A Quasi-Exactly Solvable Travel Guide

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Early explorers of the new continent of quantum mechanics soon discovered the vast regions inhabited by the followers of the group theoretical teachings, or "Gruppenpesten" as they came to be called. Within these regions, a more remote kingdom, first tentatively contacted around 40 years ago, and now the focus of active exploration, is populated by the practitioners of the mystical art of hidden symmetry groups, applied to to quantum mechanical systems. In this brief overview, I will concentrate on two particularly active provinces in this new land — the Lie algebraic and so-called "quasi-exactly solvable" principalities. This "travel guide" is not meant to be complete, or impartial, but will, I hope, motivate the casual tourist to immediately book an extended expedition to these scenic lands. A more comprehensive guide book, including many additional sights and physical applications not discussed here, was recently written by Ushveridze, [69]. Shorter, useful overviews can be found in the review chapter by Turbiner, [65], and our survey paper [32]. I should also mention that the allied principality of spectrum generating algebras is surveyed in [1,36]. Finally, I must acknowledge the essential contributions provided by my fellow explorers, Niky Kamran and Artemio González-López. Without their tireless efforts, inspiration, and devotion to detail, much of this newly discovered territory would still be just a blank on the map.

I shall begin with a brief description of the basic customs of these two principalities. A differential operator or Hamiltonian $\mathcal{H}$ is is said to be *Lie algebraic* if it lies in the universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{g}$, which is spanned by first order differential operators

$$J^a = \sum_{i=1}^{d} \xi^{ai}(x) \frac{\partial}{\partial x^i} + \eta^a(x), \qquad a = 1, \ldots, r. \tag{1}$$

In particular, a second-order differential operator is Lie algebraic if it can be written as a constant coefficient quadratic combination

$$\mathcal{H} = \sum_{a,b} c_{ab} J^a J^b + \sum_a c_a J^a + c_0. \tag{2}$$

Note that if $J \in \mathfrak{g}$, then the commutator $[J, \mathcal{H}]$, while still of the same Lie algebraic form (2), is not in general a multiple of $\mathcal{H}$, so that $\mathfrak{g}$ is a "hidden symmetry algebra" of the Hamiltonian.

A Lie algebra of differential operators $\mathfrak{g}$ is called *quasi-exactly solvable* if it possesses a finite-dimensional representation space $\mathcal{N} \subset C^\infty$ consisting of smooth functions; this means that if $\psi \in \mathcal{N}$ and $J^a \in \mathfrak{g}$, then $J^a \psi \in \mathcal{N}$. A differential operator $\mathcal{H}$ is called *quasi-exactly solvable* if it lies in the universal enveloping algebra of a quasi-exactly solvable Lie algebra of differential operators. Clearly, the module $\mathcal{N}$ is an invariant space for the Hamiltonian $\mathcal{H}$, i.e., $\mathcal{H}(\mathcal{N}) \subset \mathcal{N}$, and hence $\mathcal{H}$ restricts to a linear (matrix) operator on $\mathcal{N}$. We will call the eigenvalues and corresponding eigenfunctions for the restriction $\mathcal{H} \,|\, \mathcal{N}$ *algebraic* since they can be computed by linear algebra. If the algebraic eigenfunctions are normalizable, meaning that they lie in the appropriate Hilbert space, e.g., L^2, then the corresponding "algebraic eigenvalues" provide part of the point spectrum of the differential operator.

Example. The simplest example of a quasi-exactly solvable operator that is not exactly solvable is the one-dimensional Schrödinger operator:

$$\mathcal{H} = -D_x^2 + V(x) \tag{3}$$

with the anharmonic oscillator potential

$$V(x) = \nu^2 x^6 + \mu\nu x^4 + [\mu^2 - (4n+3)\nu]x^2. \tag{4}$$

Here μ, ν is an arbitrary real constants, and $n \geq 0$ is a nonnegative integer. This Hamiltonian can be written in Lie algebraic form

$$\mathcal{H} = -J^0 J^- + 2\nu J^+ + 2\mu J^0 - (n+1)J^- + (2n+1)\mu \tag{5}$$

with respect to the Lie algebra $\mathfrak{g}_n \simeq \mathfrak{sl}(2,\mathbb{R})$ which is spanned by the first order differential operators

$$J^- = x^{-1}D_x + \nu x^2 + \mu, \quad J^0 = xD_x + \nu x^4 + \mu x^2 - n, \quad J^+ = x^3 D_x + \nu x^6 + \mu x^4 - 2nx^2.$$

This Lie algebra admits the $(n+1)$-dimensional representation space $\mathcal{N}$ spanned by the functions

$$\psi_k(x) = x^{2k} \exp\left[-\tfrac{1}{4}\nu x^4 - \tfrac{1}{2}\mu x^2\right], \qquad 0 \leq k \leq n.$$

Therefore, the Hamiltonian (3) maps $\mathcal{N}$ to itself; indeed, as one can readily check,

$$\mathcal{H}\psi_k = -2k(2k-1)\psi_{k-1} + \mu(4k+1)\psi_k + 4\nu(k-n)\psi_{k+1}.$$

Thus, the restriction $\mathcal{H}\,|\,\mathcal{N}$ reduces the Schrödinger equation $\mathcal{H}\psi = \lambda\psi$ to a tridiagonal matrix eigenvalue problem. This enables us to determine $n+1$ eigenvalues $\lambda_0, \ldots, \lambda_n$ and corresponding eigenfunctions $\psi_0(x), \ldots, \psi_n(x) \in N$ using purely linear algebraic methods. Note finally that the resulting algebraic eigenfunctions will be L^2 normalizable provided either $\nu > 0$, or $\nu = 0$ and $\mu > 0$.

A short history of our subject would seem in order at this point in our tour. The study of Lie algebraic and quasi-exactly solvable Hamiltonians has its origins in the work of Goshen and Lipkin, [35], from the late 1950's, and Barut and Bohm, [6], and Dothan, Gell–Mann, and Ne'eman [18,9], in the 1960's. Applications of spectrum generating algebras to scattering theory, and to nuclear and molecular spectroscopy began in the late 1970's with the work of Iachello, Levine, Alhassid, Gürsey, and collaborators, cf. [2–4,36]. By the middle of the 1980's, Shifman, Turbiner, and Ushveridze, [58,60,67,69], had introduced the basic definition of quasi-exactly solvability. My own interest in the subject began with a provocative lecture given by Raphael Levine, [39], at the Institute for Mathematics and its Applications, Minnesota, in 1987. The classification problems raised by Levine seemed ideally suited to the equivalence methods that Niky Kamran and I were developing at that time. A couple of years later, we had the good fortune to also enlist Artemio González-López in this enterprise, and the resulting collaboration proved to be extraordinarily fertile, as evidenced by our collective papers, [25,27–33,37].

One of the principal research goals has been to obtain and classify new and physically important examples of quasi-exactly solvable Schrödinger operators. In the one-dimensional case, complete results are known. The real and complex classifications are identical: Up to equivalence, there is essentially just one family of one-dimensional quasi-exactly solvable Lie algebras of first order differential operators, indexed by a single quantum number $n \in \mathbb{N}$. The symmetry algebra can be identified with the spin n representation of a central extension of the unimodular Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, namely the degree n multiplier representation of the projective group action on the space of polynomials of degree at most n.

Theorem. *Every (non-singular) quasi-exactly solvable Lie algebra of differential operators on the line is isomorphic to a sub-algebra of one of the Lie algebras $\mathfrak{g}_n$, $n = 0,1,2,3,\ldots$ spanned by the first order differential operators*

$$\mathfrak{g}_n: \qquad J^- = D_z, \qquad J^0 = zD_z - \tfrac{1}{2}n, \qquad J^+ = z^2 D_z - nz, \qquad 1. \tag{6}$$

As described in [41,27,28], the parameter n can be assigned a Lie algebra cohomological interpretation. The fact that the Lie algebra (6) is quasi-exactly solvable only for non-negative integral values of n is the simplest manifestation of the general phenomenon of quantization of Lie algebra cohomology.

Substituting (6) into the Lie algebraic form (2) of the Hamiltonian, we conclude that all quasi-exactly solvable spectral problems are, in the canonical z coordinates, particular cases of Heun's equation, [52]. However, to date no-one has connected the results obtained through the Lie algebraic approach with the more classical special function theoretic results for Heun's equation. In general, the physical coordinate x is related to the canonical coordinate z in (6) via a change of variables. The associated Schrödinger operators (3) are found by applying a suitable gauge transformation so as to eliminate the first derivative terms. (This is always possible in one dimension, but not in higher dimensions, where the first derivative terms correspond to magnetic fields.) For the anharmonic oscillator potential (4), we use

$$z = x^2, \qquad \mu(z) = \exp\left[-\tfrac{1}{4}\nu x^4 - \tfrac{1}{2}\mu x^2\right].$$

as gauge factor to place the Lie algebraic operator (5) into physical form. (We have omitted an overall factor of $\frac{1}{2}$ on the resulting Lie algebra generators.)

Example. The exactly solvable harmonic oscillator $\mathcal{H} = -D_x^2 + x^2$ can be written in the usual Lie algebraic form $\mathcal{H} = -J^1 J^2 + 1$, where

$$J^1 = D_x - x, \qquad J^2 = D_x + x \tag{7}$$

are the raising and lowering operators, which, along with their commutator $\frac{1}{2}[J^1, J^2] = 1$ generate the Heisenberg algebra. However, this particular Lie algebra of differential operators is *not* quasi-exactly solvable, since it does not admit any finite-dimensional representation space consisting of smooth functions $\psi(x)$.

The harmonic oscillator can, however, be written in quasi-exactly solvable Lie algebraic form, in two distinct ways. The "even" quasi-exactly solvable form is

$$\mathcal{H} = -J^- J^0 + J^- + 2J^0 + 1,$$

where

$$J^- = x^{-1} D_x + 1, \qquad J^0 = x D_x + x^2, \tag{8}$$

generate a two-dimensional nonabelian Lie algebra having finite-dimensional representation space $\mathcal{N}_n$ spanned by the functions $x^{2j} e^{-x^2/2}$, $0 \le j \le n$. The "odd" quasi-exactly solvable form is

$$\mathcal{H} = -J^- J^0 - J^- + 2J^0 + 3,$$

where

$$J^- = x^{-1} D_x - x^{-2} + 1, \qquad J^0 = x D_x + x^2 - 1. \tag{9}$$

The associated finite-dimensional representation space $\widehat{\mathcal{N}}_n$ is spanned by the functions $x^{2j+1} e^{-x^2/2}$, $0 \le j \le n$. Note that, in both cases, the Hamiltonian admits an invariant subspace of arbitrarily large dimension; the corresponding algebraic eigenfunctions are the usual even and odd states for the harmonic oscillator. Both Lie algebras (8) and (9) are mapped under a suitable gauge transformation to the two-dimensional subalgebra spanned by D_z and zD_z of the Lie algebra (6) *for every value of the quantized cohomology parameter* n. In the realm of quasi-exactly solvability, this is the hallmark of an exactly solvable problem — that, in physical coordinates, it has no dependence on the quantized cohomology parameters, and hence admits invariant representation spaces of arbitrarily large dimension.

The complete classification of Lie algebraic and quasi-exactly solvable differential operators in one dimension appears in [60,37]. A wide variety of classical potentials, including the Gendenshtein and Morse potentials, [20], the Pöschl-Teller potentials, [50], the Natanzon potentials, [16], and others can be obtained in this manner. A curious omission is the Coulomb potential; however, as shown by Shifman, [54,56], and generalized by Auberson, [5], these and others can be obtained using the so-called Sturm representation for the Schrödinger equation. See Fushchych and Nikitin, [24], and Zhdanov, [79], for connections with the theory of conditional or nonclassical symmetry methods. A full solution to the normalizability problem for the quasi-exactly solvable operators in one dimension, based on methods from classical invariant theory, was given in [30]. Bender, Dunne, and Moshe, [8], give a striking application of higher order WKB methods, based on the semiclassical approximation, to characterize the border between the algebraic and non-algebraic parts of the spectrum of quasi-exactly solvable problems.

Another class of potentials that admit exact formulae for the eigenfunctions, but are not in the Lie algebraic class of quasi-exactly solvable problems are the multi-soliton or reflectionless potentials. These can be obtained from the one soliton or Pöschl-Teller potential through a series of Darboux-Crum or Bäcklund transformations, [17], which can also be reinterpreted within Witten's supersymmetric quantum mechanics, [72]. The Lie algebraic structure of the original potential becomes a Lie algebra of pseudo-differential operators for the transformed potential, although the classification procedure now becomes much more complicated. Details of this construction and applications appear in Shifman, [55], and Ushveridze, [69]. Extensions to periodic finite gap potentials were applied to quantum spin systems by Zaslavskii and Ulyanov, [78].

Dutra and Filho, [19], show how to use a realization of the hidden symmetry algebra $\mathfrak{sl}(2,\mathbb{R})$ by higher order differential operators to obtain polynomial potentials of degree 10. The general classification of quasi-exactly solvable Lie algebras of higher order differential operators is wide open, although apparently very difficult. For example, the classification of abelian Lie algebras of differential operators lies in the domain of classical results due to Burchnall and Chaundy, [12–14], and requires the use of general theta functions on abelian varieties to effect a complete solution! The solution includes a classification of all multi-soliton and finite gap potentials, and has played an important role in the study of the Korteweg-deVries equation, [46,38,71]. The two-dimensional nonabelian algebras give rise to Painlevé transcendent potentials and similarity reductions of soliton equations, [70].

A classification of all quasi-exactly solvable Lie algebras of first order differential operators in two complex dimensions was found in [27,28], based on Lie's classification of transformation groups in the plane, [40]; this was extended to the real plane in [29,33]. In every example, the condition of quasi-exact solvability or possessing a finite-dimensional module requires that *all* of the cohomology parameters which enter into our classification of Lie algebras of first order differential operators must assume only a discrete (integral, half-integral, etc.) set of values. In the maximal cases, the phenomenon of the "quantization of cohomology" has been given an algebro-geometric interpretation, [25], based on the Borel–Weil–Bott theory of line bundles on algebraic surfaces. See also [15] for some results on abelian algebras of higher order operators in two dimensions and [26] for applications of a multi-dimensional generalization of the Darboux transformation.

In higher dimensions, complications arise because not every second order differential operator is equivalent, under a gauge transformation, to a Schrödinger operator, and one must impose additional "closure conditions" in order to ensure that there are no additional magnetic terms. In [31,33,58,68,74], new examples of quasi-exactly solvable Schrödinger operators on both flat and curved surfaces, were found, although a complete classification of all planar quasi-exactly solvable operators appears to be very difficult. Turbiner conjectured that all flat space quasi-exactly solvable Hamiltonians admit separation of variables. A proof of Turbiner's conjecture in the case of elliptic Hamiltonians, for which the Lie algebra arises from a planar imprimitive group action, was recently found by R. Milson, [42], who also constructed a hyperbolic counterexample. Zaslavskii, [76,77], and Kamran and Milson (personal communication) have studied quasi-exactly solvable operators with magnetic terms.

Quasi-exactly solvable problems in higher dimensions are even less well catalogued. A major difficulty is that there is no classification of general transformation groups acting in three or more dimensions, so complete results are not available. My student, D. Richter, [51], has recently classified Lie algebras of first order differential operators associated with a large class of transitive, primitive Lie group actions in higher dimensions, and proved that quantization of cohomology also occurs in all cases. Examples of quasi-exactly solvable many body problems can be found in [69] and in [64,53,44]. Extensions to matrix-valued and supersymmetric differential operators can be found in work of Shifman, Turbiner and Post [58,63,65,47], Brihaye and Kosinski, [11], and Finkel, González-López, Kamran, and Rodríguez, [22,23]. Extensions to finite difference operators are discussed by Turbiner, [63,65].

Closely related is recent work of Turbiner, Post, and van den Hijligenberg, [48,49,63], classifying differential operators with polynomial invariant subspaces. Turbiner, [61–63], made the remarkable discovery that all known families of classical orthogonal polynomials arise as eigenfunctions to (higher order) quasi-exactly solvable differential operators. Coupled with the normalizability results in [30], this leads to a complete solution to the "Bochner problem" in the second order case, namely to characterize linear differential operators having an infinite sequence of orthogonal polynomial solutions to its eigenvalue problem; see also [65]. Recently, Bender and Dunne, [7], have shown that the quasi-exactly solvable wave functions appear as the generating functions of orthogonal polynomials, whose zeros are the associated algebraic eigenvalues. In addition, the Lie algebraic approach can be effectively used to understand orthogonal polynomials in multi-dimensional situations as well. See [21,75], for additional results in this area.

Finally, I should mention that there are a wide variety of intriguing and potentially significant applications to other contemporary physical theories. Connections with conformal field theory have been developed in [34,45,57], Applications to quantum spin systems and tunneling phenomena are discussed by Ulyanov and Zaslavskii, [73,66,78]. Braibant and Brihaye, [10], discuss applications to sphaleron stability in the $1+1$ abelian Higgs model. Applications to the Gaudin model arising in electromagnetism and the Bethe ansatz method appear in Ushveridze, [69]. Again, Ushveridze's book, [69], should be consulted for a more complete guide to the further reaches of the realm.

This concludes our brief tour of the intriguing dominion of quasi-exactly solvability. The reader is warmly invited to return to explore the region in more depth, and thereby help expand the already wide boundaries of this fascinating land.

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References

[1] B.G. Adams, J. Čížek and J. Paldus, "Lie algebraic methods and their applications to simple quantum systems", *Adv. Quantum Chem.* **19**, 1–85 (1988).

[2] Y. Alhassid, F. Gürsey and F. Iachello, "Group theory approach to scattering", *Ann. Phys.* **148**, 346–380 (1983).

[3] Y. Alhassid, F. Gürsey and F. Iachello, "Group theory approach to scattering II. The euclidean connection", *Ann. Phys.* **167**, 181–200 (1986).

[4] Y. Alhassid, F. Gürsey and F. Iachello, "Group theory approach to scattering III. Realistic models", *Ann. Phys.* **173**, 68–87 (1987).

[5] G. Auberson, "On a class of spectral problems admitting partial algebraization", in *Rigorous Methods in Particle Physics*, S. Ciulli, F. Scheck, W. Thirring, A. Martin, eds., Springer Tracts Modern Phys., vol. 119, Springer–Verlag, Berlin, 1990.

[6] A.O. Barut and A. Böhm, "Dynamical groups and mass formula", *Phys. Rev. B* **139**, 1107–1112 (1965).

[7] C.M. Bender and G.V. Dunne, "Quasi-exactly solvable systems and orthogonal polynomials", *J. Math.Phys.* **37**, 6–11 (1996).

[8] C.M. Bender, G.V. Dunne and M. Moshe, "Semiclassical analysis of quasi-exact solvability", preprint, Washington University, 1996.

[9] A. Bohm and Y. Ne'eman, "Dynamical groups and spectrum generating algebras", in *Dynamical Groups and Spectrum Generating Algebras*, Bohm, A., Ne'eman, Y., Barut, A.O., eds., World Scientific, Singapore, 1988, pp. 3–68.

[10] S. Braibant and Y. Brihaye, "Quasi exactly-solvable system and sphaleron stability", *J. Math. Phys.* **34**, 2107–2114 (1993).

[11] Y. Brihaye and P. Kosinski, "Quasi exactly solvable 2×2 matrix equations", *J. Math. Phys.* **35**, 3089–3098 (1994).

[12] J.L. Burchnall and T.W. Chaundy, "Commutative ordinary differential operators", *Proc. London Math. Soc.* **21**, 420–440 (1923).

[13] J.L. Burchnall and T.W. Chaundy, "Commutative ordinary differential operators", *Proc. Roy. Soc. London A* **118**, 557–583 (1928).

[14] J.L. Burchnall and T.W. Chaundy, "Commutative ordinary differential operators II", *Proc. Roy. Soc. London A* **134**, 471–485 (1932).

[15] O.A. Chalykh and A.P. Veselov, "Commutative rings of partial differential operators and Lie algebras", *Commun. Math. Phys.* **126**, 597–611 (1990).

[16] P. Cordero and S. Salamó, "Algebraic solution for Natanzon hypergeometric potentials", *J. Math. Phys.* **35**, 3301–3307 (1994).

[17] M.M. Crum, "Associated Sturm–Liouville systems", *Quart. J. Math.* **6**, 121–127 (1955).

[18] Y. Dothan, M. Gell–Mann and Y. Ne'eman, "Series of hadron energy levels as representations of non-compact groups", *Phys. Lett.* **17**, 148–151 (1965).

[19] A. de S. Dutra and H.B. Filho, "Dynamical algebra of quasi-exactly-solvable potentials", *Phys. Rev. A* **44**, 4721–4724 (1991).

[20] M.J. Englefield and C. Quesne, "Dynamic potential algebras for Gendenshtein and Morse potentials", *J. Phys. A* **24**, 3557–3574 (1991).

[21] F. Finkel, A. González-López and M.A. Rodríguez, "Quasi-exactly solvable potentials on the line and orthogonal polynomials", preprint, Universidad Complutense, Madrid, 1996.

[22] F. Finkel, A. González-López and M.A. Rodríguez, "Quasi-exactly solvable spin 1/2 Schrödinger operators", preprint, Universidad Complutense, Madrid, 1995.

[23] F. Finkel and N. Kamran, "On the equivalence of matrix differential operators to Schrödinger form", *Nonlinear Math. Phys.*, to appear.

[24] W. I. Fushchych and A.G. Nikitin, "Higher symmetries and exact solutions of linear and nonlinear Schrödinger equations", preprint, Arnold Sommerfeld Inst., ASI-TPA/9/96, 1996.

[25] A. González–López, J. Hurtubise, N. Kamran and P.J. Olver, "Quantification de la cohomologie des algèbres de Lie de champs de vecteurs et fibrés en droites sur des surfaces complexes compactes", *Comptes Rendus Acad. Sci. Paris, Série I,* **316**, 1307–1312 (1993).

[26] A. González–López and N. Kamran, "The multi-dimensional Darboux transformation", preprint, Universidad Complutense, Madrid, 1996.

[27] A. González–López, N. Kamran and P.J. Olver, "Quasi–exactly solvable Lie algebras of first order differential operators in two complex variables", *J. Phys. A* **24**, 3995–4008 (1991).

[28] A. González–López, N. Kamran and P.J. Olver, "Lie algebras of differential operators in two complex variables", *American J. Math.* **114**, 1163–1185 (1992).

[29] A. González–López, N. Kamran and P.J. Olver, "Lie algebras of vector fields in the real plane", *Proc. London Math. Soc.* **64**, 339–368 (1992).

[30] A. González–López, N. Kamran and P.J. Olver, "Normalizability of one-dimensional quasi–exactly solvable Schrödinger operators", *Commun. Math. Phys.* **153**, 117–146 (1993).

[31] A. González–López, N. Kamran and P.J. Olver, "New quasi–exactly solvable Hamiltonians in two dimensions", *Commun. Math. Phys.* **159**, 503–537 (1994).

[32] A. González–López, N. Kamran and P.J. Olver, "Quasi–exact solvability", *Contemp. Math.* **160**, 113–140 (1994).

[33] A. González–López, N. Kamran and P.J. Olver, "Real Lie algebras of differential operators, and quasi-exactly solvable potentials", *Phil. Trans. Roy. Soc. London A* **354**, 1165–1193 (1996).

[34] A.S. Gorskiĭ, "Relationship between exactly solvable and quasiexactly solvable versions of quantum mechanics with conformal-block equations in 2D theories", *JETP Lett.* **54**, 289–292 (1991).

[35] S. Goshen and H.J. Lipkin, "A simple independent-particle system having collective properties", *Ann. Phys.* **6**, 301–309 (1959).

[36] F. Iachello and R.D. Levine, *Algebraic Theory of Molecules*, Oxford Univ. Press, Oxford, 1995.

[37] N. Kamran and P.J. Olver, "Lie algebras of differential operators and Lie-algebraic potentials", *J. Math. Anal. Appl.* **145**, 342–356 (1990).

[38] I.M. Krichever, "Commutative rings of ordinary linear differential operators", *Func. Anal. Appl.* **12**, 175–185 (1978).

[39] R.D. Levine, "Lie algebraic approach to molecular structure and dynamics", in *Mathematical Frontiers in Computational Chemical Physics*, D.G. Truhlar, ed., IMA Volumes in Mathematics and its Applications, Vol. 15, Springer–Verlag, New York, 1988, pp. 245–261.

[40] S. Lie, "Gruppenregister", in *Gesammelte Abhandlungen*, vol. 5, B.G. Teubner, Leipzig, 1924, pp. 767–773.

[41] W. Miller, Jr., *Lie Theory and Special Functions*, Academic Press, New York, 1968.

[42] R. Milson, *Multi-Dimensional Lie Algebraic Operators*, Thesis, McGill University, Montréal, 1995.

[43] R. Milson, "Flat Lie-algebraic systems and separation of variables", preprint, University of Minnesota, 1995.

[44] A. Minzoni, M. Rosenbaum and A.V. Turbiner, "Quasi-exactly-solvable many-body problems", preprint, hep-th/9606092, ICN-UNAM 96-07, Mexico, 1996.

[45] A.Y. Morozov, A.M. Perelomov, A.A. Rosly, M.A. Shifman and A.V. Turbiner, "Quasi-exactly solvable quantal problems: one-dimensional analogue of rational conformal field theories", *Int. J. Mod. Phys. A* **5**, 803–832 (1990).

[46] S.P. Novikov, "The periodic problem for the Korteweg-deVries equation", *Func. Anal. Appl.* **8**, 236–246 (1974).

[47] G. Post, "Classification of graded, finite-dimensional Lie algebras of vector fields", preprint, No. 1178, Universiteit Twente, Netherlands.

[48] G. Post and A.V. Turbiner, "Classification of linear differential operators with an invariant subspace of monomials", *Russian J. Math. Phys.* **3**, 113–122 (1995).

[49] G. Post and N. van den Hijligenberg, "$\mathfrak{gl}(\lambda)$ and differential operators preserving polynomials", *Acta Appl. Math.* **44**, 257–268 (1996).

[50] C. Quesne, "An $\mathfrak{sl}(4,\mathbb{R})$ Lie algebraic treatment of the first family of Pöschl-Teller potentials", *J. Phys. A* **21**, 4487–4500 (1988).

[51] D.A. Richter, "Lie algebras of differential operators obtained from primitive non-reductive actions", preprint, University of Minnesota, 1996.

[52] R. Ronveaux (ed.), *Heun's Differential Equations*, Oxford University Press, Oxford, 1995.

[53] W. Rühl and A.V. Turbiner, "Exact solvability of the Calogero and Sutherland models", *Modern Phys. Lett. A* **10**, 2213–2221 (1995).

[54] M.A. Shifman, "New findings in quantum mechanics (partial algebraization of the spectral problem)", *Int. J. Mod. Phys. A* **4**, 2897–2952 (1989).

[55] M.A. Shifman, "Supersymmetric quantum mechanics and partial algebraization of the spectral problem", *Int. J. Mod. Phys. A* **4**, 3305–3310 (1989).

[56] M.A. Shifman, "Expanding the class of spectral problems with partial algebraization", *Int. J. Mod. Phys. A* **4**, 3311–3318 (1989).

[57] M.A. Shifman, "Quasi-exactly solvable spectral problems and conformal field theory", *Contemp. Math.* **160**, 237–262 (1994).

[58] M.A. Shifman and A.V. Turbiner, "Quantal problems with partial algebraization of the spectrum", *Commun. Math. Phys.* **126**, 347–365 (1989).

[59] Y.F. Smirnov and A.V. Turbiner, "Lie algebraic discretization of differential equations", *Modern Phys. Lett. A* **10**, 1795–1802 (1995).

[60] A.V. Turbiner, "Quasi-exactly solvable problems and $\mathfrak{sl}(2)$ algebra", *Commun. Math. Phys.* **118**, 467–474 (1988).

[61] A.V. Turbiner, "Lie algebras and polynomials in one variable", *J. Phys. A* **25**, L1087–L1093 (1992).

[62] A.V. Turbiner, "On polynomial solutions of differential equations", *J. Math. Phys.* **33**, 3989–3993 (1992).

[63] A.V. Turbiner, "Lie algebras and linear operators with invariant subspaces", *Contemp. Math.* **160**, 263–310 (1994).

[64] A.V. Turbiner, "Hidden algebra of the N-body Calogero problem", *Phys. Lett. B* **320**, 281–286 (1994).

[65] A.V. Turbiner, "Quasi-exactly-solvable differential equations", in *CRC Handbook of Lie Group Analysis of Differential Equations*, vol. 3, N.H. Ibragimov, ed., CRC Press, Boca Raton, Fl., 1996, pp. 329–364.

[66] V.V. Ulyanov and O.B. Zaslavskii, "New methods in the theory of quantum spin systems", *Phys. Rep.* **216**, 179–251 (1992).

[67] A.G. Ushveridze, "Quasi-exactly solvable models in quantum mechanics", *Sov. J. Part. Nucl.* **20**, 504–528 (1989).

[68] A.G. Ushveridze, "Quasi-exactly solvable multi-dimensional Schrödinger equations", *Mod. Phys. Lett. A* **6**, 977–979 (1991).

[69] A.G. Ushveridze, *Quasi-exactly Solvable Models in Quantum Mechanics*, Inst. of Physics Publ., Bristol, England, 1994.

[70] A.P. Veselov and A.B. Shabat, "Dressing chains and spectral theory of the Schrödinger operator", *Func. Anal. Appl.* **27**, 81–96 (1993).

[71] G. Wilson, "Bispectral commutative ordinary differential operators", *J. Reine Angew. Math.* **442**, 177–204 (1993).

[72] E. Witten, "Constraints on supersymmetry breaking", *Nuclear Phys. B* **202**, 253–316 (1982).

[73] O.B. Zaslavskii, "Effective potential for spin-boson systems and quasi-exactly solvable problems", *Phys. Lett. A* **149**, 365–368 (1990).

[74] O.B. Zaslavskii, "Quasi-exactly solvable models from finite-dimensional matrices", *J. Phys. A* **26**, 6563–6574 (1993).

[75] O.B. Zaslavskii, "Two-dimensional quasi-exactly solvable models and classical orthogonal polynomials", *J. Phys. A* **27**, L323–328 (1994).

[76] O.B. Zaslavskii, "Quasi-exactly solvable models with an inhomogeneous magnetic field", *J. Phys. A* **27**, L447–L452 (1994).

[77] O.B. Zaslavskii, "Two-dimensional quasi-exactly solvable models with a magnetic field", *Phys. Lett. A* **190**, 373–376 (1994).

[78] O.B. Zaslavskii and V.V. Ulyanov, "Periodic effective potentials for spin systems and new exact solutions of one-dimensional Schrödinger equation for the energy bands", *Theor. Math. Phys.* **71**, 520–528 (1987).

[79] R.Z. Zhdanov, "Conditional symmetry and the spectrum of the one-dimensional Schrödinger equation", preprint, Arnold Sommerfeld Inst., ASI-TPA/23/95, 1995.