Dispersive Quantization of Linear and Nonlinear Waves

Peter J. Olver

University of Minnesota

http://www.math.umn.edu/∼olver

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—, Dispersive quantization, *Amer. Math. Monthly* **117** (2010) 599–610.

Gong Chen & —, Dispersion of discontinuous periodic waves, *Proc. Roy. Soc. London A* **469** (2012), 20120407.

Gong Chen & —, Numerical simulation of nonlinear dispersive quantization, *Discrete Cont. Dyn. Syst. A* **34** (2013), 991–1008.

Dispersion

Definition. A linear partial differential equation is called dispersive if the different Fourier modes travel unaltered but at different speeds.

Substituting

$$u(t,x) = e^{\,\mathrm{i}\,(k\,x-\omega\,t)}$$

produces the dispersion relation

$$\omega = \omega(k), \qquad\qquad \omega, k \in \mathbb{R}$$

relating frequency ω and wave number k.

Phase velocity: $\quad c_p = \dfrac{\omega(k)}{k}$

Group velocity: $\quad c_g = \dfrac{d\omega}{dk}$ (stationary phase)

A Simple Linear Dispersive Wave Equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}$$

$\Longrightarrow$ linearized Korteweg–deVries equation

Dispersion relation: $\omega = k^3$

Phase velocity: $c_p = \dfrac{\omega}{k} = k^2$

Group velocity: $c_g = \dfrac{d\omega}{dk} = 3\,k^2$

Thus, wave packets (and energy) move *faster* (to the right) than the individual waves.

Linear Dispersion on the Line

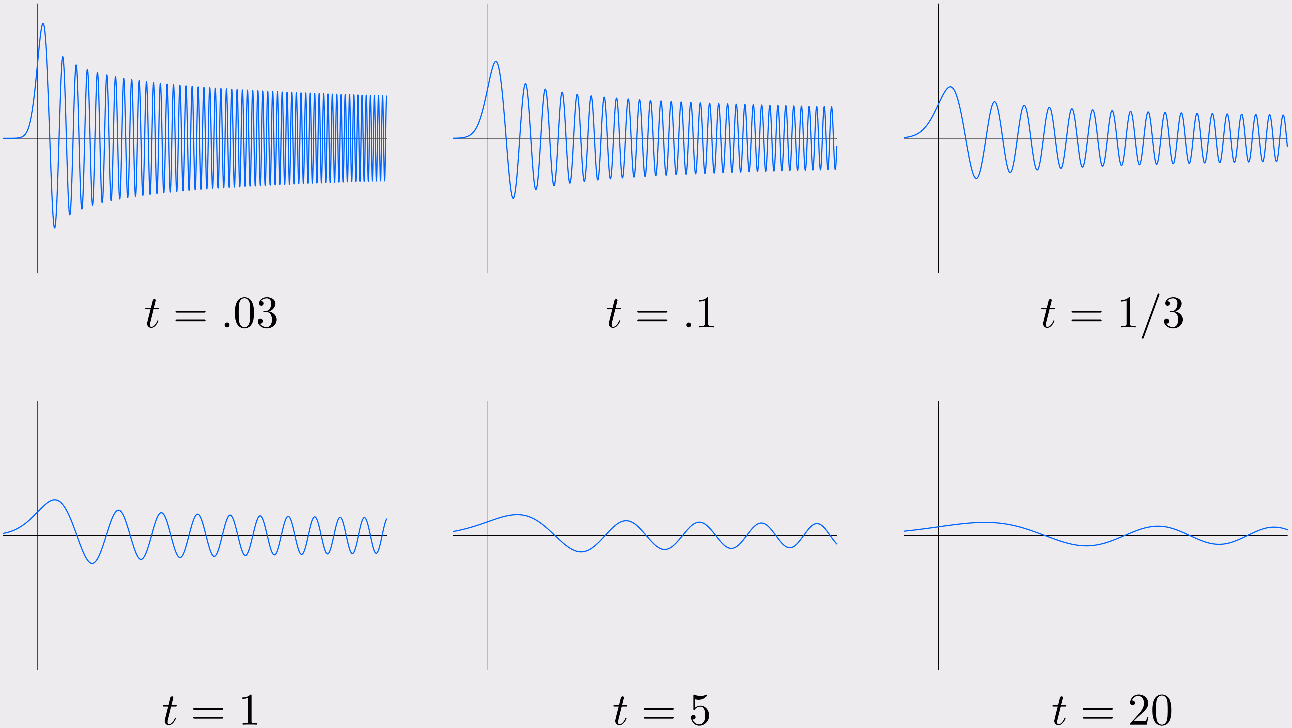
$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} \qquad u(0,x) = f(x)$$

Fourier transform solution:

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(k)\, e^{\,\mathrm{i}\,(k\,x - k^3\,t)}\, dk$$

Fundamental solution $\qquad u(0,x) = \delta(x)$

$$u(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\,\mathrm{i}\,(k\,x - k^3\,t)}\, dk = \frac{1}{\sqrt[3]{3\,t}}\ \mathrm{Ai}\left(-\frac{x}{\sqrt[3]{3\,t}} \right)$$



t = .03
t = .1
t = 1/3
t = 1
t = 5
t = 20

Linear Dispersion on the Line

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} \qquad u(0,x) = f(x)$$

Superposition solution formula:

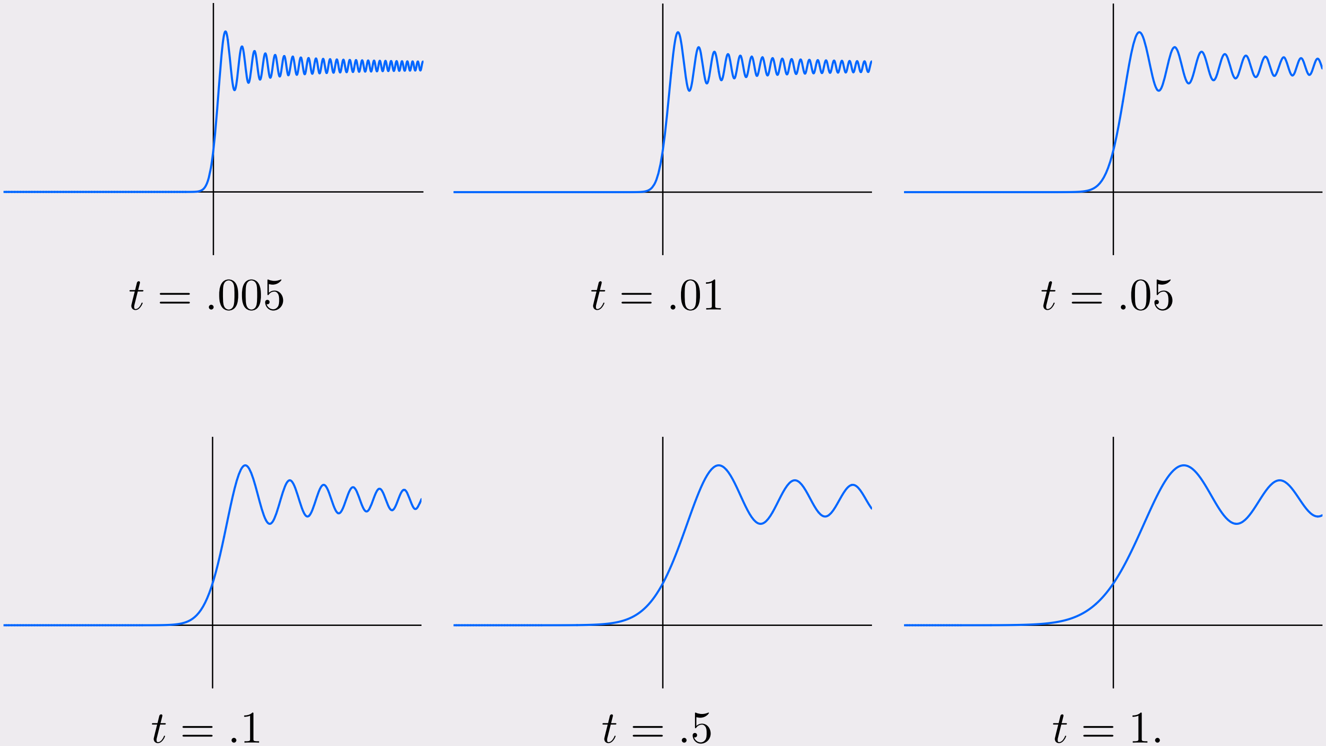
$$u(t,x) = \frac{1}{\sqrt[3]{3t}} \int_{-\infty}^{\infty} f(\xi)\, \mathrm{Ai}\left(\frac{\xi - x}{\sqrt[3]{3t}}\right) d\xi$$

Step function initial data: $\quad u(0,x) = \sigma(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$

$$u(t,x) = \frac{1}{3} - H\left(-\frac{x}{\sqrt[3]{3t}}\right)$$

$$H(z) = \frac{z\,\Gamma\left(\frac{2}{3}\right)\,{}_1F_2\left(\frac{1}{3};\,\frac{2}{3},\,\frac{4}{3};\,\frac{1}{9}z^3\right)}{3^{5/3}\,\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)} - \frac{z^2\,\Gamma\left(\frac{2}{3}\right)\,{}_1F_2\left(\frac{2}{3};\,\frac{4}{3},\,\frac{5}{3};\,\frac{1}{9}z^3\right)}{3^{7/3}\,\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)}$$

$\Longrightarrow$ Mathematica — via Meijer G functions


t = .005
t = .01
t = .05
t = .1
t = .5
t = 1.

Periodic Linear Dispersion

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}$$

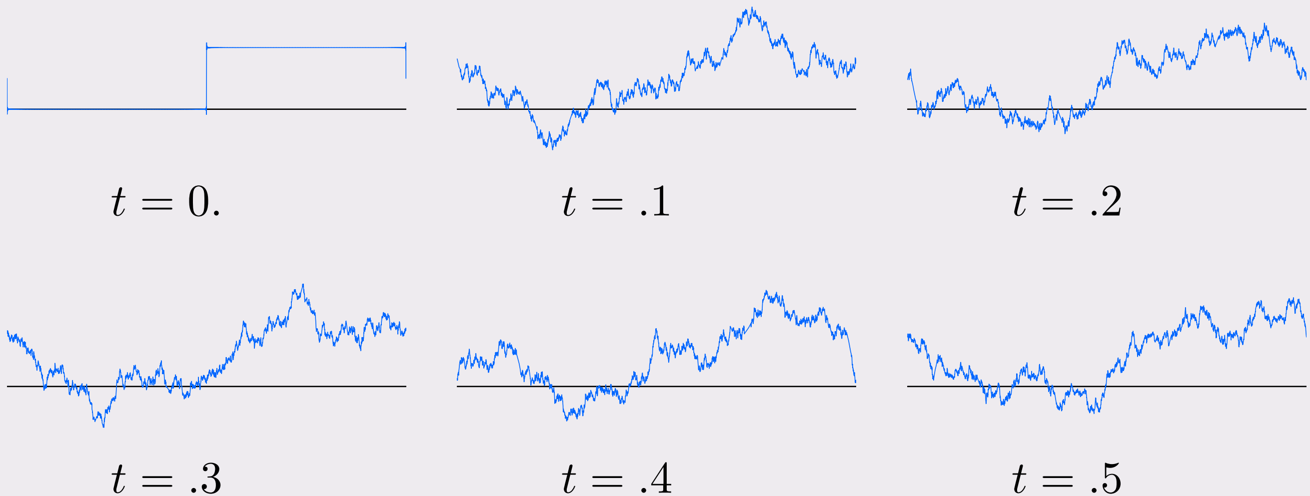
$$u(t,-\pi) = u(t,\pi) \qquad \frac{\partial u}{\partial x}(t,-\pi) = \frac{\partial u}{\partial x}(t,\pi) \qquad \frac{\partial^2 u}{\partial x^2}(t,-\pi) = \frac{\partial^2 u}{\partial x^2}(t,\pi)$$

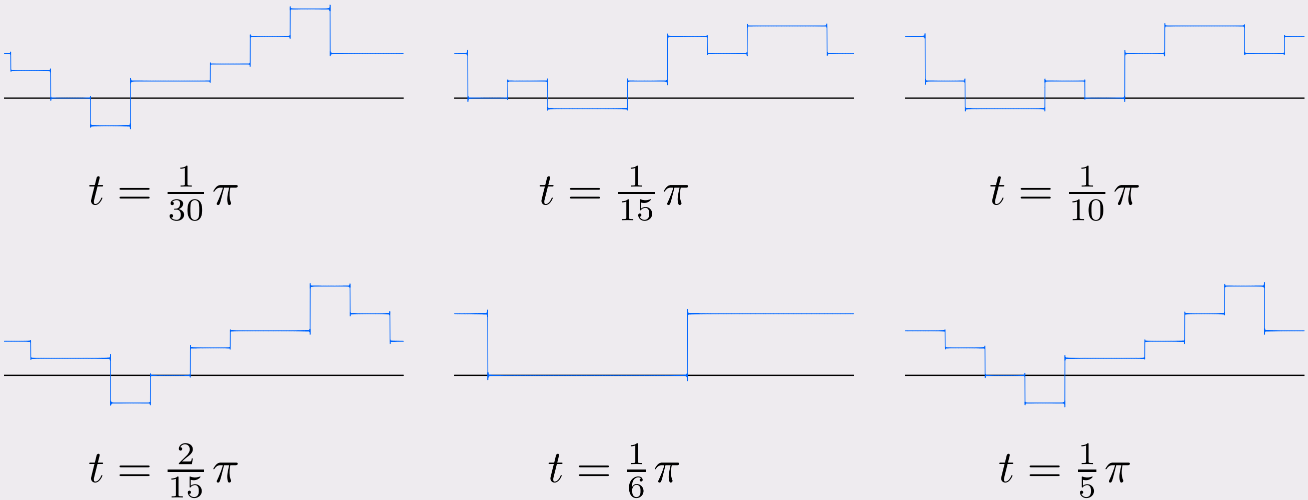
Step function initial data:

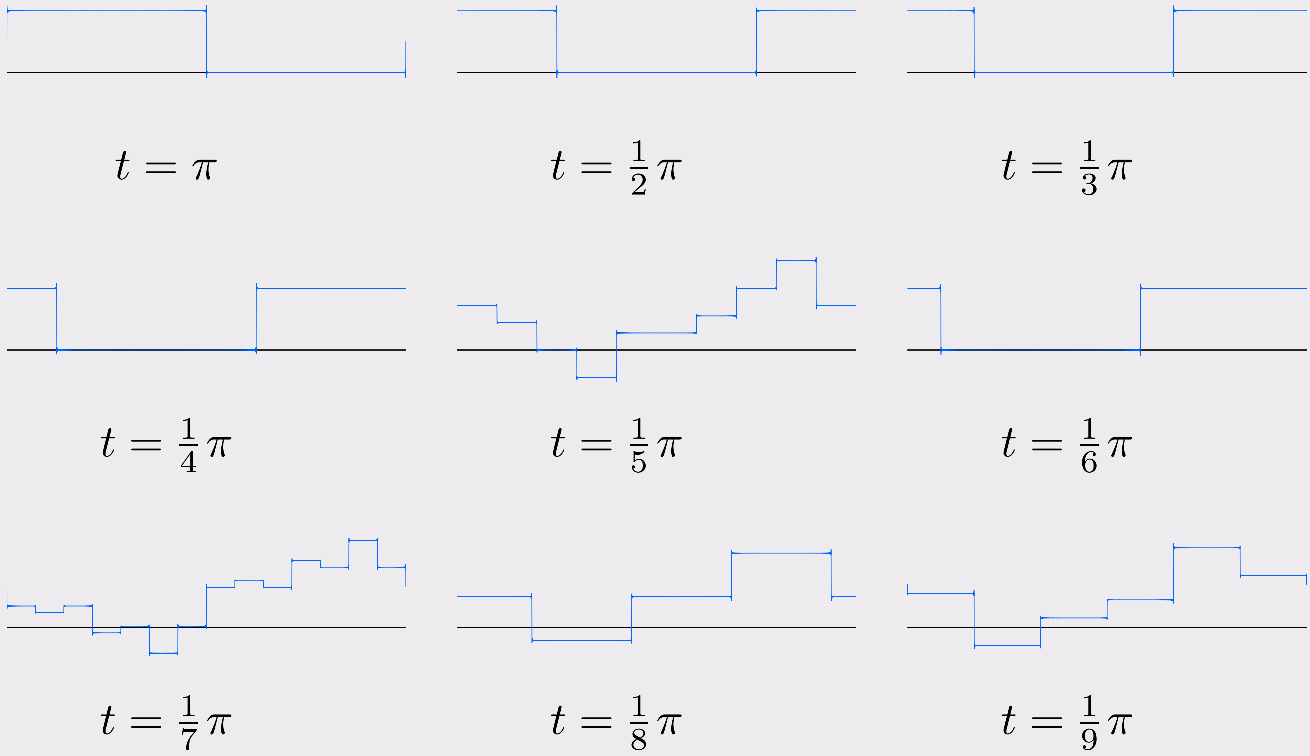
$$u(0,x) = \sigma(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

Fourier series solution formula:

$$u^\star(t,x) \sim \frac{1}{2} + \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\sin(\,(2\,j+1)\,x - (2\,j+1)^3\,t\,)}{2\,j+1}\,.$$


$t = 0.$
$t = .1$
$t = .2$
$t = .3$
$t = .4$
$t = .5$


$t = \frac{1}{30}\pi$
$t = \frac{1}{15}\pi$
$t = \frac{1}{10}\pi$
$t = \frac{2}{15}\pi$
$t = \frac{1}{6}\pi$
$t = \frac{1}{5}\pi$



$t = \pi$
$t = \frac{1}{2}\pi$
$t = \frac{1}{3}\pi$
$t = \frac{1}{4}\pi$
$t = \frac{1}{5}\pi$
$t = \frac{1}{6}\pi$
$t = \frac{1}{7}\pi$
$t = \frac{1}{8}\pi$
$t = \frac{1}{9}\pi$

Theorem. At rational time $t = 2\pi p/q$, the solution $u^\star(t,x)$ is constant on every subinterval $2\pi j/q < x < 2\pi(j+1)/q$. At irrational time $u^\star(t,x)$ is a non-differentiable continuous function.

Lemma.

$$f(x) \;\sim\; \sum_{k=-\infty}^{\infty} c_k \, e^{\,\mathrm{i}\,k\,x}$$

is piecewise constant on intervals $2\pi\,j/q < x < 2\pi\,(j+1)/q$ if and only if

$$\widehat{c}_k = \widehat{c}_l, \quad k \equiv l \not\equiv 0 \mod q, \qquad \widehat{c}_k = 0, \quad 0 \neq k \equiv 0 \mod q.$$

where

$$\widehat{c}_k = \frac{2\pi\,k\,c_k}{\mathrm{i}\,q\,(e^{-2\,\mathrm{i}\,\pi\,k/q} - 1)} \qquad k \not\equiv 0 \mod q.$$

$\Longrightarrow$ DFT

The Fourier coefficients of the solution $u^\star(t,x)$ at rational time $t = 2\pi\, p/q$ are

$$c_k = b_k\, e^{-2\,\pi\,\mathrm{i}\,k^3\,p/q} \qquad (*)$$

where, for the step function initial data,

$$b_k = \begin{cases} -\,\mathrm{i}\,/(\pi\,k), & k \text{ odd}, \\ 1/2, & k = 0, \\ 0, & 0 \neq k \ \text{ even}. \end{cases}$$

Crucial observation: $\boxed{\text{if } k \equiv l \mod q \text{ then } k^3 \equiv l^3 \mod q}$

which implies

$$e^{-2\,\pi\,\mathrm{i}\,k^3\,p/q} = e^{-2\,\pi\,\mathrm{i}\,l^3\,p/q}$$

and hence the Fourier coefficients $(*)$ satisfy the condition in the Lemma. *Q.E.D.*

The Fundamental Solution: $F(0,x) = \delta(x)$

Theorem. At rational time $t = 2\pi p/q$, the fundamental solution $F(t,x)$ is a linear combination of finitely many periodically extended delta functions, based at $2\pi j/q$ for integers $-\frac{1}{2}q < j \leq \frac{1}{2}q$.

Corollary. At rational time, any solution profile $u(2\pi p/q, x)$ to the periodic initial-boundary value problem is a linear combination of $\leq q$ translates of the initial data, namely $f(x + 2\pi j/q)$, and hence its value depends on only finitely many values of the initial data.

★★ The same quantization/fractalization phenomenon appears in any linearly dispersive equation with "integral polynomial" dispersion relation:

$$\omega(k) = \sum_{m=0}^{n} c_m k^m$$

where

$$c_m = \alpha\, n_m \qquad n_m \in \mathbb{Z}$$

Linear Free-Space Schrödinger Equation

$$\mathrm{i}\,\frac{\partial u}{\partial t} = -\,\frac{\partial^2 u}{\partial x^2}$$

Dispersion relation: $\omega = k^2$

Phase velocity: $c_p = \dfrac{\omega}{k} = k$

Group velocity: $c_g = \dfrac{d\omega}{dk} = 2\,k$

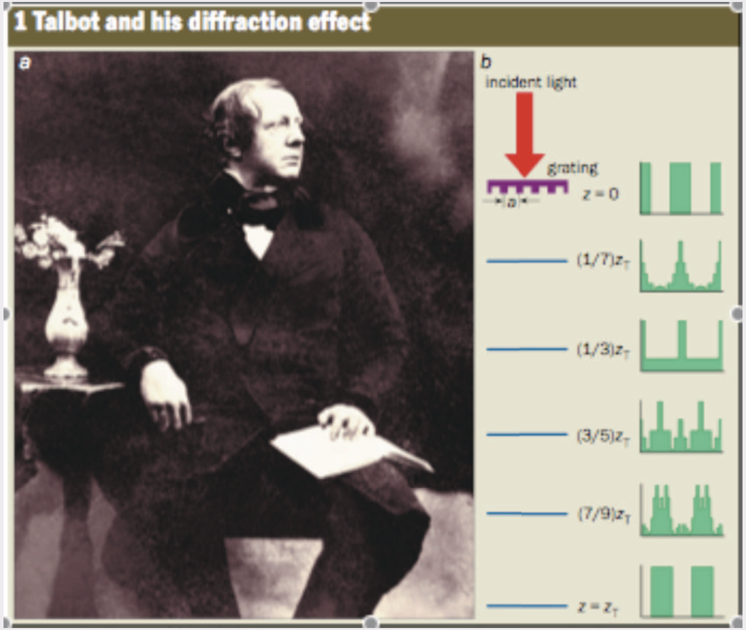
The Talbot Effect

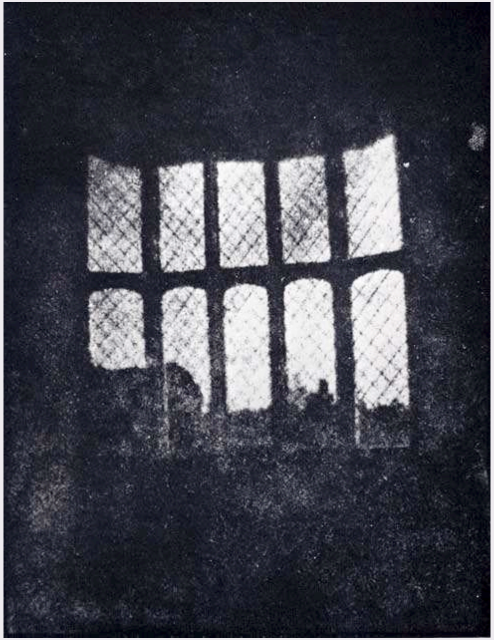
$$\mathrm{i}\,\frac{\partial u}{\partial t} = -\,\frac{\partial^2 u}{\partial x^2}$$

$$u(t,-\pi) = u(t,\pi) \qquad \frac{\partial u}{\partial x}(t,-\pi) = \frac{\partial u}{\partial x}(t,\pi)$$

- Michael Berry, et. al.
- Bernd Thaller, *Visual Quantum Mechanics*
- Oskolkov
- Kapitanski, Rodnianski
  "Does a quantum particle know the time?"
- Michael Taylor

William Henry Fox Talbot (1800–1877)





★ Talbot's 1835 image of a latticed window in Lacock Abbey

$\Longrightarrow$ oldest photographic negative in existence.

A Talbot Experiment

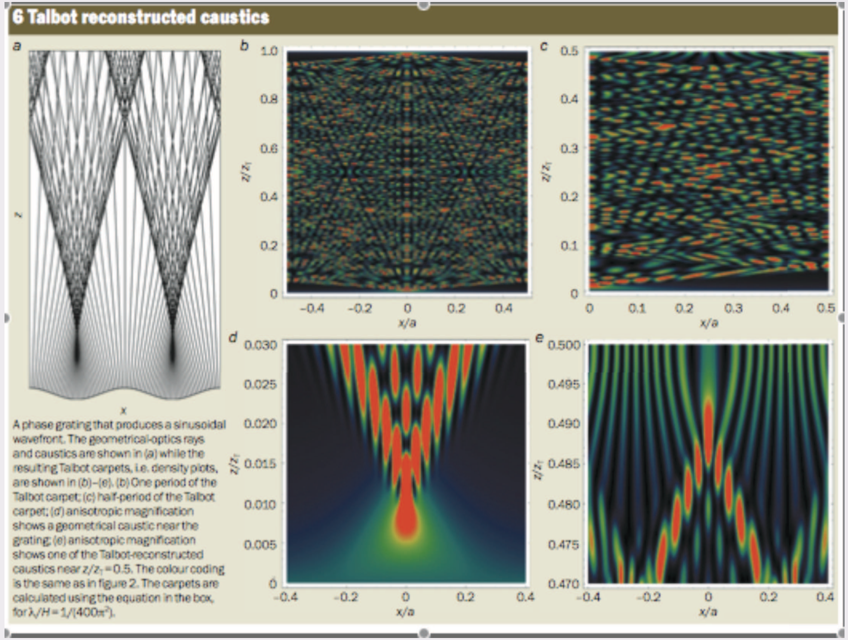
Fresnel diffraction by periodic gratings (1836):

"It was very curious to observe that though the grating was greatly out of the focus of the lens . . . the appearance of the bands was perfectly distinct and well defined . . . the experiments are communicated in the hope that they may prove interesting to the cultivators of optical science."

— *Fox Talbot*

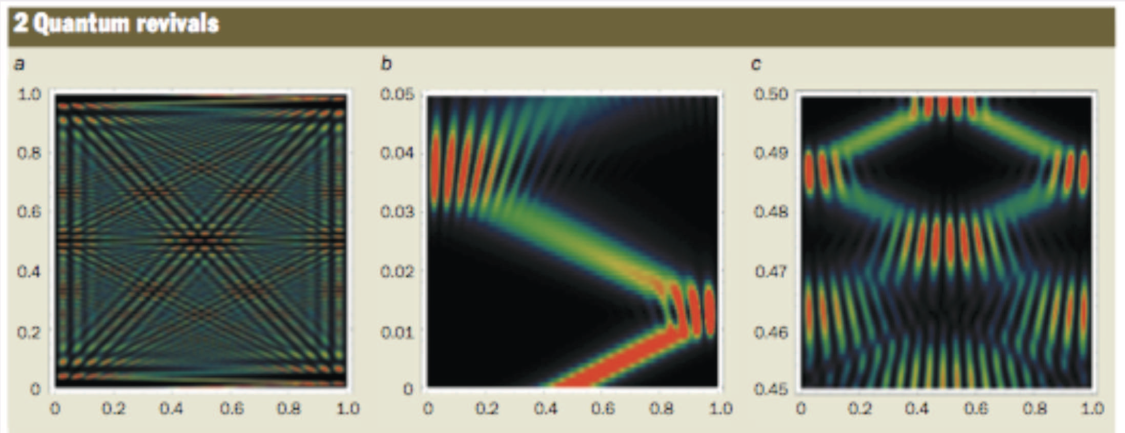
$\Longrightarrow$ Lord Rayleigh calculates the Talbot distance (1881)

The Quantized/Fractal Talbot Effect



- Optical experiments — Berry & Klein
- Diffraction of matter waves (helium atoms) — Nowak et. al.

Quantum Revival



- Electrons in potassium ions — Yeazell & Stroud
- Vibrations of bromine molecules — Vrakking, Villeneuve, Stolow

Periodic Linear Schrödinger Equation

$$\mathrm{i}\,\frac{\partial u}{\partial t} = -\,\frac{\partial^2 u}{\partial x^2}$$

$$u(t,-\pi) = u(t,\pi) \qquad \frac{\partial u}{\partial x}(t,-\pi) = \frac{\partial u}{\partial x}(t,\pi)$$

Integrated fundamental solution:

$$u(t,x) = \frac{1}{2\pi} \sum_{0 \neq k = -\infty}^{\infty} \frac{e^{\,\mathrm{i}\,(k\,x - k^2 t)}}{k}\,.$$

For $x/t \in \mathbb{Q}$, this is known as a Gauss sum (or, more generally, Weyl sum), of great importance in number theory

★★ The Riemann Hypothesis!

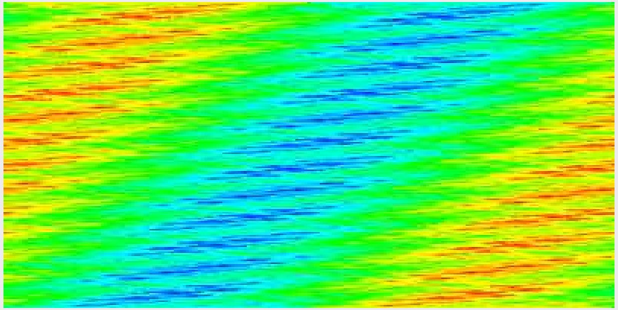
$\Longrightarrow$ Hardy, Littlewood, Weil, I. Vinogradov, etc.

Integrated fundamental solution:

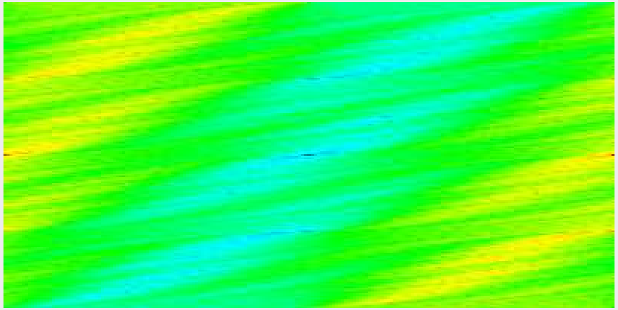
$$u(t,x) = \frac{1}{2\pi} \sum_{0 \neq k = -\infty}^{\infty} \frac{e^{\,\mathrm{i}\,(k\,x - k^2 t)}}{k}\,.$$

Theorem.

- The fundamental solution $\partial u/\partial x$ is a Jacobi theta function. At rational times $t = 2\pi\, p/q$, it linear combination of delta functions concentrated at rational nodes $x_j = 2\pi\, j/q$.
- At irrational times t, the integrated fundamental solution is a continuous but nowhere differentiable function.



Dispersive Carpet



Schrödinger Carpet

Periodic Linear Dispersion

$$\frac{\partial u}{\partial t} = L(D_x)\, u, \qquad u(t, x + 2\pi) = u(t, x)$$

Dispersion relation:

$$u(t,x) = e^{\,\mathrm{i}\,(k\,x - \omega\, t)} \quad \Longrightarrow \quad \omega(k) = -\,\mathrm{i}\, L(-\,\mathrm{i}\, k) \qquad \text{assumed real}$$

Riemann problem: step function initial data

$$u(0,x) = \sigma(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

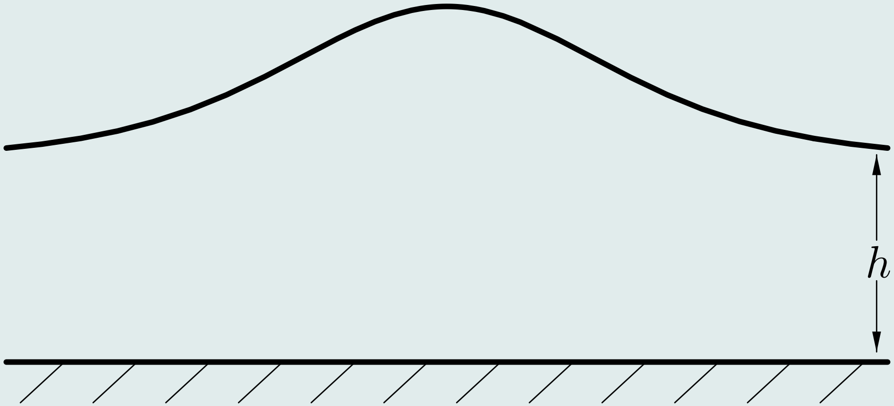
Solution:

$$u(t,x) \;\sim\; \frac{1}{2} + \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\sin[\,(2\,j+1)\,x - \omega(k)\,t\,]}{2\,j+1}\,.$$

★ ★ $\omega(-\,k) = -\,\omega(k)$ odd

Polynomial dispersion, rational t $\Longrightarrow$ Weyl exponential sums

2D Water Waves



$y = h + \eta(t, x)$ surface elevation

$\phi(t, x, y)$ velocity potential

2D Water Waves

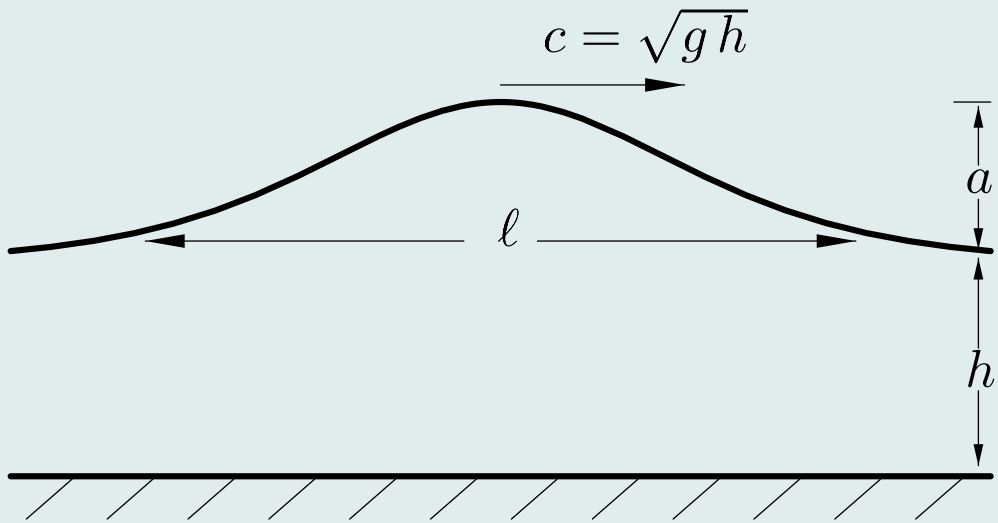
- Incompressible, irrotational fluid.
- No surface tension

$$\left.\begin{aligned} \phi_t + \tfrac{1}{2}\,\phi_x^2 + \tfrac{1}{2}\,\phi_y^2 + g\,\eta = 0 \\ \eta_t = \phi_y - \eta_x \phi_x \end{aligned}\right\} \qquad y = h + \eta(t,x)$$

$$\phi_{xx} + \phi_{yy} = 0 \qquad 0 < y < h + \eta(t,x)$$

$$\phi_y = 0 \qquad y = 0$$

- Wave speed (maximum group velocity): $c = \sqrt{g\,h}$
- Dispersion relation: $\sqrt{g\,k\ \tanh(h\,k)} = c\,k - \frac{1}{6}\,c\,h^2 k^3 + \ \cdots$



Small parameters — long waves in shallow water (KdV regime)

$$\alpha = \frac{a}{h} \qquad \beta = \frac{h^2}{\ell^2} = O(\alpha)$$

Rescale:

$$x \longmapsto \ell\, x \qquad y \longmapsto h\, y \qquad t \longmapsto \frac{\ell\, t}{c}$$

$$\eta \longmapsto a\, \eta \qquad \phi \longmapsto \frac{g\, a\, \ell\, \phi}{c} \qquad c = \sqrt{g\, h}$$

Rescaled water wave system:

$$\left.\begin{aligned} \phi_t + \frac{\alpha}{2}\,\phi_x^2 + \frac{\alpha}{2\,\beta}\,\phi_y^2 + \eta = 0 \\ \eta_t = \frac{1}{\beta}\,\phi_y - \alpha\,\eta_x\,\phi_x \end{aligned}\right\} \qquad y = 1 + \alpha\,\eta$$

$$\beta\,\phi_{xx} + \phi_{yy} = 0 \qquad 0 < y < 1 + \alpha\,\eta$$

$$\phi_y = 0 \qquad y = 0$$

Boussinesq expansion

Set

$$\psi(t,x) = \phi(t,x,0) \qquad u(t,x) = \phi_x(t,x,\theta) \qquad 0 \le \theta \le 1$$

Solve Laplace equation:

$$\phi(t,x,y) = \psi(t,x) - \tfrac{1}{2}\beta^2 y^2 \psi_{xx} + \tfrac{1}{4!}\beta^4 y^4 \psi_{xxxx} + \cdots$$

Plug expansion into free surface conditions: To first order

$$\psi_t + \eta + \tfrac{1}{2}\alpha \psi_x^2 - \tfrac{1}{2}\beta \psi_{xxt} = 0$$

$$\eta_t + \psi_x + \alpha (\eta\psi_x)_x - \tfrac{1}{6}\beta \psi_{xxxx} = 0$$

Bidirectional Boussinesq systems:

$$u_t + \eta_x + \alpha\, u\, u_x - \tfrac{1}{2}\beta\,(\theta^2 - 1)\, u_{xxt} = 0$$

$$\eta_t + u_x + \alpha\,(\eta\, u)_x - \tfrac{1}{6}\beta\,(3\,\theta^2 - 1)\, u_{xxx} = 0$$

★ ★ at $\theta = 1$ this system is integrable

$\Longrightarrow$ Kaup, Kupershmidt

Boussinesq equation

$$u_{tt} = u_{xx} + \tfrac{1}{2}\alpha\,(u^2)_{xx} - \tfrac{1}{6}\beta\, u_{xxxx}$$

Regularized Boussinesq equation

$$u_{tt} = u_{xx} + \tfrac{1}{2}\alpha\,(u^2)_{xx} - \tfrac{1}{6}\beta\, u_{xxtt}$$

$\Longrightarrow$ DNA dynamics (Scott)

Unidirectional waves:

$$u = \eta - \tfrac{1}{4}\,\alpha\,\eta^2 + \left(\tfrac{1}{3} - \tfrac{1}{2}\,\theta^2\right)\beta\,\eta_{xx} + \cdots$$

Korteweg-deVries (1895) equation:

$$\eta_t + \eta_x + \tfrac{3}{2}\,\alpha\,\eta\,\eta_x + \tfrac{1}{6}\,\beta\,\eta_{xxx} = 0$$

$\Longrightarrow$ Due to Boussinesq in 1877!

Benjamin–Bona–Mahony (BBM) equation:

$$\eta_t + \eta_x + \tfrac{3}{2}\,\alpha\,\eta\,\eta_x - \tfrac{1}{6}\,\beta\,\eta_{xxt} = 0$$

Shallow Water Dispersion Relations

Water waves	$\pm\sqrt{k\,\tanh k}$
Boussinesq system	$\pm\dfrac{k}{\sqrt{1+\frac{1}{3}k^2}}$
Boussinesq equation	$\pm k\sqrt{1+\frac{1}{3}k^2}$
Korteweg–deVries	$k-\frac{1}{6}k^3$
BBM	$\dfrac{k}{1+\frac{1}{6}k^2}$

Dispersion Asymptotics

★ The qualitative behavior of the solution to the periodic problem depends crucially on the asymptotic behavior of the dispersion relation $\omega(k)$ for large wave number $k \to \pm\infty$.

$$\omega(k) \sim k^{\alpha}$$

- $\alpha = 0$ — large scale oscillations
- $0 < \alpha < 1$ — dispersive oscillations
- $\alpha = 1$ — traveling waves
- $1 < \alpha < 2$ — oscillatory becoming fractal
- $\alpha \geq 2$ — fractal/quantized

Periodic Korteweg–deVries equation

$$\frac{\partial u}{\partial t} = \alpha \, \frac{\partial^3 u}{\partial x^3} + \beta \, u \, \frac{\partial u}{\partial x} \qquad u(t, x + 2\ell) = u(t, x)$$

Zabusky–Kruskal (1965)

$$\alpha = 1, \qquad \beta = .000484, \qquad \ell = 1, \qquad u(0, x) = \cos \pi x.$$

Lax–Levermore (1983) — small dispersion

$$\alpha \longrightarrow 0, \qquad \beta = 1.$$

Gong Chen (2011)

$$\alpha = 1, \qquad \beta = .000484, \qquad \ell = 1, \qquad u(0, x) = \sigma(x).$$

Periodic Korteweg–deVries Equation

Analysis of non-smooth initial data:

Estimates, existence, well-posedness, stability, ...

- Kato
- Bourgain
- Kenig, Ponce, Vega
- Colliander, Keel, Staffilani, Takaoka, Tao
- Oskolkov
- D. Russell, B–Y Zhang
- Erdoğan, Tzirakis

Operator Splitting

$$u_t = \alpha\, u_{xxx} + \beta\, u u_x = L[u] + N[u]$$

Flow operators: $\Phi_L(t)$, $\Phi_N(t)$

Godunov scheme:

$$u_\Delta^G(t_n) \simeq \left(\Phi_L(\Delta t)\, \Phi_N(\Delta t) \right)^n u_0$$

Strang scheme:

$$u_\Delta^S(t_n) \simeq \left(\Phi_N(\tfrac{1}{2}\Delta t)\, \Phi_L(\Delta t)\, \Phi_N(\tfrac{1}{2}\Delta t) \right)^n u_0$$

Numerical implementation:

- FFT for Φ_L — linearized KdV
- FFT + convolution for Φ_N — conservative version of inviscid Burgers', using Backward Euler + fixed point iteration to overcome mild stiffness. Shock dynamics doesn't complicate due to small time stepping.

Convergence of Operator Splitting

★ Holden, Karlsen, Risebro and Tao prove:

First order convergence of the Godunov scheme

$$u_\Delta^G(t_n) \simeq \left(\Phi_L(\Delta t)\, \Phi_N(\Delta t) \right)^n u_0$$

for initial data $u_0 \in H^s$ for $s \geq 5$:

$$\| u(t_n) - u_\Delta^G(t_n) \| \leq C\, \Delta t$$

Second order convergence of the Strang scheme

$$u_\Delta^S(t_n) \simeq \left(\Phi_N(\tfrac{1}{2}\Delta t)\, \Phi_L(\Delta t)\, \Phi_N(\tfrac{1}{2}\Delta t) \right)^n u_0$$

for initial data $u_0 \in H^s$ for $s \geq 17$:

$$\| u(t_n) - u_\Delta^S(t_n) \| \leq C\, (\Delta t)^2$$

Convergence for Rough Data?

However, subtle issues prevent us from establishing convergence of the operator splitting method for rough initial data.

- Bourgain proves well-posedness of the periodic KdV flow in L^2
- Conservation of the L^2 norm establishes well-posedness in L^2 of the linearized flow Φ_L
- Thus, if the solution has bounded L^∞ norm, then the linearized flow is L^1 contractive
- Oskolkov proves that is the initial data is has bounded BV norm, then the resulting solution to the periodic linearized KdV equation is uniformly bounded in L^∞
- Unfortunately, Oskolkov's bound depends on the BV and L^∞ norms of the initial data. Moreover, at irrational times, the solution is nowhere differentiable and has unbounded BV norm
- Also, we do not have good control of the BV norm of the nonlinear inviscid Burgers' flow Φ_N
- ????

Periodic Nonlinear Schrödinger Equation

$$\mathrm{i}\, u_t + u_{xx} + |\, u\,|^p\, u = 0, \qquad x \in \mathbb{R}/\mathbb{Z}, \qquad u(0,x) = g(x).$$

Theorem. (Erdoğan, Tzirakis)

Suppose $p = 2$ (the integrable case) and $g \in \mathrm{BV}$. Then

(i) $u(t,\cdot)$ is continuous at irrational times $t \notin \mathbb{Q}$

(ii) $u(t,\cdot)$ is bounded with at most countably many discontinuities at rational times $t \in \mathbb{Q}$

(iii) When the initial data is sufficiently "rough", i.e., $g \notin \bigcup_{\epsilon>0} H^{1/2+\epsilon}$ then, at almost all t, the real or imaginary part of the graph of $u(t,\,\cdot\,)$ has fractal (upper Minkowski) dimension $\frac{3}{2}$.

Periodic Linear Dispersive Equations

$\Longrightarrow$ Chousionis, Erdoğan, Tzirakis

Theorem. Suppose $3 \le k \in \mathbb{Z}$ and

$$\mathrm{i}\, u_t + (-\mathrm{i}\, \partial_x)^k u = 0, \qquad x \in \mathbb{R}/\mathbb{Z}, \qquad u(0,x) = g(x) \in \mathrm{BV}$$

(i) $u(t,\cdot)$ is continuous for almost all t

(ii) When $g \notin \bigcup_{\epsilon>0} H^{1/2+\epsilon}$, then, at almost all t, the real and imaginary parts of the graph of $u(t,\,\cdot\,)$ has fractal dimension $1 + 2^{1-k} \le D \le 2 - 2^{1-k}$.

Theorem. For the periodic Korteweg–deVries equation

$$u_t + u_{xxx} + u\, u_x = 0, \qquad x \in \mathbb{R}/\mathbb{Z}, \qquad u(0,x) = g(x) \in \mathrm{BV}$$

(i) $u(t,\cdot)$ is continuous for almost all t

(ii) When $g \notin \bigcup_{\epsilon>0} H^{1/2+\epsilon}$, then, at almost all t, the real and imaginary parts of the graph of $u(t,\,\cdot\,)$ has fractal dimension $\frac{5}{4} \le D \le \frac{7}{4}$.

The Vortex Filament Equation

$\Longrightarrow$ Da Rios (1906)

Localized Induction Approximation (LIA) or binormal flow

$$\boxed{\gamma_t = \gamma_s \times \gamma_{ss} = \kappa\, \mathbf{b}}$$

$\gamma(t,s) \in \mathbb{R}^3$ at time t represents the vortex filament — a space curve parametrized by arc length — that moves in an incompressible fluid flow with vorticity concentrated on the filament.

Frenet frame: $\mathbf{t}, \mathbf{n}, \mathbf{b}$ — unit tangent, normal, binormal

κ — curvature $\qquad$ τ — torsion

$$\boxed{\gamma_t = \gamma_s \times \gamma_{ss}}$$

Hasimoto transformation:

$$u = \kappa \exp\left(\mathrm{i} \int \tau \, ds \right)$$

solves the integrable nonlinear Schrödinger equation:

$$\mathrm{i}\, u_t = u_{xx} + |\, u \,|^2 \, u$$

de la Hoz and Vega (2013): If the initial data is a closed polygon, then at rational times the curve is a polygon, whereas at irrational times it is a fractal.

Chousionis, Erdoğan, Tzirakis (2014): further results on fractal behavior for some smooth initial data

Vortex Filament Polygons

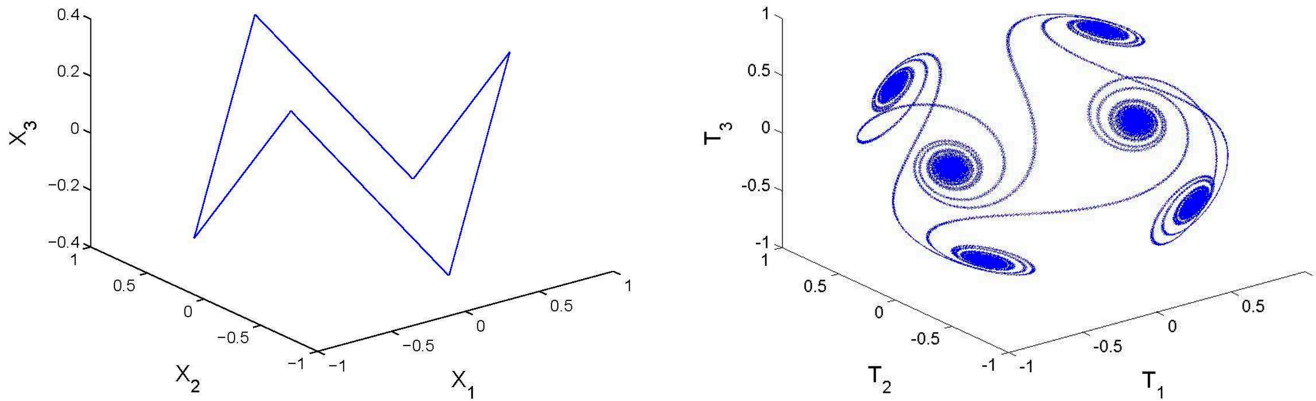


Figure 7: $\mathbf{X}_{alg}$ and $\mathbf{T}_{alg}$, at $t = \frac{2\pi}{9}(\frac{1}{4} + \frac{1}{49999})$.

Vortex Filament Polygons

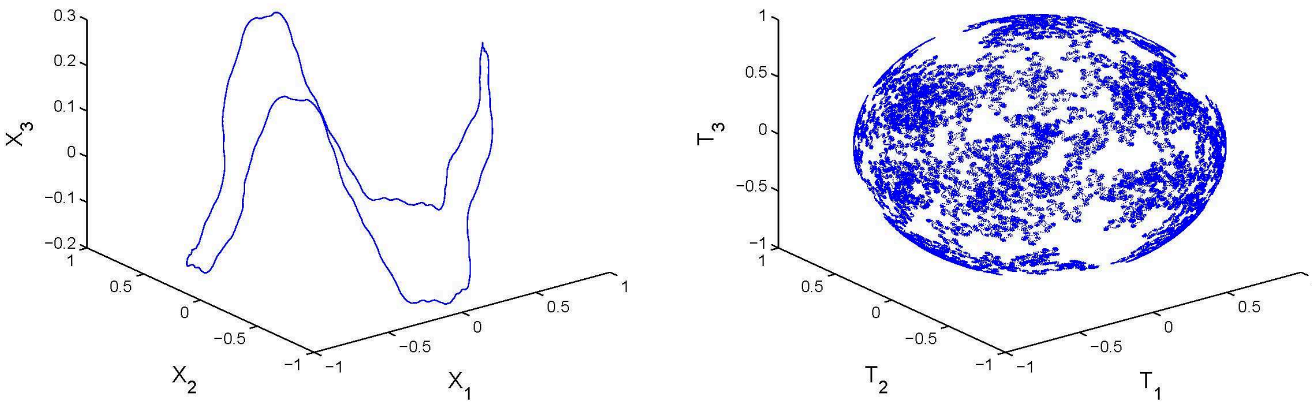


Figure 8: $\mathbf{X}_{alg}$ and $\mathbf{T}_{alg}$, at $t = \frac{2\pi}{9}(\frac{1}{4} + \frac{1}{41} + \frac{1}{401}) = \frac{2\pi}{9} \cdot \frac{18209}{65764}$.

Future Directions

- General dispersion behavior
- Other boundary conditions (Fokas' Method)
- Higher space dimensions and other domains
  (tori, spheres, . . .)
- Dispersive nonlinear partial differential equations
- Discrete systems: Fermi–Pasta–Ulam
- Numerical solution techniques?
- Experimental verification in dispersive media?