JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **145**, 342–356 (1990)

Lie Algebras of Differential Operators and Lie-Algebraic Potentials

NIKY KAMRAN*

School of Mathematics, The Institute for Advanced Study, Princeton, New Jersey 08540

AND

PETER J. OLVER†

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455

Submitted by G.-C. Rota

Received June 20, 1988

An explicit characterisation of all second order differential operators on the line which can be written as bilinear combinations of the generators of a finite-dimensional Lie algebra of first order differential operators is found, solving a problem arising in the Lie-algebraic approach to scattering theory and molecular dynamics. One-dimensional potentials corresponding to these Lie algebras are explicitly classified, which include the harmonic oscillator, Morse, one-soliton (Pöschl–Teller), Mathieu, Lamé, confluent hypergeometric, and Bessel potentials.

1. INTRODUCTION

The use of Lie-algebraic methods is a well-established, powerful tool in molecular dynamics and quantum chemistry [2]. The solution of the Schrödinger equation is particularly simple if the Hamiltonian differential operator lies in a finite-dimensional Lie algebra; however, these Hamiltonians are the exception as far as complicated molecular behavior is concerned. A far more interesting case, which occurs surprisingly often for physically important Hamiltonians in both nuclear and molecular physics, is when the Hamiltonian is a bilinear combination of the generators of a finite-dimensional Lie algebra. Indeed there are now a number of well-

* Research supported in part by NSF and NSERC grants.
† Research supported in part by the NSF under Grant DMS 86-02004.

0022-247X/90 $3.00

established methods for the calculation of eigenvalues, spectra, and dynamics for such operators (cf. [1]). In a survey lecture delivered at the Institute for Mathematics and Its Applications in Minneapolis, R. D. Levine [6] noted that although many examples of such Hamiltonians are known, there is no general systematic method for finding such operators. He therefore posed the general problem of classifying all second order Hamiltonians on the line which admit such a decomposition. In this paper we provide a complete solution to Levine's problem. In Theorem 1 we present the complete classification, due to W. Miller [8], of all finite-dimensional Lie algebras of first order differential operators on the line, generalizing Lie's classification [7] of all finite-dimensional Lie algebras of vector fields on the line. We then apply the solution to the equivalence problem for second order differential operators on the line found in [5], to determine all differential operators admitting a bilinear decomposition in terms of the generators of one of the Lie algebras, leading to the notion of a "Lie-algebraic potential." Many of the physically important potentials of interest in scattering theory and molecular dynamics appear among the Lie-algebraic potentials, the complete list of which appears in Theorem 7.

2. Lie Algebras of Differential Operators

Consider the space **D** of all first order differential operators

$$\mathscr{D} = f(x)\,D + g(x), \tag{1}$$

where f, g are analytic functions of the real variable $x \in \mathbb{R}$, and $D = d/dx$. There is a natural Lie bracket, provided by the commutator

$$[\mathscr{D}, \mathscr{E}] = \mathscr{D} \cdot \mathscr{E} - \mathscr{E} \cdot \mathscr{D},$$

which makes **D** into a Lie algebra. The basic problem to be solved in this section is the determination of all possible finite-dimensional Lie subalgebras of the Lie algebra **D**.

There are two basic pseudogroups of coordinate changes which can act on the algebra of differential operators. The first, and more restrictive, consists of all (locally) invertible changes of independent variable

$$\bar{x} = \varphi(x). \tag{2}$$

According to the chain rule, the derivative operators $D = d/dx$ and $\bar{D} = d/d\bar{x}$ transform according to the basic formula

$$\bar{D} = \frac{1}{\varphi'(x)}\,D.$$

Thus the change of variable (2) maps $\mathscr{D}$ to the differential operator

$$\bar{\mathscr{D}} = \bar{f}(\bar{x})\, \bar{D} + \bar{g}(\bar{x}), \tag{3}$$

where

$$\bar{f}(\varphi(x)) = \varphi'(x)\, f(x), \qquad \bar{g}(\varphi(x)) = g(x).$$

A second, larger pseudogroup includes further rescalings of the field variable by smooth functions. In order to preserve the Lie algebra structure on the space **D**, they must take the form

$$\bar{\mathscr{D}} = \psi(x) \cdot \mathscr{D} \cdot \frac{1}{\psi(x)}. \tag{4}$$

In general, we will call two differential operators $\mathscr{D}$ and $\bar{\mathscr{D}}$ *equivalent* if there is a change of variables (2) and a function $\psi(x)$ such that they are related by the formula (4). For first order operators (1), (3), equivalence implies that the coefficient functions are related by the more general formulae

$$\bar{f}(\varphi(x)) = \varphi'(x)\, f(x), \qquad \bar{g}(\varphi(x)) = g(x) - f(x)\, \frac{\psi'(x)}{\psi(x)}. \tag{5}$$

Two Lie subalgebras of differential operators are *equivalent* if they can be mapped to each other by a coordinate change (2), (4). W. Miller [8, Chap. 8] provides a complete classification, up to equivalence, of all finite-dimensional Lie subalgebras of the full Lie algebra **D**. We begin by stating his result and, for the reader's convenience, presenting a simplified proof of this result.

THEOREM 1. *Let $\mathfrak{g} \subset \mathbf{D}$ be a finite-dimensional Lie algebra of differential operators on the line. Then $\mathfrak{g}$ is equivalent to one of the following Lie algebras*:

$$\text{(a)} \qquad \mathfrak{g}_{\mathbf{a}}^{(0)} = \{f_1(x), ..., f_n(x)\},$$

where $f_1(x), ..., f_n(x)$ are any linearly independent functions;

$$\text{(b)} \quad \mathfrak{g}_{\mathbf{a}}^{(1)} = \{D, f_1(x), ..., f_n(x)\},$$

where $\mathbf{a} = (a_0, a_1, ..., a_{n-1})$ is an n-tuple of real numbers and $\{f_1(x), ..., f_n(x)\}$ forms a basis for the solution space to the nth order, homogeneous, constant coefficient, linear ordinary differential equation

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0; \tag{6}$$

in particular, each function $f_j(x)$ is a combination of polynomials, exponentials, sines, and cosines.

(c) $\mathfrak{g}^{(2)}_{-1,k} = \{D, x D + k\}, \qquad k \in \mathbb{R},$

$\mathfrak{g}^{(2)}_n = \{D, x D, x^n, x^{n-1}, ..., x, 1\}, \qquad n \geqslant 0,$

(d) $\mathfrak{g}^{(3)}_{k,1} = \{D, x D, x^2 D + 2kx, 1\}, \qquad k \in \mathbb{R},$

$\mathfrak{g}^{(3)}_{k,0} = \{D, x D + k, x^2 D + 2kx\}, \qquad k \in \mathbb{R}.$

The structure of these particular Lie algebras is easily determined. Besides the trivial abelian subalgebras of type (a), the only possible abelian subalgebras are the special cases $\{D\}$, $\{D, 1\}$ of those of type (b). The other Lie algebras of type (b) or (c) are always solvable. (These are classified according to Jacobson [3] for dimension $\leqslant 3$, and Patera *et al.* [9] for dimension $\leqslant 5$.) Algebras of type (d) are isomorphic to either $\mathfrak{sl}(2,\mathbb{R})$ or the central extension $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$. In fact, we note that if $\mathfrak{g}$ is any Lie algebra on our list, and $1 \notin \mathfrak{g}$, then the central extension $\mathfrak{g} \oplus \{1\} \cong \mathfrak{g} \oplus \mathbb{R}$ is also a finite-dimensional Lie algebra on our list.

There are two important infinite-dimensional subalgebras of the Lie algebra **D**. The first is the subalgebra $\mathbf{V} \subset \mathbf{D}$ of vector fields

$$\mathbf{v} = f(x) D \tag{7}$$

on the line, where we identify the differential operator D with the basis tangent vector $\partial/\partial x$. The second subalgebra is the space **M** of multiplication operators, which are differential operators (1) having no derivative term, i.e., $f \equiv 0$, which forms an abelian subalgebra $\mathbf{M} \subset \mathbf{D}$. Note that the vector fields act naturally on the multiplication operators, giving **D** the structure of a semi-direct product

$$\mathbf{D} = \mathbf{V} \ltimes \mathbf{M}.$$

LEMMA 2. *The projection $\pi: \mathbf{D} \to \mathbf{V}$ taking the differential operator $\mathscr{D} = f(x)D + g(x)$ to the vector field $\mathbf{v} = \pi(\mathscr{D}) = f(x)D$ is a Lie algebra isomorphism. Moreover, the change of variables* (2) *preserves the subalgebra* **V**.

Consequently, if $\mathfrak{g} \subset \mathbf{D}$ is a finite-dimensional Lie subalgebra, its image $\pi(\mathfrak{g}) \subset \mathbf{V}$ must be a finite-dimensional Lie subalgebra of the algebra of vector fields. We now invoke a result due to Lie [7] classifying all such subalgebras.

THEOREM 3. *Let $\mathfrak{h} \subset \mathbf{V}$ be a non-zero finite-dimensional Lie algebra of* *vector fields on the line. Then* $\mathfrak{h}$ *is equivalent under a change of variables* (2) *to one of the three Lie algebras*

$$\mathfrak{h}_1 = \{D\}, \qquad \mathfrak{h}_2 = \{D, x\,D\}, \qquad \mathfrak{h}_3 = \{D, x\,D, x^2\,D\}.$$

Note that $\mathfrak{h}_1$ is the unique one-dimensional Lie algebra, $\mathfrak{h}_2$ is the unique solvable two-dimensional Lie algebra, and $\mathfrak{h}_3$ is isomorphic to the semi-simple algebra $\mathfrak{sl}(2, \mathbb{R})$.

Proof of Theorem 1. Let $\mathfrak{g} \subset \mathbf{D}$ be a finite-dimensional subalgebra, and let $\mathfrak{h} = \pi(\mathfrak{g})$. If $\mathfrak{h} = \{0\}$, then the Lie algebra $\mathfrak{g}$ consists solely of multiplication operators, and we are in the trivial case (a) of the theorem. Otherwise, since $\mathfrak{h}$ is finite-dimensional, it is equivalent under the change of independent variables (2) to one of the three subalgebras listed in Theorem 3. Moreover, Lemma 2 implies that $\mathfrak{g}$ itself has the semi-direct product structure $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{s}$, where $\mathfrak{s}$ is a finite-dimensional space of multiplication operators. The proof of Theorem 1 accordingly splits into three cases, depending on the isomorphism class of $\mathfrak{h}$.

Case 1. Here $\pi(\mathfrak{g}) = \mathfrak{h}_1$, hence $\mathfrak{g}$ is spanned by the operators

$$D + g(x), f_1(x), ..., f_n(x). \tag{8}$$

Since $[D + g, f_j] = f_j'$ must lie in $\mathfrak{g}$, the functions f_j must satisfy the conditions of the following elementary lemma:

LEMMA 4. *Suppose* $f_1(x), ..., f_n(x)$ *are linearly independent real-valued functions with the property that each derivative* f_j' *lies in the span of* $f_1, ..., f_n$. *Then* $\{f_1, ..., f_n\}$ *forms a basis for the solution space to a constant coefficient* n*th order homogeneous linear ordinary differential equation*; *see* (6).

Case 2. Here $\pi(\mathfrak{g}) = \mathfrak{h}_2$, hence $\mathfrak{g}$ is spanned by the operators

$$D + g(x), x\,D + h(x), f_1(x), ..., f_n(x).$$

Note that we can use the standard filtration on the space of analytic functions on the line, so that we can take, without loss of generality,

$$f_j(x) = cx^{m_j} + O(x^{m_j + 1}),$$

where $0 \leqslant m_1 < m_2 < \cdots < m_n$. In order that the Lie bracket $[D + g, f_j] = f_j'$ lie in $\mathfrak{g}$, hence in the linear span of the f_j, we must have $m_j = j - 1$, i.e., $f_j(x) = cx^{j-1} + O(x^j)$. Moreover, the Lie bracket $[x\,D + h, f_j] = xf_j'$ must also lie in $\mathfrak{g}$. If we compare the power series expansions we find that

$$xf_j' = (j-1)f_j + \sum_{j<k} c_{jk} f_k,$$

for certain constants c_{jk}. In particular, $xf'_n = (n-1)f_n$, so f_n is a multiple of x^{n-1}. Proceeding by reverse induction on n, we easily deduce that

$$f_j(x) = \sum_{j \leqslant k} \tilde{c}_{jk} x^k,$$

and so we can replace $f_1(x), \ldots, f_n(x)$ by the monomials $1, x, \ldots, x^{n-1}$. Finally, since

$$[x\,D + h, D + g] = -D + (xg' - h') = -(D+g) + (xg' + g - h'),$$

the expression $xg' + g - h'$ must be a polynomial of degree at most $n-1$, hence $h(x) = xg(x) + q(x)$, where $q(x)$ is a polynomial of degree at most n. We can subtract off the terms of degree $< n$ from q using the basis elements x^j, $j \leqslant n-1$, and so deduce that $\mathfrak{g}$ is spanned by the operators

$$D + g,\ x(D+g) + kx^n,\ x^{n-1}, \ldots, x, 1.$$

For $n = 0$, we have the generating set

$$D + g,\ x(D+g) + k, \tag{9a}$$

while for $n > 0$, we can replace $g(x)$ by $g(x) + kx^{n-1}$, and thereby reduce to the simpler generating set

$$D + g,\ x(D+g),\ x^{n-1}, \ldots, x, 1. \tag{9b}$$

Case 3. Here $\pi(\mathfrak{g}) = \mathfrak{h}_3$, so that $\mathfrak{g}$ is spanned by the operators

$$D + g(x),\ x\,D + h(x),\ x^2\,D + r(x),\ f_1(x), \ldots, f_n(x).$$

Everything said in Case 2 still holds; in particular, we can take $f_j(x) = x^{j-1}$. However, the Lie bracket

$$[x^2\,D + k(x), x^{n-1}] = nx^n$$

is not in our subalgebra unless $n = 1$ or 0. Therefore, $\mathfrak{g}$ is spanned either by the operators

$$D + g(x),\ x(D + g(x)) + k,\ x^2\,D + r(x),$$

or by the operators

$$D + g(x),\ x(D + g(x)),\ x^2\,D + r(x), 1.$$

In the first subcase, since both Lie brackets

$$[D+g, x^2 D+r] = 2(x(D+g)+k) + [r' - x^2 g' - 2xg - 2k],$$
$$[x(D+g)+k, x^2 D+r] = (x^2 D+r) + [xr' - r - x^3 g' - x^2 g]$$

must lie in the subalgebra, the functions in square brackets must vanish. The first implies that $r(x) = x^2 g(x) + 2kx + l$, and the second requires that $l = 0$. Thus $\mathfrak{g}$ is spanned by the operators

$$D+g,\ x(D+g)+k,\ x^2(D+g)+2kx. \tag{10}$$

In the second subcase, we have similar bracket formulae

$$[D+g, x^2 D+r] = 2x(D+g) + [r' - x^2 g' - 2xg],$$
$$[x(D+g), x^2 D+r] = (x^2 D+r) + [xr' - r - x^3 g' - x^2 g].$$

In this case the functions in square brackets must be constant. This implies that $r(x) = x^2 g(x) + lx + m$. Since 1 is already a spanning element, we can take $m = 0$ without loss of generality. Thus $\mathfrak{g}$ is spanned by the operators

$$D+g,\ x(D+g),\ x^2(D+g)+lx,\ 1. \tag{11}$$

This completes our classification of finite-dimensional Lie subalgebras of **D** under the more restricted notion of equivalence induced solely by the changes of independent variable (2). The four classes of Lie algebras (8), (9), (10), (11) have their obvious counterparts in Theorem 1. We need only use an appropriate rescaling of the field variable, prescribed by (4), to change the differential operator $D+g$ in each of the algebras to a pure derivative D, which is done by choosing $\psi(x) = \exp\{\int^x g(t)\, dt\}$. This completes the proof of Theorem 1.

It would be interesting, and not too difficult, to extend this result to Lie algebras of differential operators in the plane. Lie [7] has classified all finite-dimensional Lie algebras of vector fields in the plane, so the differential operator classification would follow by use of analogous techniques. Miller [8] has some examples corresponding to semi-simple Lie algebras, but does not attempt to do the full classification.

3. Lie-Algebraic Potentials

Now that we know all the possible finite-dimensional Lie algebras of first order differential operators, we are able to solve Levine's problem of which second order differential operators can be written as bilinear expressions. The proof ultimately rests on the solution to the equivalence problem for second order differential operators based on Cartan's equivalence method, which is presented in [5] (although we will not require the full machinery used there). The main fact required here is that every second order differential operator can be transformed into an operator of the form second derivative plus potential,

$$\mathscr{S} = \pm D^2 + w(x), \tag{12}$$

by the complete pseudogroup (2), (4). The differential operator (12) is of elementary Sturm–Liouville type, and of fundamental importance in quantum mechanics, scattering theory, and the theory of the Korteweg–deVries equation. For a periodic potential $w(x)$, the operator (12) is known as Hill's operator.

THEOREM 5. *Let*

$$\mathscr{D} = f(x)\, D^2 + g(x)\, D + h(x) \tag{13}$$

be a second order differential operator, where f, g, h, are analytic functions of the real variable $x \in \mathbb{R}$. Then $\mathscr{D}$ is equivalent to an operator $\mathscr{S} = \bar{D}^2 + w(\bar{x})$ with potential

$$w(\bar{x}) = \frac{8gf' - 3f'^2 - 4g^2}{16f} + h - \frac{1}{2}g' + \frac{1}{4}f''. \tag{14}$$

The change of independent variable (2) *is explicitly given by*

$$\bar{x} = \varphi(x) = \int^x \frac{dy}{\sqrt{f(y)}}, \tag{15}$$

and the scaling factor $\psi(x)$, cf. (4), *is*

$$\psi(x) = \exp\left\{\int^x \chi(y)\, dy\right\}, \qquad \text{where} \quad \chi = \frac{2g - f'}{4f}. \tag{16}$$

Moreover, two differential operators are equivalent if and only if their potentials, as determined by (14), *are translates of each other: $w(\bar{x}) = \tilde{w}(\bar{x} + \delta)$.*

We therefore need only classify all operators of the form (12) which can be written as a bilinear function of the generators $\mathscr{D}_i$ of a finite-dimensional Lie algebra of first order differential operators,

$$\mathscr{S} = \sum_{i,j} c_{ij}\, \mathscr{D}_i \cdot \mathscr{D}_j, \tag{17}$$

where the coefficients c_{ij} are constants. (More generally, we might also add in linear terms in the generators $\mathscr{D}_i$, but since either the multiplication operator 1 is in the Lie algebra or it can be added by making a one-dimensional central extension, this form is not any more general.)

DEFINITION 6. The *potential* $w(x)$ of a second order differential operator is called a *Lie-algebraic potential* if the operator itself is a bilinear function of the generators of a finite-dimensional Lie algebra of first order differential operators.

THEOREM 7. *Every Lie-algebraic potential is equivalent to one of the following potentials under translations and scalings of the independent variable*: $x \to cx + d$.

(a) $w(x)$ *is a solution to a linear, homogeneous, constant coefficient ordinary differential equation, cf.* (6),

(b) $w(x) = kx^{-2} + P(x^2)$,

(c) $w(x) = k \sec^2 x + l \sec x \tan x + P(\cos x)$,

(d) $w(x) = k \operatorname{sech}^2 x + l \operatorname{sech} x \tanh x + P(\cosh x)$,

(e) $w(x) = k\mathscr{P}(x) + l + Q[\mathscr{P}(x)]\, \mathscr{P}'(x)^{-2}$,

where $\mathscr{P}(x)$ *denotes the Weierstrass elliptic function.*

In all cases k, l *denote arbitrary constants,* $P(z)$ *denotes an arbitrary polynomial, and* $Q(z)$ *is an arbitrary quadratic polynomial.*

Note that our classification includes the one-dimensional harmonic oscillator, Mathieu equation, and Morse potential (case (a)), the Lamé equation (case (e)), radial Laplace and harmonic oscillator (case (b)), and the one-soliton or Pöschl–Teller potential (case (d)), as well as several versions of the confluent hypergeometric equation and Bessel's equation. (See Kamke [4, Eq. (2.12), (2.22), (2.25), (2.26), (2.46), (2.87), (2.113), (2.148), (2.153), (2.155), (2.162)(7), (2.273)(14)].) The fact that we recover the defining equations for many of the classical special functions should not be surprising in view of the well-known relationship between special functions and representation theory of Lie groups [8–10]. A significant potential which does not appear among our Lie-algebraic potentials is the Coulomb potential $w(x) = ax^{-1} + bx^{-2}$; cf. [1].

To prove Theorem 7, we need only determine all the potentials w corresponding to some bilinear expression in the generators of one of our Lie algebras. We do this in order, saving the hardest case until last.

Case 1. For the Lie algebra $\mathfrak{g}_{\mathbf{a}}^{(1)}$, the most general bilinear expression (17) is a constant multiple (which we always ignore) of an operator of the form

$$\mathscr{D} = D^2 + g(x)\,D + h(x), \tag{18}$$

where $g(x)$ is any solution to the corresponding ordinary differential equation (6), and $h = \sum c_{ij} f_i f_j$ is a bilinear combination of solutions to the same ordinary differential equation. If $g = 0$, then we are already in the correct form (12), with potential as given by case (a) of Theorem 7. More generally, since the coefficient of D^2 is 1, we do not need to effect a change of independent variable, but can just apply a suitable transformation of type (4) to eliminate the coefficient g of D. According to (14), the potential corresponding to the operator (18) is

$$w(x) = h - \tfrac{1}{2}g' - \tfrac{1}{4}g^2,$$

which is also a bilinear expression in terms of the basis f_i of the ordinary differential equation (6). However, an elementary result states that any bilinear function of the solutions of a linear, constant coefficient, homogeneous ordinary differential equation is the solution to a (higher order) linear, constant coefficient, homogeneous ordinary differential equation. Therefore, even the operators (18) with $g \neq 0$ can be transformed into an operator with Lie-algebraic potential of type (a).

Case 2. For Lie algebras of type (c) in the classification theorem, we can safely ignore the cases $\mathfrak{g}_{-1,k}^{(2)}$ by replacing them by their central extension $\mathfrak{g}_0^{(2)}$. For the Lie algebra $\mathfrak{g}_n^{(2)}$, the bilinear expression (17) has the form

$$\mathscr{D} = f_2(x)\,D^2 + g_{n+1}(x)\,D + h_{2n}(x), \tag{19}$$

where the coefficient functions f_2, g_{n+1}, h_{2n} are polynomials whose degrees are indicated by their subscripts. Substituting into (14), we find that, as a function of x, the potential is given by a rational function of the form

$$w(x) = \frac{r_{2n+2}(x)}{f_2(x)},$$

where r_{2n+2} is another polynomial of degree $2n+2$, which depends on the previous polynomials. However, we also need to rewrite the formula for $w(x)$ in terms of the new independent variable $\bar{x} = \varphi(x)$, which is obtained by inverting the integral (15) with $f = f_2$ a non-zero quadratic polynomial. There are four subcases. If f_2 is a constant, then $x = \bar{x}$, and the potential is a polynomial, so we are back in the type (a) potentials. If f_2 is a linear function of x, then $x = \bar{x}^2 + k$, which leads to potentials of type (b). If f_2 is a perfect square, then, up to translation and scaling, $x = e^{\bar{x}}$, which leads to potentials of type (a). Finally, if f_2 is a general quadratic polynomial, then, again up to translation and scaling, $x = \cos(\bar{x})$, or $x = \cosh(\bar{x})$, depending on the sign of the discriminant of f_2, which leads to potentials of types (c) and (d).

Case 3. For the Lie algebra $\mathfrak{g}_{k,1}^{(3)}$, the most general bilinear expression (17) is

$$\mathscr{D} = f_4(x)\, D^2 + g_3(x)\, D + h_2(x), \tag{20}$$

where the coefficients are polynomials of the form

$$f_4(x) = ax^4 + bx^3 + cx^2 + dx + e,$$
$$g_3(x) = 2(2k+1)ax^3 + b'x^2 + c'x + d',$$
$$h_2(x) = 2k(2k+1)ax^2 + 2k(b' - (2k+1)b)x + c''.$$

Substituting into (14), we find that, as a function of x, the potential is given by a rational function of the form

$$w(x) = \frac{r_4(x)}{f_4(x)},$$

where r_4 is another quartic polynomial. (Some of the terms cancel due to the form of the generators of the Lie algebra.) We must re-express w in terms of the new independent variable $\bar{x} = \varphi(x)$, which is found by inverting the integral (15) with $f = f_4$ a non-zero quartic polynomial. There are various subcases, depending on the geometric configuration of the roots of f. If f is of degree 2 or less, or has a repeated root, then the integral can be done in terms of elementary functions, and leads to special versions of one of the cases (a)–(d). Otherwise, the integral in an elliptic integral, whose inverse $x = \chi(\bar{x})$ can be written in terms of the Weierstrass elliptic $\mathscr{P}$-function; this leads to the final case (e). In fact, if we view the polynomial $f_4(z)$ on the Riemann sphere, $\mathbb{P}^1$, then the class of the corresponding Lie-algebraic potentials is uniquely determined by the structure of the roots of f_4: a quadruple root leads to a polynomial potential, which is in case (a), a triple root leads to a potential of type (b), two double roots lead to an exponential, type (a), one double and two simple roots lead to types (c) and (d), while four simple roots lead to potentials of type (e).

Note that a given Lie-algebraic potential might correspond to more than one Lie algebra on our list. We illustrate this result by displaying different Lie-algebraic realizations of three important potentials.

EXAMPLE 8. For the Morse potential,

$$w(x) = ae^{-2x} + be^{-x} + c, \tag{21}$$

there are three possible realizations. For the Lie algebras of class 1, the representation is trivial. We set

$$\mathscr{D}_1 = D, \qquad \mathscr{D}_2 = e^{-2x}, \qquad \mathscr{D}_3 = e^{-x},$$

which generate the Lie algebra $\mathfrak{g}_{\mathbf{a}}^{(1)}$, $\mathbf{a} = (3, 2)$, corresponding to the ordinary differential equation

$$y'' + 3y' + 2y = 0.$$

Clearly,

$$\mathscr{S} = \mathscr{D}_1^2 + a\mathscr{D}_2 + b\mathscr{D}_3 + c,$$

and no change of variables (2) is required.

For the Lie algebras of class 2, we take

$$\mathscr{D}_1 = x\,D, \qquad \mathscr{D}_2 = D, \tag{22}$$

which generate the Lie algebra $\mathfrak{g}_{-1,0}^{(2)}$, with central extension $\mathfrak{g}_0^{(2)}$. Consider the operator

$$\mathscr{D} = x^2 D^2 + (\alpha x + \beta)\,D + \gamma = \mathscr{D}_1^2 + (\alpha - 1)\mathscr{D}_1 + \beta\mathscr{D}_2 + \gamma. \tag{23}$$

Under the change of variables of Theorem 5, which is given by

$$\bar{x} = \log x, \qquad \psi(x) = |x|^{(\alpha-1)/2} e^{-\beta x/2},$$

the operator $\mathscr{D}$ is transformed into an operator with potential

$$w(\bar{x}) = -\tfrac{1}{4}\beta^2 e^{-2\bar{x}} + \beta(1 - \tfrac{1}{2}\alpha)e^{-\bar{x}} + \gamma - \tfrac{1}{4}(\alpha - 1)^2.$$

Thus to get the Morse potential (21), we set

$$a = -\tfrac{1}{4}\beta^2, \qquad b = \beta(1 - \tfrac{1}{2}\alpha), \qquad c = \gamma - \tfrac{1}{4}(\alpha - 1)^2.$$

(If $a > 0$, we just replace $\mathscr{D}$ by $-\mathscr{D}$.) The transformed operators generating the solvable two-dimensional Lie algebra equivalent to $\mathfrak{g}_{-1,0}^{(2)}$ are now

$$\bar{\mathscr{D}}_1 = \bar{D} - \frac{\alpha - 1}{2} - \frac{\beta}{2} e^{-\bar{x}}, \qquad \bar{\mathscr{D}}_2 = e^{-\bar{x}}\,\bar{D} - \frac{\alpha - 1}{2} e^{-\bar{x}} - \frac{\beta}{2} e^{-2\bar{x}},$$

and

$$\mathscr{S} = \bar{D}^2 + w(\bar{x}) = \bar{\mathscr{D}}_1^2 + (\alpha - 1)\bar{\mathscr{D}}_1 + \beta\bar{\mathscr{D}}_2 + \gamma.$$

For the semi-simple Lie algebras $\mathfrak{g}_{k,0}^{(3)}$, we can use the same representation (23), with the additional operator

$$\mathscr{D}_3 = x^2 D + 2kx.$$

Essentially, since $\mathfrak{g}^{(2)}_{-1,0}$ is a subalgebra of $\mathfrak{g}_{k,0}^{(3)}$, the same change of variables goes through, but we have one additional transformed operator

$$\bar{\mathscr{D}}_3 = e^{\bar{x}} \bar{D} - \frac{4k - \alpha + 1}{2} e^{\bar{x}} - \frac{\beta}{2}$$

to generate the copy of $\mathfrak{sl}(2, \mathbb{R})$.

EXAMPLE 9. The radial harmonic oscillator potential,

$$w(x) = ax^{-2} + b + cx^2,$$

is not the solution to a linear constant coefficient ordinary differential equation (unless $a = 0$), so there is no realization using Lie algebras of class 1. For the Lie algebras of class 2, we again take the generators (22) of the Lie algebra $\mathfrak{g}^{(2)}_{-1,0}$, and consider the operator

$$\mathscr{D} = x\, D^2 + (\alpha x + \beta)\, D + \gamma = \mathscr{D}_1 \cdot \mathscr{D}_2 + \alpha \mathscr{D}_1 + \beta \mathscr{D}_2 + \gamma.$$

Under the change of variables of Theorem 5,

$$\bar{x} = 2\sqrt{|x|}, \qquad \psi(x) = |x|^{(2\beta - 1)/4}\, e^{\alpha x/2},$$

the operator $\mathscr{D}$ gets transformed into an operator with potential

$$w(\bar{x}) = a\bar{x}^{-2} + b + c\bar{x}^2,$$

where

$$a = -\beta^2 + 2\beta - \tfrac{3}{4}, \qquad b = \gamma - \tfrac{1}{2}\alpha\beta, \qquad c = -\tfrac{1}{4}\alpha^2.$$

The transformed operators generating the solvable two-dimensional Lie algebra are now

$$\bar{\mathscr{D}}_1 = \frac{1}{2} \bar{x}\, \bar{D} - \frac{2\beta - 1}{4} - \frac{\alpha}{8} \bar{x}^2, \qquad \bar{\mathscr{D}}_2 = 2\bar{x}^{-1}\, \bar{D} - \frac{\alpha}{2} - (2\beta - 1)\bar{x}^{-2}.$$

We can thus express the radial harmonic operator as

$$\mathscr{S} = \bar{D}^2 + w(\bar{x}) = \bar{\mathscr{D}}_1 \cdot \bar{\mathscr{D}}_2 + \alpha \bar{\mathscr{D}}_1 + \beta \bar{\mathscr{D}}_2 + \gamma.$$

For the semi-simple Lie algebras $\mathfrak{g}_{k,0}^{(3)}$, we can use the same representation, with the additional operator

$$\mathscr{D}_3 = x^2 D + 2kx.$$

As above, since $\mathfrak{g}_{-1,0}^{(2)}$ is a subalgebra of $\mathfrak{g}_{k,0}^{(3)}$, the same change of variables goes through, but we have one additional transformed operator,

$$\bar{\mathscr{D}}_3 = \frac{\bar{x}^3}{8}\bar{D} - \frac{\alpha}{32}\bar{x}^4 + \frac{(8k-2\beta+1)}{16}\bar{x}^2,$$

to generate the copy of $\mathfrak{sl}(2, \mathbb{R})$.

EXAMPLE 10. For the one-soliton (Pöschl–Teller) potential,

$$w(x) = a\,\mathrm{sech}^2\, x + b,$$

we again have only Lie algebras of classes 2 and 3. For class 2 we again take the Lie algebra $\mathfrak{g}_{-1,0}^{(2)}$ (22), and consider the operator

$$\mathscr{D} = (x^2+1)\, D^2 + \alpha x\, D + \beta = \mathscr{D}_1^2 + \mathscr{D}_2^2 + (\alpha - 1)\mathscr{D}_1 + \beta.$$

The change of variables of Theorem 7 is

$$x = \sinh \bar{x}, \qquad \psi(x) = (x^2+1)^{\alpha/4} e^{-x/4},$$

and the operator $\mathscr{D}$ gets transformed into the operator with potential

$$w(\bar{x}) = a\,\mathrm{sech}^2\, \bar{x} + b,$$

where

$$a = -\tfrac{1}{4}\alpha^2 + \alpha - \tfrac{3}{4}, \qquad b = \beta - \tfrac{1}{2}\alpha + \tfrac{1}{2}.$$

The transformed operators generating the solvable two-dimensional Lie algebra are now

$$\bar{\mathscr{D}}_1 = \sinh \bar{x}\, \bar{\mathscr{D}}_2, \qquad \bar{\mathscr{D}}_2 = \mathrm{sech}\, \bar{x}(\bar{D} - \tfrac{1}{2}\alpha \sinh \bar{x}) - \tfrac{1}{4},$$

and

$$\mathscr{S} = \bar{D}^2 + w(\bar{x}) = \bar{\mathscr{D}}_1^2 + \bar{\mathscr{D}}_2^2 + (\alpha - 1)\bar{\mathscr{D}}_1 + \beta.$$

We can also realize it using the semi-simple Lie algebra $\mathfrak{g}_{k,0}^{(3)}$, with the additional operator

$$\mathscr{D}_3 = x^2 D + 2kx,$$

which transforms to

$$\bar{\mathscr{D}}_3 = \sinh^2 \bar{x}\ \bar{\mathscr{D}}_2 + 2k \sinh \bar{x}.$$

Clearly, we are far from exhausting the range of examples provided in Theorem 7, but these should serve to illustrate the basic techniques used to construct the Lie algebra of first order differential operators corresponding to any Lie-algebraic potential on our list.

ACKNOWLEDGMENTS

We thank Artemio González-López and Willard Miller, Jr., for helpful comments on an earlier draft of this paper.

Note added in proof. See a recent paper of A. V. Turbiner, *Commun. Math. Phys.* **118** (1988), 467, for related results.

The classification of Lie algebras of differential operators on $\mathbb{C}^2$ has now been completed by the authors and A. González-López.

REFERENCES

1. Y. ALHASSID, J. ENGEL, AND J. WU, Algebraic approach to the scattering matrix, *Phys. Rev. Lett.* **53** (1984), 17–20.
2. C. D. H. CHISHOLM, "Group Theoretical Techniques in Quantum Chemistry," Academic Press, New York, 1976.
3. N. JACOBSON, "Lie Algebras," Interscience, New York, 1962.
4. E. KAMKE, "Differentialgleichungen Lösungsmethoden und Lösungen," Chelsea, New York, 1974.
5. N. KAMRAN AND P. J. OLVER, The equivalence problem for differential operators, *SIAM J. Math. Anal.* **20** (1989), 1172–1185.
6. R. D. LEVINE, Lie algebraic approach to molecular structure and dynamics, *in* "Mathematical Frontiers in Computational Chemical Physics" (D. G. Truhlar, Ed.), IMA Volumes in Mathematics and Its Applications, Vol. 15, Springer-Verlag, New York, 1988, pp. 245–261.
7. S. LIE, Theorie der Transformationsgruppen, *Math. Ann.* **16** (1880), 441–528; also "Gesammelte Abhandlungen," Vol. 6, pp. 1–94, Teubner, Leipzig, 1927; see M. ACKERMAN AND R. HERMANN, "Sophus Lie's 1880 Transformation Group Paper," Math. Sci. Press, Brookline, MA, 1975, for an English translation.
8. W. MILLER, JR., "Lie Theory and Special Functions," Academic Press, New York, 1968.
9. J. PATERA, R. T. SHARP, P. WINTERNITZ, AND H. ZASSENHAUS, Invariants of real low dimensional Lie algebras, *J. Math. Phys.* **17** (1976), 986–994.
10. N. J. VILENKIN, "Special Functions and the Theory of Group Representations," Amer. Math. Soc., Providence, RI, 1968.