Ghost Symmetries

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Jet Notation

$(x^1, \ldots, x^p)$ — independent variables

$(u^1, \ldots, u^q)$ — dependent variables

Total derivative

$$D_i(x^j) = \delta_i^j, \qquad D_i(u_J^\alpha) = u_{J+e_i}^\alpha,$$
$$D^J = (D_1)^{j_1} \ \cdots \ (D_p)^{j_p}$$

$u_J^\alpha = D^J(u^\alpha)$ — derivative coordinates

Differential polynomial:

$$Q[u] = \sum_{K,\mathcal{J}} c_K(x)\, u_{J_1}^{\alpha_1} \ \cdots \ u_{J_k}^{\alpha_k}$$

Local algebra

$$J = (j_1, \ldots, j_p) \ \geq \ 0$$

Nonlocal algebra

$$J \in \mathbb{Z}^p \qquad \text{i.e.} \qquad j_\nu < 0 \qquad \text{allowed}$$

or, more generally

$$D^J Q \qquad \text{is defined for all} \qquad J \in \mathbb{Z}^p$$

The Kadomtsev–Petviashvili equation

$$u_{xt} = u_{yy} - 6\,u\,u_{xx} - 6\,u_x^2 - u_{xxxx}$$

Evolutionary form

$$\begin{aligned} u_t &= D_x^{-1} u_{yy} - 6\,u\,u_x - u_{xxx} \\ &= u_{-1,2} - 6\,u_{0,0}\,u_{1,0} - u_{3,0} \end{aligned}$$

Symmetries and loop algebras

- Chen, Lee, Lin: 1982
- David, Kamran, Levi, Winternitz: 1985

Nonlocal Symmetries

- Vinogradov, Krasil'shchik: 1980, 1984

  $\Longrightarrow$ "coverings"
- Kapcov: 1982
- Vladimirov, Volovich: 1985
- Fushchych, Nikitin: 1987
- Bluman, Kumei, Reid: 1988
- Akhatov, Gazizov, Ibragimov: 1991
- Galas: 1992
- Guthrie, Hickman: 1993
- Dodd: 1994
- Anco, Bluman: 1995
- Sanders, Wang: 2000
- Leo, Leo, Soliani, Tempesta: 2002
- Mikhailov, V. Novikov: 2002

Generalized vector field

$$\mathbf{v} = \mathbf{v}_Q = \sum_{\alpha, J} D^J Q^\alpha \frac{\partial}{\partial u_J^\alpha}$$

Characteristic

$$Q^\alpha = \mathbf{v}_Q(u^\alpha) \qquad \alpha = 1, \ldots, q$$

$$\mathbf{v}_Q(u_J^\alpha) = D_J(Q^\alpha)$$

$$[\, \mathbf{v}_Q,\, D_i \,] = 0$$

$$\mathbf{v}_Q(P) = \mathrm{D}_P(Q) = \sum_{\alpha=1}^{q} \mathrm{D}_P^\alpha(Q^\alpha)$$

Fréchet derivative

$$\mathrm{D}_P^\alpha = \sum_J \frac{\partial P}{\partial u_J^\alpha} \, D^J$$

Commutator

Lie bracket or *commutator*

$$[\,\mathbf{v}_P,\,\mathbf{v}_Q\,] = \mathbf{v}_{[\,P,Q\,]}$$

where

$$[\,P,\,Q\,] = \mathbf{v}_P(Q) - \mathbf{v}_Q(P) = \mathrm{D}_Q(P) - \mathrm{D}_P(Q)$$

Theorem. The generalized vector field $\mathbf{v}_Q$ is a symmetry of the evolution equation

$$u_t = P$$

if and only if

$$\frac{\partial Q}{\partial t} \;+\; [\,P,\,Q\,] = 0.$$

A Jacobi Paradox?

$$[\,1,\,[\,u_x,\,D_x^{-1}u\,]\,] + [\,u_x,\,[\,D_x^{-1}u,\,1\,]\,] + [\,D_x^{-1}u,\,[\,1,\,u_x\,]\,] \;= 1$$

$$[\,1,\,u_x\,] = \mathrm{D}_{u_x}(1) - \mathrm{D}_1(u_x) = 0 \implies [\,\partial_u,\,\partial_x\,] = 0$$

$$\begin{aligned}[\,u_x,\,D_x^{-1}u\,] &= \mathrm{D}_{D_x^{-1}u}(u_x) - \mathrm{D}_{u_x}(D_x^{-1}u)\\ &= D_x^{-1}u_x - D_x(D_x^{-1}u)\\ &= (u+c) - u = c\end{aligned}$$

$$[\,1,\,[\,u_x,\,D_x^{-1}u\,]\,] = [\,1,\,c\,] = 0$$

$$[\,D_x^{-1}u,\,1\,] = -D_x^{-1}(1) = -x+d$$

$$\begin{aligned}[\,u_x,\,[\,D_x^{-1}u,\,1\,]\,] &= [\,u_x,\,-x+d\,]\\ &= -D_x(-x+d) = 1\end{aligned}$$

General Framework

$\mathcal{A}$ — algebra of functions $f(x)$

$\mathcal{U}_0$ — u–dependent differential functions

$\mathcal{U} = \mathcal{A} \oplus \mathcal{U}_0$ — full differential algebra

$$F(x, u^{(n)}) = f(x) + P(x, u^{(n)})$$

$$\Longrightarrow \quad P(x, 0) \equiv 0$$

D_i — total derivatives on $\mathcal{U}$

$$\ker D_i = \{\, f(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^p) \,\} \subset \mathcal{A}$$

Definition. A *evolutionary vector field* $\mathbf{v}$ is a derivation $\mathbf{v}: \mathcal{U} \to \mathcal{U}$, with

$$\mathbf{v}(P + Q) = \mathbf{v}(P) + \mathbf{v}(Q)$$

$$\mathbf{v}(P \cdot Q) = \mathbf{v}(P) \cdot Q + P \cdot \mathbf{v}(Q)$$

$$\mathbf{v}(x^i) = 0 \qquad \Longrightarrow \qquad \mathcal{A} \subset \ker \mathbf{v}$$

$$[\, \mathbf{v},\, D_i \,] = 0$$

Commutator:

$$[\, \mathbf{v},\, \mathbf{w} \,](P) = \mathbf{v}(\mathbf{w}(P)) - \mathbf{w}(\mathbf{v}(P))$$

Theorem. The Jacobi identity holds!

$$[\, \mathbf{u},\, [\, \mathbf{v},\, \mathbf{w} \,]\,] + [\, \mathbf{v},\, [\, \mathbf{w},\, \mathbf{u} \,]\,] + [\, \mathbf{w},\, [\, \mathbf{u},\, \mathbf{v} \,]\,] = 0$$

Characteristic:

$$\mathbf{v}(u^\alpha) = Q^\alpha \qquad \alpha = 1, \ldots, q.$$

Note

$$\mathbf{v}(u^\alpha_J) = \mathbf{v}(D_J u^\alpha) = D_J \mathbf{v}(u^\alpha) = D_J Q^\alpha$$

Key observation:

For local differential algebras, an evolutionary vector field is uniquely determined by its characteristic. This is *not* true in nonlocal differential algebras. There are nonzero evolutionary vector fields with zero characteristic — *ghosts*.

Ghosts

Definition. An evolutionary vector field $\boldsymbol{\gamma}$ is called a K-*ghost* if

$$\boldsymbol{\gamma}(u_L^\alpha) = 0$$

for all $L \geq K$ and $\alpha = 1, \ldots, q$.

Example. $p = 1$ $u_n = D_x^n u$

The vector field $\mathbf{v}_1$ with characteristic $Q = 1$ is a 1-ghost:

$$\mathbf{v}_1(u_x) = D_x(1) = 0$$

$$\mathbf{v}_1(u_n) = D_x^n(1) = 0 \qquad n \geq 1$$

$\Longrightarrow$ positive (local) ghost

Example. A non-local ghost:

$$\boldsymbol{\gamma}(u_k) = \chi_{k+1}(x) = D_x^{k+1}(1)$$
$$= \begin{cases} \dfrac{x^{-k-1}}{(-k-1)!} & k \le -1 \\ 0 & \text{otherwise} \end{cases}$$

Characteristic:

$$\boldsymbol{\gamma}(u) = Q = 0$$
$$\boldsymbol{\gamma}(u_n) = D_x^n(Q) = 0 \qquad n \ge 0$$
$$\boldsymbol{\gamma}(u_{-1}) = 1$$
$$\boldsymbol{\gamma}(u_{-n}) = D_x^{-n+1}(1)$$

$\Longrightarrow$ Only polynomial u-independent vector fields can be ghosts!

$$\mathbf{v}(u_J^\alpha) = q_J^\alpha(x) \in \mathcal{A}$$

Jacobi Revisited

$$[\,1,\,[\,u_x,\,D_x^{-1}u\,]\,] + [\,u_x,\,[\,D_x^{-1}u,\,1\,]\,] + [\,D_x^{-1}u,\,[\,1,\,u_x\,]\,]$$

Generalized vector fields $\sim$ characteristics

$$\mathbf{v} \;\sim\; 1, \qquad \mathbf{w} \;\sim\; u_x, \qquad \mathbf{z} \;\sim\; D_x^{-1}u.$$

Surprise: The problem is with the local commutator

$$[\,\partial_u,\,\partial_x\,] \;\sim\; [\,1,\,u_x\,] \;\sim\; [\,\mathbf{v},\,\mathbf{w}\,] \;=\; \boldsymbol{\gamma} \;\neq\; 0$$

On the nonlocal differential algebra, $\boldsymbol{\gamma}$ is a ghost vector field!

$$\mathbf{v}(u_k) = D_x^k(1) = \chi_k(x) = \begin{cases} \dfrac{x^{-k}}{(-k)!}, & k \leq 0, \\[2ex] 0 & \text{otherwise,} \end{cases}$$

$$\mathbf{w}(u_k) = D_x(u_k) = u_{k+1}$$

$$\mathbf{w}(\mathbf{v}(u_k)) = 0,$$

$$\mathbf{v}(\mathbf{w}(u_k)) = \mathbf{v}(u_{k+1}) = \chi_{k+1}(x)$$

$$\boldsymbol{\gamma}(u_k) = \mathbf{v}(\mathbf{w}(u_k)) - \mathbf{w}(\mathbf{v}(u_k)) = \chi_{k+1}(x)$$

$$[\,1,\,[\,u_x,\,D_x^{-1}u\,]\,]+[\,u_x,\,[\,D_x^{-1}u,\,1\,]\,]+[\,D_x^{-1}u,\,[\,1,\,u_x\,]\,]$$

$$\mathbf{v}\ \sim\ 1,\qquad \mathbf{w}\ \sim\ u_x,\qquad \mathbf{z}\ \sim\ D_x^{-1}u.$$

$$[\,1,\,u_x\,]\ \sim\ [\,\mathbf{v},\,\mathbf{w}\,]=\boldsymbol{\gamma}$$

This ghost provides the missing terms in the Jacobi identity:

$$[\,\mathbf{z},\,\boldsymbol{\gamma}\,]=-\,\mathbf{v}$$

because

$$[\,\mathbf{z},\,\boldsymbol{\gamma}\,](u)=-\,\boldsymbol{\gamma}(\mathbf{z}(u))=-\,\boldsymbol{\gamma}(D_x^{-1}u)=-\,1.$$

$$[\,\mathbf{z},\,[\,\mathbf{v},\,\mathbf{w}\,]\,]=[\,\mathbf{z},\,\boldsymbol{\gamma}\,]=-\,\mathbf{v}$$

$$[\,\mathbf{v},\,[\,\mathbf{w},\,\mathbf{z}\,]\,]=[\,\mathbf{v},\,\mathbf{v}_c\,]=0$$

$$[\,\mathbf{w},\,[\,\mathbf{z},\,\mathbf{v}\,]\,]=[\,\mathbf{w},\,\mathbf{v}_{-x+d}\,]=\mathbf{v}$$

The Ghost Calculus

Assume $q = 1$ — one dependent variable u

Restrict to polynomial vector fields.

Only vector fields that do not depend on u_K can be ghosts!

Define

$$\chi_K = D^K(1) = \begin{cases} \dfrac{x^{-K}}{(-K)!}, & -K \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

Basis ghost vector field:

$$\boldsymbol{\gamma}_J(u_K) = \chi_{J+K},$$

Theorem. Every ghost vector field is a constant coefficient linear combination of the basis ghosts

$$\boldsymbol{\gamma} = \sum_J c_J \, \boldsymbol{\gamma}_J.$$

$\Longrightarrow$ The summation can be infinite, but with certain restrictions.

Theorem. Any evolutionary vector field on a polynomial differential algebra can be written a linear combination of basis ghosts and a polynomial u-dependent vector field with characteristic Q

$$\mathbf{v} = \mathbf{v}_Q + \sum_J c_J \boldsymbol{\gamma}_J$$

$$\mathbf{v}(u) = Q + \sum_J c_J \, \chi_J$$

$$= Q + \sum_{-J \geq 0} c_J \, \frac{x^{-J}}{(-J)!}$$

$$\mathbf{v}(u_K) = D^K Q + \sum_J c_J \, \chi_{K+J}$$

$$= D^K Q + \sum_{-J-K \geq 0} c_J \, \frac{x^{-J-K}}{(-J-K)!}$$

$\Longrightarrow$ sums must be finite

Ghost calculus rules

Uniform notation: Replace

$$x^J \Longrightarrow J!\,\chi_{-J} \qquad J \geq 0$$

$$\boldsymbol{\gamma}_K \Longrightarrow \chi_K$$

Every differential polynomial is a sum of monomials

$$\chi_L u_{K_1} \cdots u_{K_m} \sim \frac{x^{-L}}{(-L)!}\, u_{K_1} \cdots u_{K_m}$$

$$\chi_L\, \chi_K = \binom{-K-L}{-L}\, \chi_{K+L} \qquad -L \geq 0$$

$$[\,\chi_J,\, \chi_K\,] = 0 \qquad [\,\chi_J,\, u_K\,] = \chi_{J+K}$$

$$[\,\chi_J,\, \chi_L u_{K_1} \cdots u_{K_m}\,]$$

$$= \sum \binom{-K_\nu - L}{-L}\, \chi_{J+K_\nu+L}\, u_{K_1} \cdots \widehat{u_{K_\nu}} \cdots u_{K_m}$$

$$[\,Q,\, R\,] = \mathbf{v}_Q(R) - \mathbf{v}_R(Q)$$

Jacobi paradox revisited

$$[\,1,\,[\,u_x,\,D_x^{-1}u\,]\,]+[\,u_x,\,[\,D_x^{-1}u,\,1\,]\,]$$
$$+[\,D_x^{-1}u,\,[\,1,\,u_x\,]\,]$$

The three ghost characteristics are

$$1 \Longrightarrow \chi_0 \qquad u_x \Longrightarrow u_1 \qquad D_x^{-1}u \Longrightarrow u_{-1}$$

$$[\,\chi_0,\,[\,u_1,\,u_{-1}\,]\,]=0$$

$$[\,u_1,\,[\,u_{-1},\,\chi_0\,]\,]=-\,[\,u_1,\,\chi_{-1}\,]=\chi_0$$

$$[\,u_{-1},\,[\,\chi_0,\,u_1\,]\,]=[\,u_{-1},\,\chi_1\,]=-\,\chi_0$$

The sum of these three terms is 0.

The Original Jacobi Paradox

Independent variables — x, y
Dependent variable — u

Characteristics: $y,\ yu_x,\ u_x D_x^{-1} u_y$.

Without ghost terms:

$$[\, y,\, [\, u_x D_x^{-1} u_y,\, yu_x \,]\,] + [\, yu_x,\, [\, y,\, u_x D_x^{-1} u_y \,]\,]$$

$$+ [\, u_x D_x^{-1} u_y,\, [\, yu_x,\, y \,]\,] = -2yu_x \neq 0$$

Ghost characteristics

$$y \Longrightarrow \chi_{0,-1} \qquad yu_x \Longrightarrow \chi_{0,-1}\, u_{1,0} \qquad u_x D_x^{-1} u_y \Longrightarrow u_{1,0}\, u_{-1,1}$$

The three terms are

$$[\, \chi_{0,-1},\, \chi_{0,-1}\, u_{1,0} \,] = \chi_{0,-1}\chi_{1,-1} = 2\,\chi_{1,-2}$$

$$[\, u_{1,0}\, u_{-1,1},\, 2\,\chi_{1,-2} \,] = -2\,\chi_{0,-1}\, u_{1,0},$$

$$[\, \chi_{0,-1}\, u_{1,0},\, u_{1,0}\, u_{-1,1} \,] = u_{0,0} u_{1,0}$$

$$[\, \chi_{0,-1},\, u_{0,0} u_{1,0} \,] = \chi_{0,-1}\, u_{1,0}$$

$$[\, u_{1,0}\, u_{-1,1},\, \chi_{0,-1} \,] = -\chi_{1,-1}\, u_{-1,1} - \chi_{-1,0}\, u_{1,0} = -\chi_{-1,0}\, u_{1,0}$$

$$[\, \chi_{0,-1}\, u_{1,0},\, -\chi_{-1,0}\, u_{1,0} \,] = \chi_{0,-1}\, u_{1,0}$$

The sum of these three terms is 0, and so the Jacobi identity is valid in the ghost framework.

The Kadomtsev–Petviashvili equation

$$u_t = D_x^{-1} u_{yy} - 6\,u\,u_x - u_{xxx}$$

The following known KP symmetries

$$M = \tfrac{1}{12}\,D_x^{-2} u_{yyy} - \tfrac{1}{4}\,u_{xxy} - u u_y - \tfrac{1}{2}\,u_x D_x^{-1} u_y$$

$$N = y$$

$$H = y u_y + \tfrac{1}{2}\,x u_x + u,$$

span an $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra:

$$[\,M,\,N\,] = H, \;\; [\,H,\,M\,] = 2M, \;\; [\,H,\,N\,] = -2N.$$

Paradox:

$$[\,M,\,u_x\,] = [\,N,\,u_x\,] = 0, \qquad \text{but} \qquad [\,H,\,u_x\,] = \tfrac{1}{2}\,u_x$$

$\Longrightarrow$ The vector space spanned by u_x is a one-dimensional representation of $\mathfrak{sl}(2,\mathbb{R})$, but representation theory requires that

$$[\,H,\,u_x\,] = 0.$$

Ghosts to the Rescue!

The symmetries do not, in fact, form a Lie algebra, but must be modified by appending a ghost to

$$[\, M,\, N\,] = y u_y + \tfrac{1}{2}\, x u_x + u + \tfrac{1}{4}\, \boldsymbol{\gamma}_{2,0} = H + \tfrac{1}{4}\, \boldsymbol{\gamma}_{2,0} \equiv \widehat{H}$$

Now, their Lie brackets are correct:

$$[\, M,\, N\,] = \widehat{H}, \quad [\, \widehat{H},\, M\,] = 2M, \quad [\, \widehat{H},\, N\,] = -2N.$$

We have

$$[\, \widehat{H},\, u_x\,] = \tfrac{1}{2}\, u_x + \tfrac{1}{4}\, \boldsymbol{\gamma}_{3,0}$$

Thus u_x is *not* an eigenvector for H, but actually belongs to a two-dimensional $\mathfrak{sl}(2,\mathbb{R})$ representation space spanned by u_x and $\boldsymbol{\gamma}_{3,0}$.