_x) = u_{tt} - \kappa(x)u_{xx} - \kappa'(x)u_x = 0,$$

which is used to model vibrations in a non-uniform one-dimensional medium.

The defining attributes of linear operators (1.11) imply the key properties shared by all homogeneous linear (differential) equations.

**Proposition 1.3.** *The sum of two solutions to a homogeneous linear differential equation is again a solution, as is the product of a solution with any constant.*

*Proof:* Let  $u_1, u_2$  be solutions, meaning that  $L[u_1] = 0$  and  $L[u_2] = 0$ . Then, thanks to linearity,

$$L[u_1 + u_2] = L[u_1] + L[u_2] = 0,$$

and hence their sum  $u_1 + u_2$  is a solution. Similarly, if  $c$  is any constant, and  $u$  any solution, then

$$L[cu] = cL[u] = c0 = 0,$$

and so the scalar multiple  $cu$  is also a solution.

*Q.E.D.*

As a result, starting with a handful of solutions to a homogeneous linear differential equation, by repeating these operations of adding solutions and multiplying by constants, we are able to build up large families of solutions. In the case of the heat equation (1.5), we are already in possession of two solutions, namely (1.6) and (1.7). Multiplying each by a constant produces two infinite families of solutions:

$$u(t, x) = c_1\left(t + \frac{1}{2}x^2\right), \quad \text{and} \quad u(t, x) = \frac{c_2 e^{-x^2/(4t)}}{2\sqrt{\pi t}},$$

where  $c_1, c_2$  are arbitrary constants. Moreover, one can add the latter solutions together, producing a two-parameter family of solutions

$$u(t, x) = c_1\left(t + \frac{1}{2}x^2\right) + \frac{c_2 e^{-x^2/(4t)}}{2\sqrt{\pi t}}$$

valid for any choice of the constants  $c_1, c_2$ .

The preceding construction is a special case of the general *Superposition Principle* for homogeneous linear equations.

**Theorem 1.4.** *If  $u_1, \dots, u_k$  are solutions to a common homogeneous linear equation  $L[u] = 0$ , then the linear combination or superposition  $u = c_1u_1 + \dots + c_ku_k$  is a solution for any choice of constants  $c_1, \dots, c_k$ .*

*Proof:* Repeatedly applying the linearity requirements (1.11), we find

$$\begin{aligned} L[u] &= L[c_1u_1 + \dots + c_ku_k] = L[c_1u_1 + \dots + c_{k-1}u_{k-1}] + L[c_ku_k] \\ &= \dots = L[c_1u_1] + \dots + L[c_ku_k] = c_1L[u_1] + \dots + c_kL[u_k]. \end{aligned} \tag{1.14}$$

In particular, if the functions are solutions, so  $L[u_1] = 0, \dots, L[u_k] = 0$ , then the right hand side of (1.14) vanishes, proving that  $u$  also solves the equation  $L[u] = 0$ . *Q.E.D.*

In the linear algebraic language of Appendix B, Theorem 1.4 tells us that the solutions to a homogeneous linear partial differential equation form a vector space. The same holds true for linear algebraic equations, [108], and linear ordinary differential equations, [17, 23, 40]. In these two situations, once one finds a sufficient number of independent solutions, the general solution is obtained as a linear combination thereof. In the language of linear algebra, the solution space is finite-dimensional. In contrast, most partial differential equations admit an infinite number of independent solutions, meaning that the solution space is infinite-dimensional, and, as a consequence, one cannot hope to build the general solution by taking *finite* linear combinations. Instead, one requires the far more delicate operation of forming infinite series involving the basic solutions. Such considerations will soon lead us into the heart of Fourier analysis, and require spending an entire chapter developing the required analytical tools.

In physical applications, homogeneous linear equations model unforced systems that are solely responding to their own internal attributes. External forcing is represented by an additional term that does not involve the dependent variable.

**Definition 1.5.** An *inhomogeneous linear differential equation* has the form

$$L[v] = f, \tag{1.15}$$

where  $L$  is a linear differential operator,  $v$  is the dependent variable, and  $f$  is a given non-zero function of the independent variables alone.

For example, the inhomogeneous form of the heat equation (1.13) is

$$L[v] = \partial_t v - \partial_x^2 v = v_t - v_{xx} = f(t, x), \tag{1.16}$$

and serves to model the thermodynamics of a one-dimensional medium subject to an external heat source.

You already learned the basic technique for solving inhomogeneous linear equations in your study of elementary ordinary differential equations. Step one is to determine the general solution to the homogeneous equation. Step two is to find a particular solution to the inhomogeneous version. The general solution to the inhomogeneous equation is then obtained by adding the two together. Here is the general version of this procedure:

**Theorem 1.6.** Let  $v_\star$  be a particular solution to the inhomogeneous linear equation  $L[v_\star] = f$ . Then the general solution to  $L[v] = f$  is given by  $v = v_\star + u$ , where  $u$  is the general solution to the corresponding homogeneous equation  $L[u] = 0$ .

*Proof:* Let us first show that  $v = v_\star + u$  is also a solution whenever  $L[u] = 0$ . By linearity,

$$L[v] = L[v_\star + u] = L[v_\star] + L[u] = f + 0 = f.$$

To show that every solution to the inhomogeneous equation can be expressed in this manner, suppose  $v$  satisfies  $L[v] = f$ . Set  $u = v - v_\star$ . Then, by linearity,

$$L[u] = L[v - v_\star] = L[v] - L[v_\star] = 0,$$

and hence  $u$  is a solution to the homogeneous differential equation. Thus,  $v = v_\star + u$  has the required form. *Q.E.D.*

In physical applications, one can interpret the particular solution  $v_*$  as a possible response of the system to the external forcing function. The solution  $u$  to the homogeneous equation represents the system's internal, unforced motion. The general solution to the inhomogeneous linear equation is thus a combination,  $v = v_* + u$ , of the external and internal responses.

Finally, the *Superposition Principle* for inhomogeneous linear equations allows one to combine the responses of the system to different external forcing functions. The proof of this result is left to the reader as Exercise ■.

**Theorem 1.7.** *Let  $v_1, \dots, v_k$  be solutions to the inhomogeneous linear systems:  $L[v_1] = f_1, \dots, L[v_k] = f_k$  involving the same linear operator  $L$ . Then, for constants  $c_1, \dots, c_k$ , the linear combination  $v = c_1 v_1 + \dots + c_k v_k$  solves the inhomogeneous system  $L[v] = f$  for the combined forcing function  $f = c_1 f_1 + \dots + c_k f_k$ .*

The two general superposition principles provide us with powerful tools for solving linear partial differential equations, that we shall repeatedly exploit throughout this text. In contrast, nonlinear partial differential equations are much tougher nuts to crack, and, typically, knowledge of several solutions is of scant help constructing others. Indeed, even finding one solution to a nonlinear partial differential equation can be quite a challenge. This introductory book will concentrate on analyzing some of the most basic and most important linear partial differential equations. But we will have occasion to briefly foray into the nonlinear realm, to appreciate some recent developments in this fascinating arena of contemporary research.