# *Object Recognition, Symmetry Detection, Jigsaw Puzzles, and Cancer*

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# Symmetry

**Definition.** A symmetry of a set $S$ is a transformation that preserves it:

$$g \cdot S = S$$

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$\star\star$ The set of symmetries forms a group, called the symmetry group of the set $S$.

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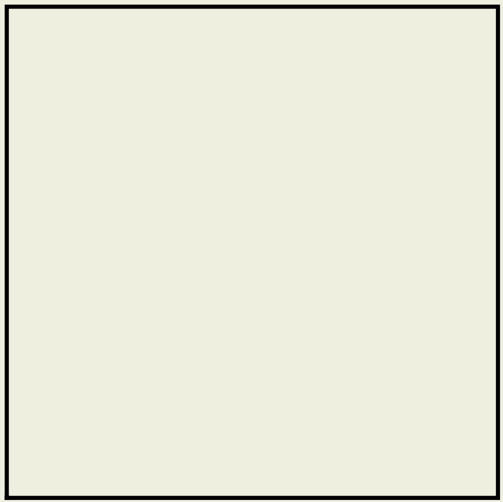
# Discrete Symmetry Group



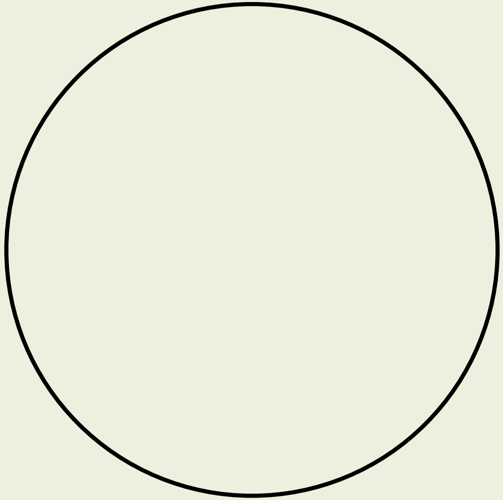
Rotations by 90°:

$$G_S = \mathbb{Z}_4$$

Rotations + reflections:

$$G_S = \mathbb{Z}_4 \ltimes \mathbb{Z}_4$$

# Continuous Symmetry Group



Rotations:

$$G_S = \mathrm{SO}(2)$$

Rotations + reflections:

$$G_S = \mathrm{O}(2)$$

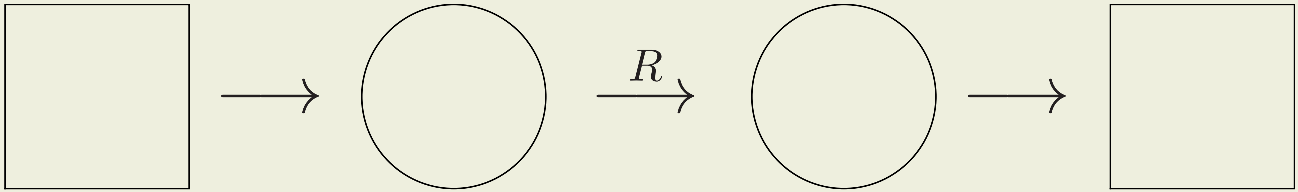
Conformal Inversions:

$$\overline{x} = \frac{x}{x^2 + y^2} \qquad \overline{y} = \frac{y}{x^2 + y^2}$$

⋆ A continuous group is known as a Lie group

— in honor of Sophus Lie.

# Continuous Symmetries of a Square



# Symmetry

⋆ To define the set of symmetries requires a priori specification of the allowable transformations or, equivalently, the underlying geometry.

$G$ — transformation group or pseudo-group of allowable transformations of the ambient space $M$

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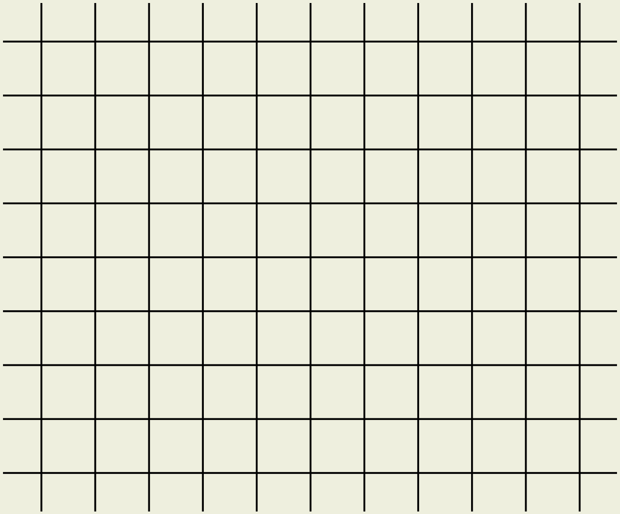
**Definition.** A symmetry of a subset $S \subset M$ is an allowable transformation $g \in G$ that preserves it:

$$g \cdot S = S$$

# What is the Symmetry Group?



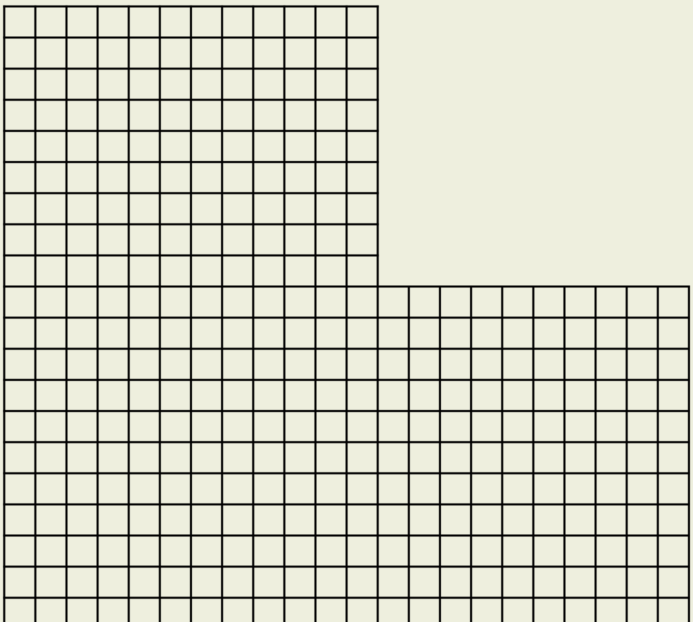
Allowable transformations:

Rigid motions

$$G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$$

$$G_S = \mathbb{Z}_4 \ltimes \mathbb{Z}^2$$

# What is the Symmetry Group?



Allowable transformations:

Rigid motions

$G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$

$G_S = \{e\}$

# Local Symmetries

**Definition.** $g \in G$ is a local symmetry of $S \subset M$ based at a point $z \in S$ if there is an open neighborhood $z \in U \subset M$ such that

$$g \cdot (S \cap U) = S \cap (g \cdot U)$$

$G_z \subset G$ — the set of local symmetries based at $z$.

Global symmetries are local symmetries at all $z \in S$:

$$G_S \subset G_z \qquad G_S = \bigcap_{z \in S} G_z$$

⋆⋆ The set of all local symmetries forms a groupoid!

# Groupoids

**Definition.** A groupoid is a small category such that every morphism has an inverse.

$\Longrightarrow$ Brandt (quadratic forms), Ehresmann (Lie pseudo-groups)

Mackenzie, R. Brown, A. Weinstein

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Groupoids form the appropriate framework for studying objects with variable symmetry.

# Groupoids

Double fibration:

$$\begin{array}{ccc} & \mathcal{G} & \\ {}^{\boldsymbol{\sigma}}\swarrow & & \searrow^{\boldsymbol{\tau}} \\ M & & M \end{array}$$

$\boldsymbol{\sigma}$ — source map $\qquad$ $\boldsymbol{\tau}$ — target map

---

$\star\star$ You are only allowed to multiply $\alpha \cdot \beta \in \mathcal{G}$ if

$$\boldsymbol{\sigma}(\alpha) = \boldsymbol{\tau}(\beta)$$

# Groupoids

- *Source and target of products:*

$$\boldsymbol{\sigma}(\alpha \cdot \beta) = \boldsymbol{\sigma}(\beta) \qquad \boldsymbol{\tau}(\alpha \cdot \beta) = \boldsymbol{\tau}(\alpha) \qquad \text{when} \qquad \boldsymbol{\sigma}(\alpha) = \boldsymbol{\tau}(\beta)$$

- *Associativity:*

$$\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma \qquad \text{when defined}$$

- *Identity section:* $e : M \to \mathcal{G} \qquad \boldsymbol{\sigma}(e(x)) = x = \boldsymbol{\tau}(e(x))$

$$\alpha \cdot e(\boldsymbol{\sigma}(\alpha)) = \alpha = e(\boldsymbol{\tau}(\alpha)) \cdot \alpha$$

- *Inverses:* $\boldsymbol{\sigma}(\alpha) = x = \boldsymbol{\tau}(\alpha^{-1}), \qquad \boldsymbol{\tau}(\alpha) = y = \boldsymbol{\sigma}(\alpha^{-1}),$

$$\alpha^{-1} \cdot \alpha = e(x), \qquad \alpha \cdot \alpha^{-1} = e(y)$$

# Jet Groupoids

$\Longrightarrow$ Ehresmann

The set of Taylor polynomials of degree $\leq n$, or Taylor series $(n = \infty)$ of local diffeomorphisms $\Psi : M \to M$ forms a groupoid.

$\diamondsuit$ Algebraic composition of Taylor polynomials/series is well-defined only when the source of the second matches the target of the first.

# The Symmetry Groupoid

**Definition.** The symmetry groupoid of $S \subset M$ is

$$\mathcal{G}_S = \{\, (g, z) \mid \; z \in S, \; g \in G_z \,\} \subset G \times S$$

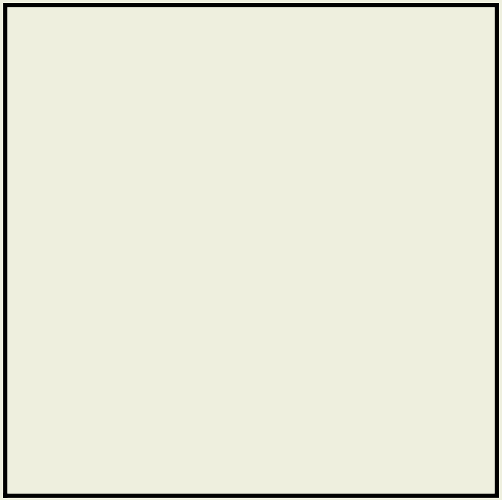
Source and target maps: $\boldsymbol{\sigma}(g, z) = z, \qquad \boldsymbol{\tau}(g, z) = g \cdot z.$

Groupoid multiplication and inversion:

$$(h, g \cdot z) \cdot (g, z) = (g \cdot h, z) \qquad (g, z)^{-1} = (g^{-1}, g \cdot z)$$

Identity map: $e(z) = (z, e) \in \mathcal{G}_S$

# What is the Symmetry Groupoid?



$G = \mathrm{SE}(2)$

Corners:

$$G_z = G_S = \mathbb{Z}_4$$

Sides: $G_z$ generated by

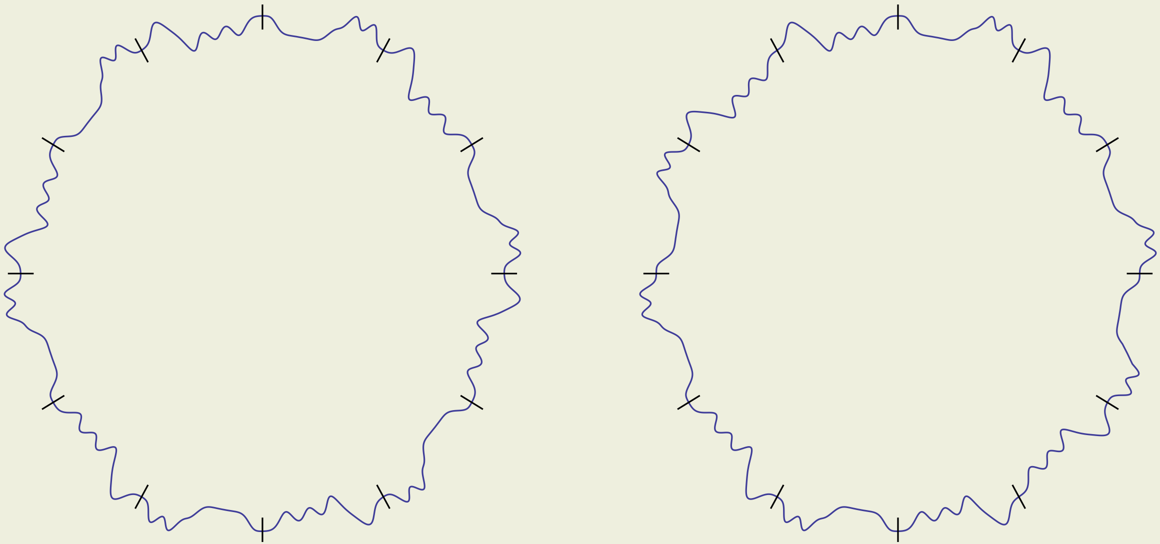
$G_S = \mathbb{Z}_4$

some translations

$180^\circ$ rotation around $z$

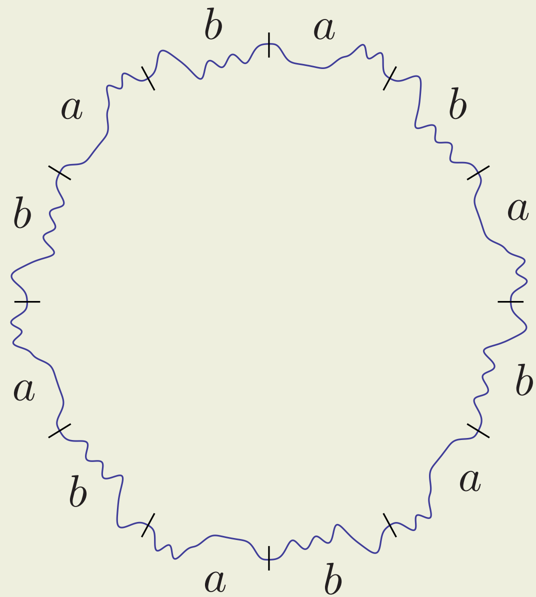
# What is the Symmetry Groupoid?

**Cogwheels** $\Longrightarrow$ Musso–Nicoldi

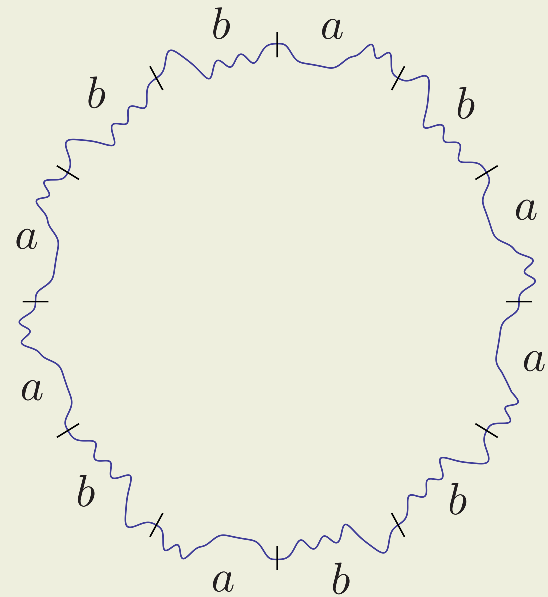


# What is the Symmetry Groupoid?

**Cogwheels** $\Longrightarrow$ Musso–Nicoldi




$G_S = \mathbb{Z}_6$




$G_S = \mathbb{Z}_2$

**Geometry = Group Theory**

Felix Klein's Erlanger Programm (1872):

Each type of geometry is founded on
an underlying transformation group.

# Plane Geometries/Groups

## Euclidean geometry:

SE(2) — rigid motions (rotations and translations)

$$\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

E(2) — plus reflections?

## Equi-affine geometry:

SA(2) — area-preserving affine transformations:

$$\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \qquad \alpha\,\delta - \beta\,\gamma = 1$$

## Projective geometry:

PSL(3) — projective transformations:

$$\overline{x} = \frac{\alpha\,x + \beta\,y + \gamma}{\rho\,x + \sigma\,y + \tau} \qquad \overline{y} = \frac{\lambda\,x + \mu\,y + \nu}{\rho\,x + \sigma\,y + \tau}$$

# The Equivalence Problem

$\Longrightarrow$ É Cartan

$G$ — transformation group acting on $M$

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## Equivalence:

Determine when two subsets

$$S \quad \text{and} \quad \overline{S} \subset M$$

are congruent:

$$\overline{S} = g \cdot S \qquad \text{for} \qquad g \in G$$

## Symmetry:

Find all symmetries or self-congruences:

$$S = g \cdot S$$

# Tennis, Anyone?

# Invariants

The solution to an equivalence problem rests on understanding its invariants.

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**Definition.** If $G$ is a group acting on $M$, then an invariant is a real-valued function $I : M \to \mathbb{R}$ that does not change under the action of $G$:

$$I(g \cdot z) = I(z) \qquad \text{for all} \qquad g \in G, \quad z \in M$$

---

★ If $G$ acts transtively, there are no (non-constant) invariants.

# Differential Invariants

Given a submanifold (curve, surface, . . . )

$$S \subset M$$

a differential invariant is an invariant of the prolonged action of $G$ on its Taylor coefficients (jets):

$$\boxed{I(g \cdot z^{(k)}) \;=\; I(z^{(k)})}$$

# Euclidean Plane Curves

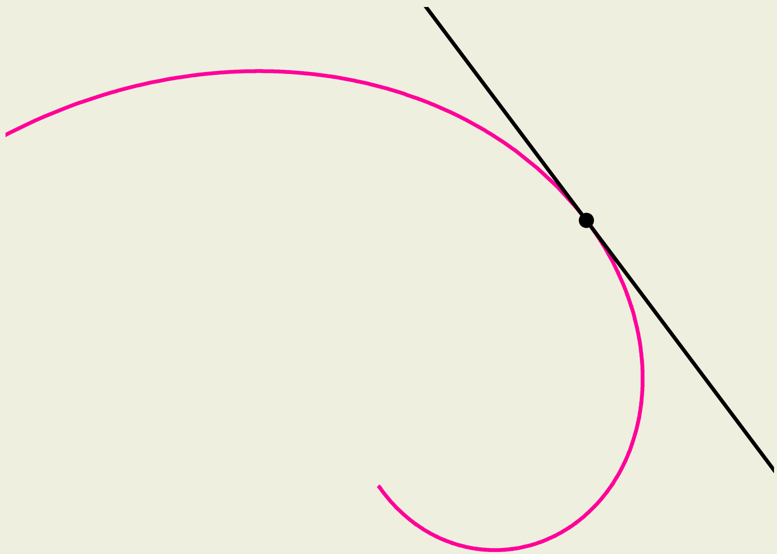
$$G = \mathrm{SE}(2) \qquad \text{acts on curves} \qquad C \subset M = \mathbb{R}^2$$

The simplest differential invariant is the curvature, defined as the reciprocal of the radius of the osculating circle:

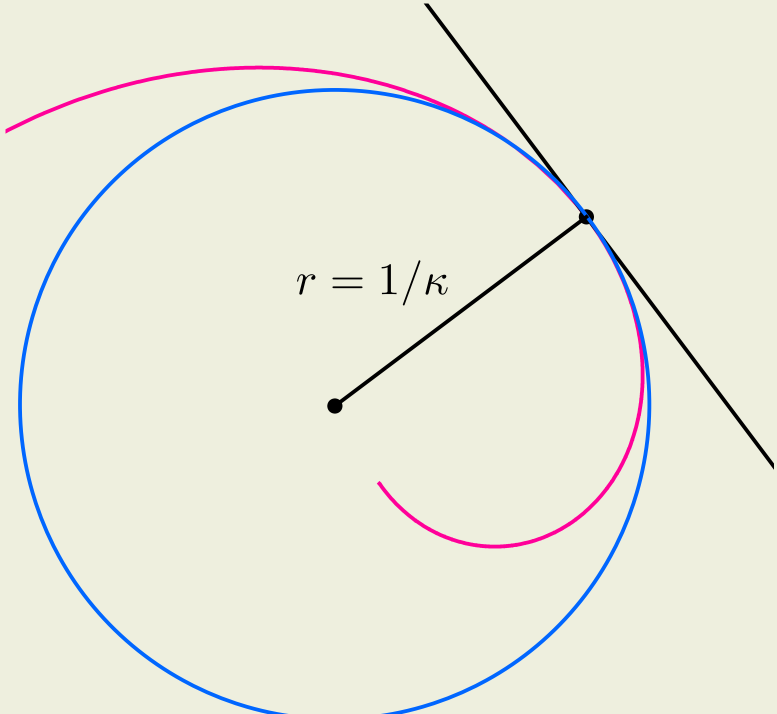
$$\kappa = \frac{1}{r}$$

# Curvature

# Curvature



# Curvature



# Euclidean Plane Curves: $G = \mathrm{SE}(2)$

Differentiation with respect to the Euclidean-invariant arc length element $ds$ is an invariant differential operator, meaning that it maps differential invariants to differential invariants.

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Thus, starting with curvature $\kappa$, we can generate an infinite collection of higher order Euclidean differential invariants:

$$\kappa, \quad \frac{d\kappa}{ds}, \quad \frac{d^2\kappa}{ds^2}, \quad \frac{d^3\kappa}{ds^3}, \quad \cdots$$

---

**Theorem.** All Euclidean differential invariants are functions of the derivatives of curvature with respect to arc length: $\kappa, \ \kappa_s, \ \kappa_{ss}, \ \cdots$

# Euclidean Plane Curves: $G = \mathrm{SE}(2)$

Assume the curve $C \subset M$ is a graph: $\quad y = u(x)$

Differential invariants:

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}, \qquad \frac{d\kappa}{ds} = \frac{(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1+u_x^2)^3}, \qquad \frac{d^2\kappa}{ds^2} = \cdots$$

Arc length (invariant one-form):

$$ds = \sqrt{1+u_x^2}\; dx, \qquad \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\,\frac{d}{dx}$$

## Equi-affine Plane Curves: $G = \mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$

Equi-affine curvature:

$$\kappa = \frac{5\, u_{xx} u_{xxxx} - 3\, u_{xxx}^2}{9\, u_{xx}^{8/3}} \qquad \frac{d\kappa}{ds} = \cdots$$

Equi-affine arc length:

$$ds = \sqrt[3]{u_{xx}}\; dx \qquad \frac{d}{ds} = \frac{1}{\sqrt[3]{u_{xx}}}\, \frac{d}{dx}$$

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**Theorem.** All equi-affine differential invariants are functions of the derivatives of equi-affine curvature with respect to equi-affine arc length: $\kappa, \quad \kappa_s, \quad \kappa_{ss}, \quad \cdots$

# Plane Curves

**Theorem.** Let $G$ be an ordinary$^\star$ Lie group acting on $M = \mathbb{R}^2$. Then for curves $C \subset M$, there exists a unique (up to functions thereof) lowest order differential invariant $\kappa$ and a unique (up to constant multiple) invariant differential form $ds$. Every other differential invariant can be written as a function of the "curvature" invariant and its derivatives with respect to "arc length": $\kappa, \quad \kappa_s, \quad \kappa_{ss}, \quad \cdots \quad .$

$^\star$ ordinary = transitive + no pseudo-stabilization.

# Moving Frames

The equivariant method of moving frames provides a systematic and algorithmic calculus for determining complete systems of differential invariants, invariant differential forms, invariant differential operators, etc., and the structure of the non-commutative differential algebra they generate.

# Equivalence & Invariants

- Equivalent submanifolds $S \approx \overline{S}$ must have the same invariants: $I = \overline{I}$.

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Constant invariants provide immediate information:

$$\text{e.g.} \qquad \kappa = 2 \quad \Longleftrightarrow \quad \overline{\kappa} = 2$$

Non-constant invariants are not useful in isolation, because an equivalence map can drastically alter the dependence on the submanifold parameters:

$$\text{e.g.} \qquad \kappa = x^3 \qquad \text{versus} \qquad \overline{\kappa} = \sinh x$$

# Syzygies

However, a functional dependency or syzygy among the invariants *is* intrinsic:

$$\text{e.g.} \qquad \kappa_s = \kappa^3 - 1 \quad \Longleftrightarrow \quad \overline{\kappa}_{\overline{s}} = \overline{\kappa}^3 - 1$$

- Universal syzygies — Gauss–Codazzi
- Distinguishing syzygies.

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**Theorem.** (Cartan)
Two regular submanifolds are locally equivalent if and only if they have identical syzygies among *all* their differential invariants.

# Finiteness of Generators and Syzygies

♠ There are, in general, an infinite number of differential invariants and hence an infinite number of syzygies must be compared to establish equivalence.

♡ But the higher order differential invariants are always generated by invariant differentiation from a finite collection of basic differential invariants, and the higher order syzygies are all consequences of a finite number of low order syzygies!

## **Example — Plane Curves**

If non-constant, both $\kappa$ and $\kappa_s$ depend on a single parameter, and so, locally, are subject to a syzygy:

$$\kappa_s = H(\kappa) \qquad (*)$$

But then

$$\kappa_{ss} = \frac{d}{ds} H(\kappa) = H'(\kappa)\,\kappa_s = H'(\kappa)\,H(\kappa)$$

and similarly for $\kappa_{sss}$, etc.

Consequently, all the higher order syzygies are generated by the fundamental first order syzygy $(*)$.

Thus, for Euclidean (or equi-affine or projective or . . . ) plane curves we need only know a single syzygy between $\kappa$ and $\kappa_s$ in order to establish equivalence!

# Signature Curves

**Definition.** The signature curve $\Sigma \subset \mathbb{R}^2$ of a plane curve $C \subset \mathbb{R}^2$ is parametrized by the two lowest order differential invariants

$$\chi \; : \; C \longrightarrow \Sigma = \left\{ \left( \kappa \, , \, \frac{d\kappa}{ds} \right) \right\} \subset \mathbb{R}^2$$

$\Longrightarrow$ *Calabi, PJO, Shakiban, Tannenbaum, Haker*

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**Theorem.** Two regular curves $C$ and $\overline{C}$ are locally equivalent:

$$\overline{C} = g \cdot C$$

if and only if their signature curves are identical:

$$\overline{\Sigma} = \Sigma$$

$\Longrightarrow$ regular: $(\kappa_s, \kappa_{ss}) \neq 0$.

# Continuous Symmetries of Curves

**Theorem.** For a connected curve, the following are equivalent:

- All the differential invariants are constant on $C$:
$$\kappa = c, \qquad \kappa_s = 0, \qquad \ldots$$
- The signature $\Sigma$ degenerates to a point: $\dim \Sigma = 0$
- $C$ is a piece of an orbit of a 1-dimensional subgroup $H \subset G$
- $C$ admits a one-dimensional local symmetry group

# Discrete Symmetries of Curves

**Definition.** The index of a completely regular point $\zeta \in \Sigma$ equals the number of points in $C$ which map to it:

$$i_\zeta = \# \, \chi^{-1}\{\zeta\}$$

Regular means that, in a neighborhood of $\zeta$, the signature is an embedded curve — no self-intersections.

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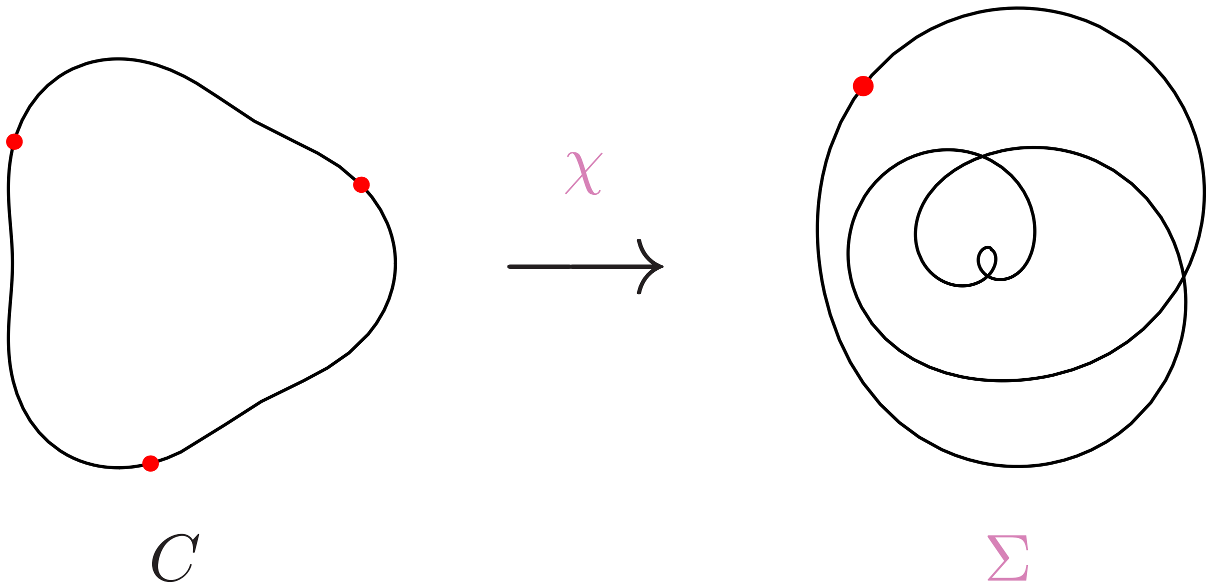
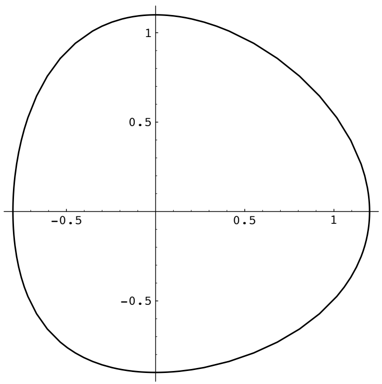
**Theorem.** If $\chi(z) = \zeta$ is completely regular, then its index counts the number of discrete local symmetries of $C$.

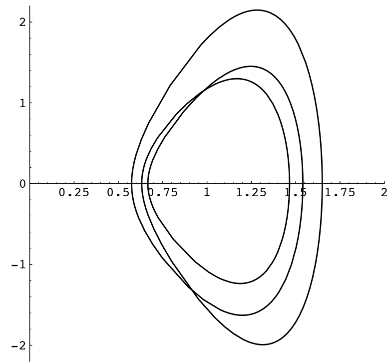
# The Index



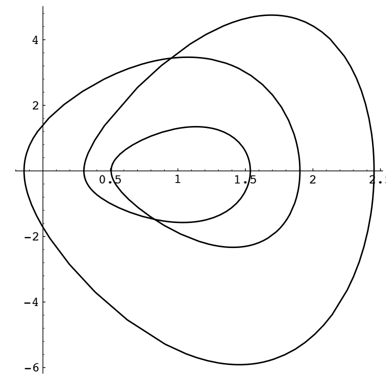
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \;\; y = \sin t + \frac{1}{10}\sin^2 t$



The Original Curve

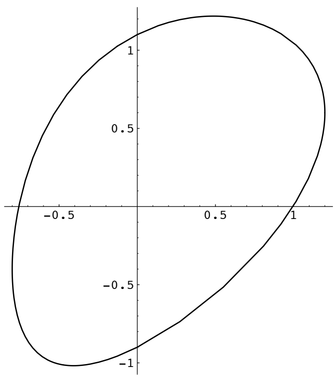


Euclidean Signature

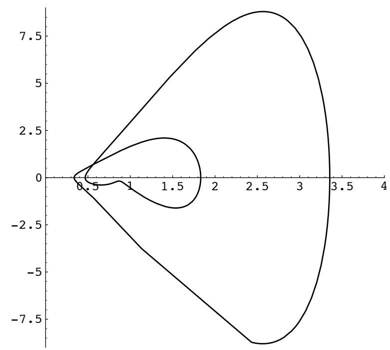


Equi-affine Signature

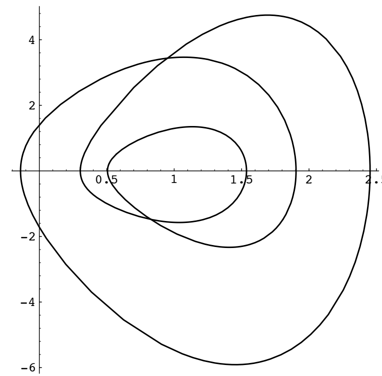
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \;\; y = \frac{1}{2}x + \sin t + \frac{1}{10}\sin^2 t$



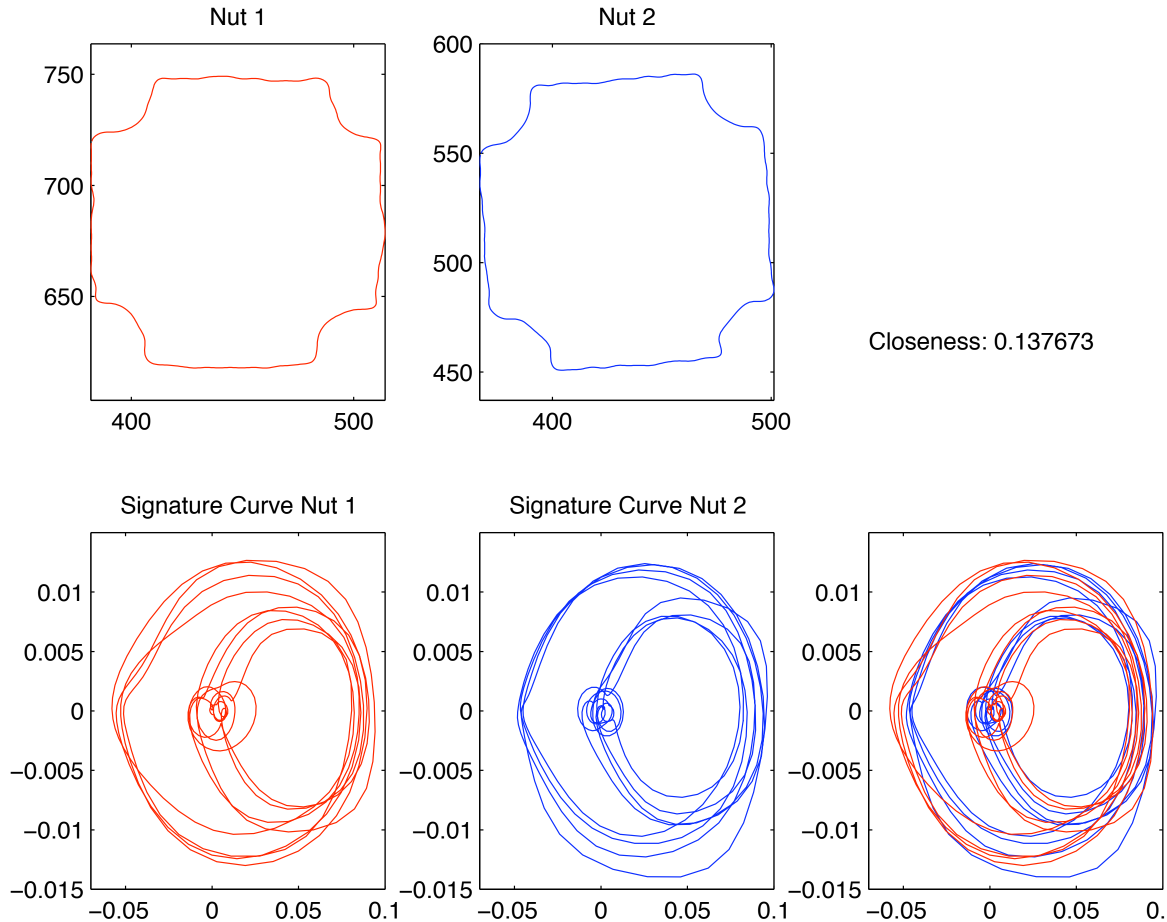

The Original Curve

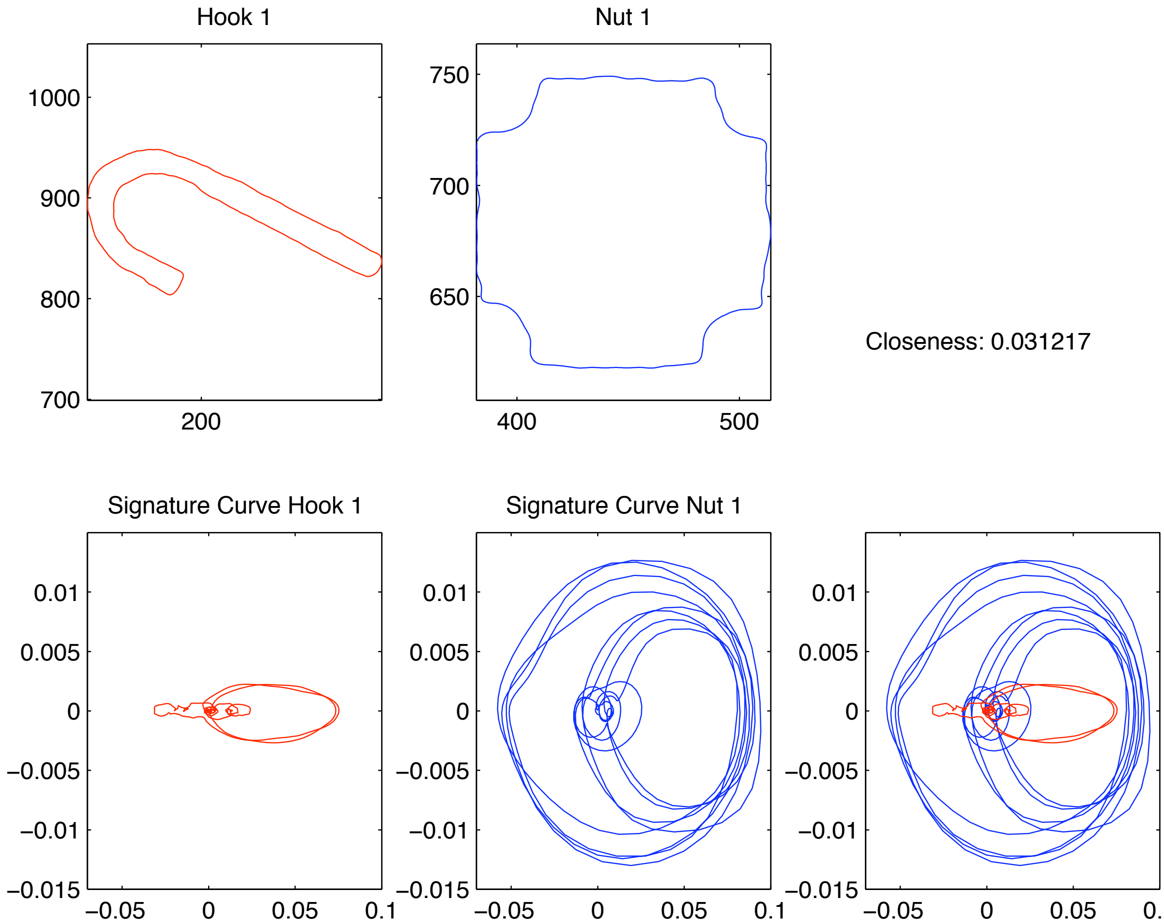



Euclidean Signature

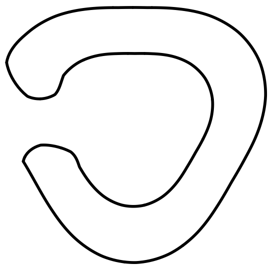



Equi-affine Signature


Nut 1
Nut 2
Closeness: 0.137673
Signature Curve Nut 1
Signature Curve Nut 2
750
700
650
400
500
600
550
500
450
400
500
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1

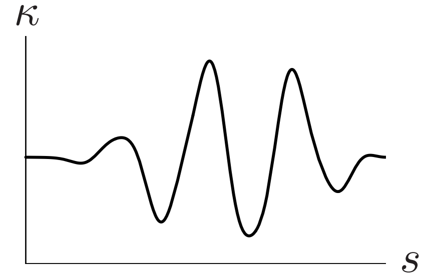

Hook 1
Nut 1
Closeness: 0.031217
Signature Curve Hook 1
Signature Curve Nut 1
1000
900
800
700
200
750
700
650
400
500
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1

# Signatures

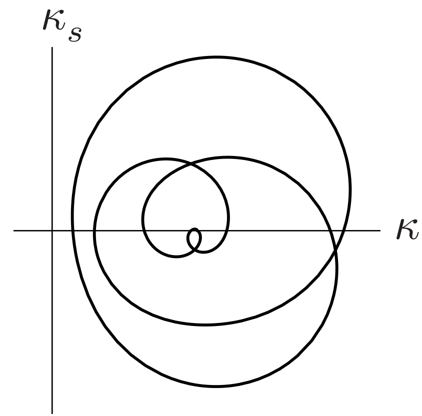


Original curve

$\longrightarrow$

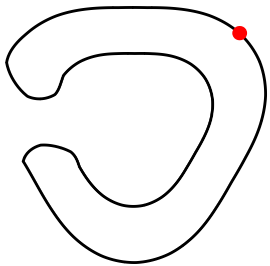



Classical Signature



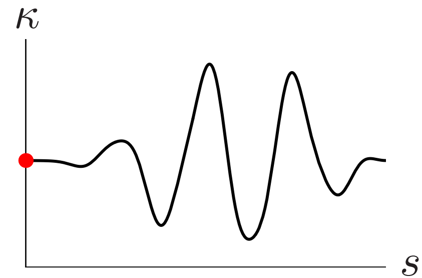

Differential invariant signature

# Signatures

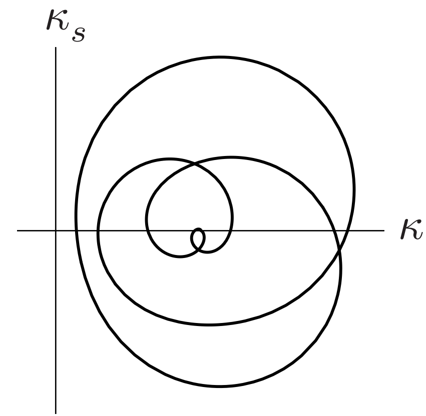


Original curve

$\longrightarrow$

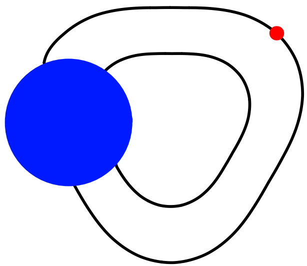



Classical Signature

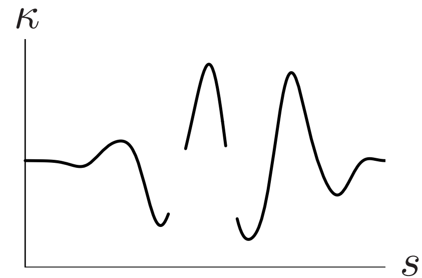



Differential invariant signature

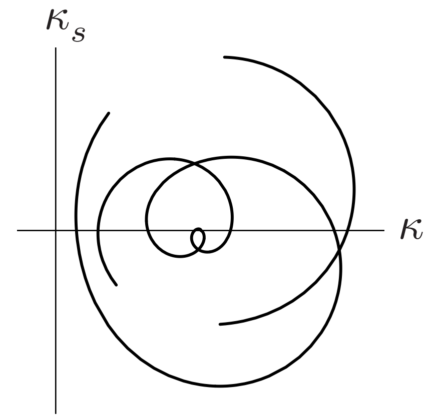
# Occlusions



Original curve




Classical Signature




Differential invariant signature

# 3D Differential Invariant Signatures

**Euclidean space curves:** $C \subset \mathbb{R}^3$

$$\Sigma = \{ \, ( \, \kappa \, , \, \kappa_s \, , \, \tau \, ) \, \} \; \subset \; \mathbb{R}^3$$

- $\kappa$ — curvature, $\tau$ — torsion

---

**Euclidean surfaces:** $S \subset \mathbb{R}^3$ (generic)

$$\Sigma = \left\{ \left( H \, , \, K \, , \, H_{,1} \, , \, H_{,2} \, , \, K_{,1} \, , \, K_{,2} \right) \right\} \subset \mathbb{R}^6$$

or $$\widehat{\Sigma} = \left\{ \left( H \, , \, H_{,1} \, , \, H_{,2} \, , \, H_{,11} \right) \right\} \subset \mathbb{R}^4$$

- $H$ — mean curvature, $K$ — Gauss curvature

---

**Equi–affine surfaces:** $S \subset \mathbb{R}^3$ (generic)

$$\Sigma = \left\{ \left( P \, , \, P_{,1} \, , \, P_{,2} , \, P_{,11} \right) \right\} \subset \mathbb{R}^4$$

- $P$ — Pick invariant

# Vertices of Euclidean Curves

Ordinary vertex: local extremum of curvature

Generalized vertex: $\kappa_s \equiv 0$

- critical point
- circular arc
- straight line segment

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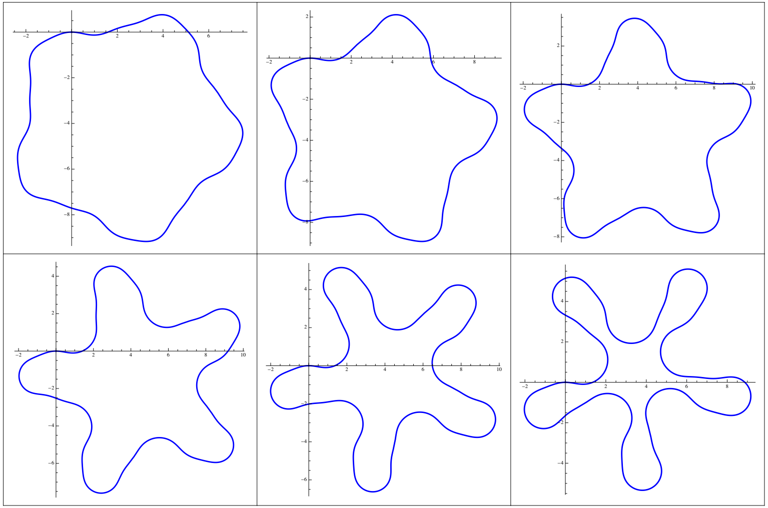
**Mukhopadhya's Four Vertex Theorem**:

A simple closed, non-circular plane curve has $n \geq 4$ generalized vertices.

# “Counterexamples”

⋆ Generalized vertices map to a single point of the signature.

Hence, the (degenerate) curves obtained by replace ordinary vertices with circular arcs of the same radius all have *identical* signature:



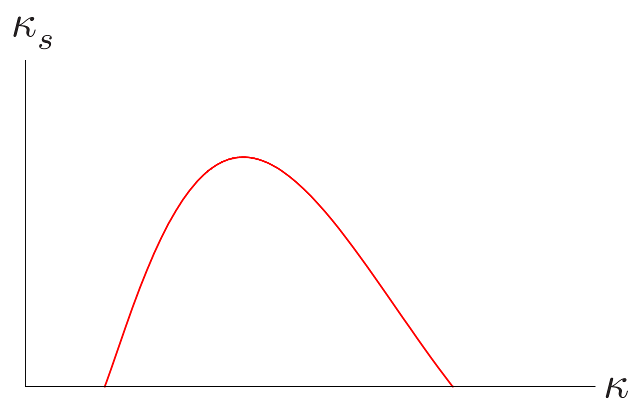
$\Longrightarrow$ Musso–Nicoldi

# Bivertex Arcs

Bivertex arc: $\kappa_s \neq 0$ everywhere on the arc $B \subset C$
*except* $\kappa_s = 0$ at the two endpoints

---

The signature $\Sigma = \chi(B)$ of a bivertex arc is a single arc that starts and ends on the $\kappa$–axis.



# Bivertex Decomposition

v-regular curve — finitely many generalized vertices

$$C = \bigcup_{j=1}^{m} B_j \cup \bigcup_{k=1}^{n} V_k$$

$B_1, \ldots, B_m$ — bivertex arcs

$V_1, \ldots, V_n$ — generalized vertices: $n \geq 4$

---

Main Idea: Compare individual bivertex arcs, and then decide whether the rigid equivalences are (approximately) the same.

D. Hoff & PJO, Extensions of invariant signatures for object recognition, *J. Math. Imaging Vision* **45** (2013), 176–185.

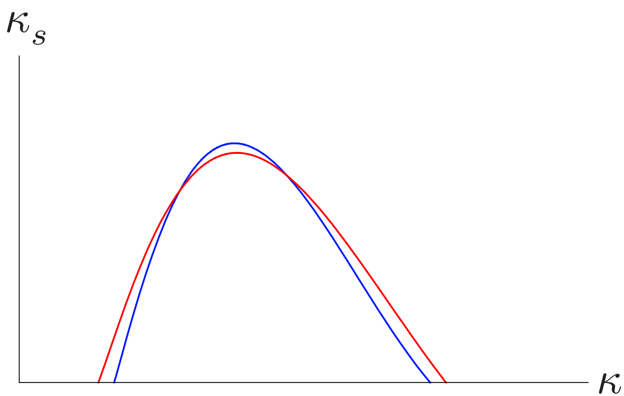
# Signature Metrics

Used to compare signatures:

- Hausdorff
- Monge–Kantorovich transport
- Electrostatic/gravitational attraction
- Latent semantic analysis
- Histograms
- Geodesic distance
- Diffusion metric
- Gromov–Hausdorff & Gromov–Wasserstein

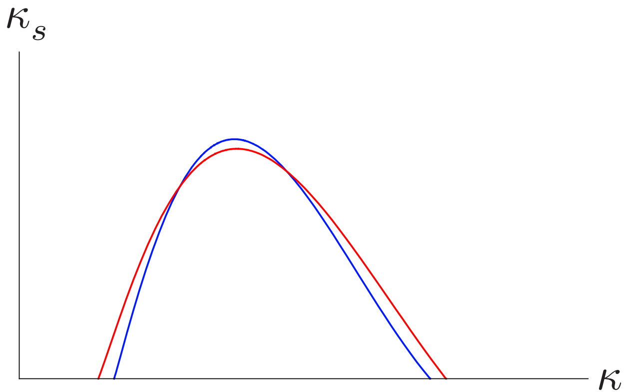
# Gravitational/Electrostatic Attraction

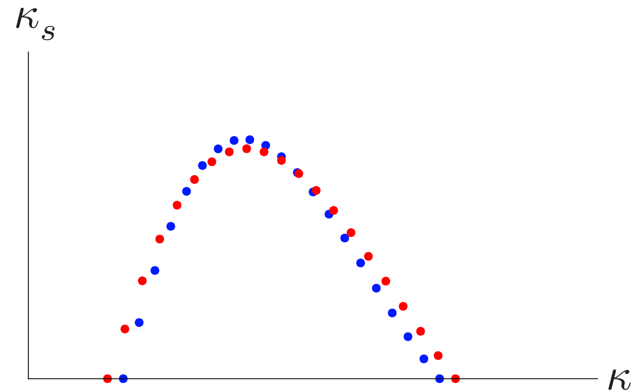
- ♡ Treat the two signature curves as masses or as oppositely charged wires. The higher their mutual attraction, the closer they are together.



# Gravitational/Electrostatic Attraction

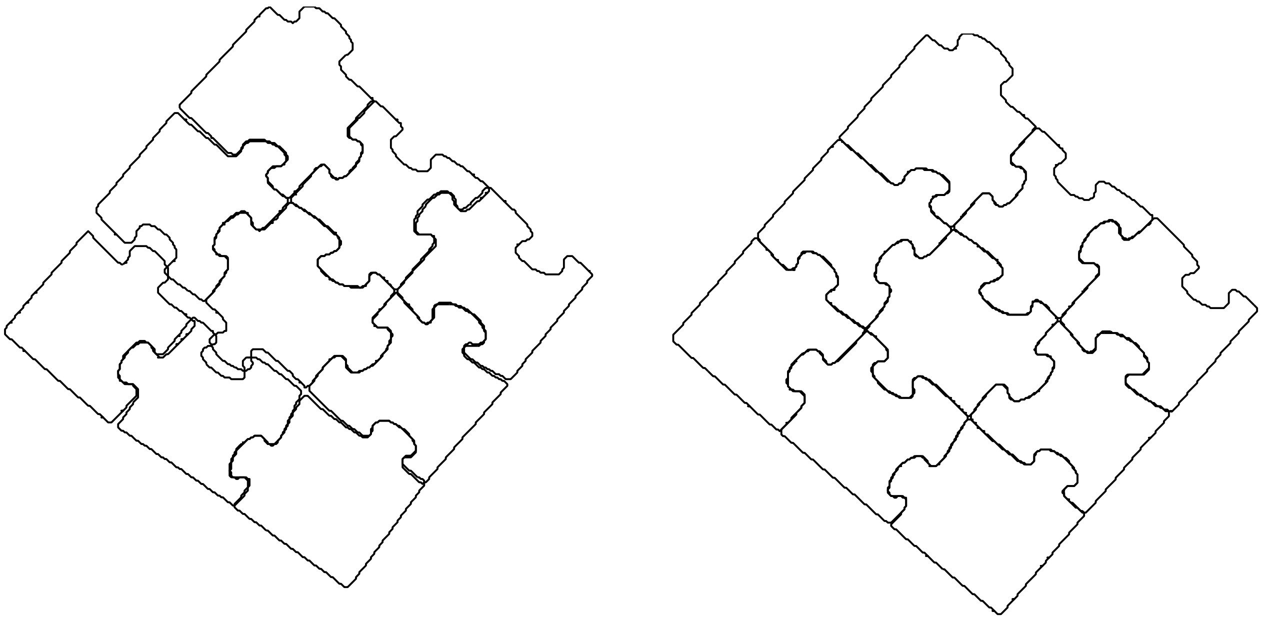
- ♡ Treat the two signature curves as masses or as oppositely charged wires. The higher their mutual attraction, the closer they are together.
- ♠ In practice, we are dealing with discrete data (pixels) and so treat the curves and signatures as point masses/charges.



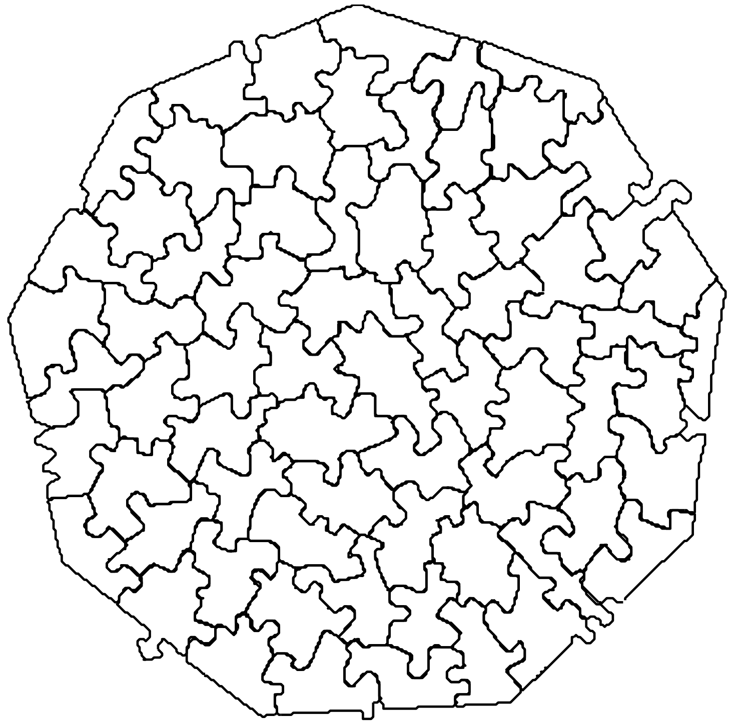
# The Baffler Jigsaw Puzzle

# Piece Locking

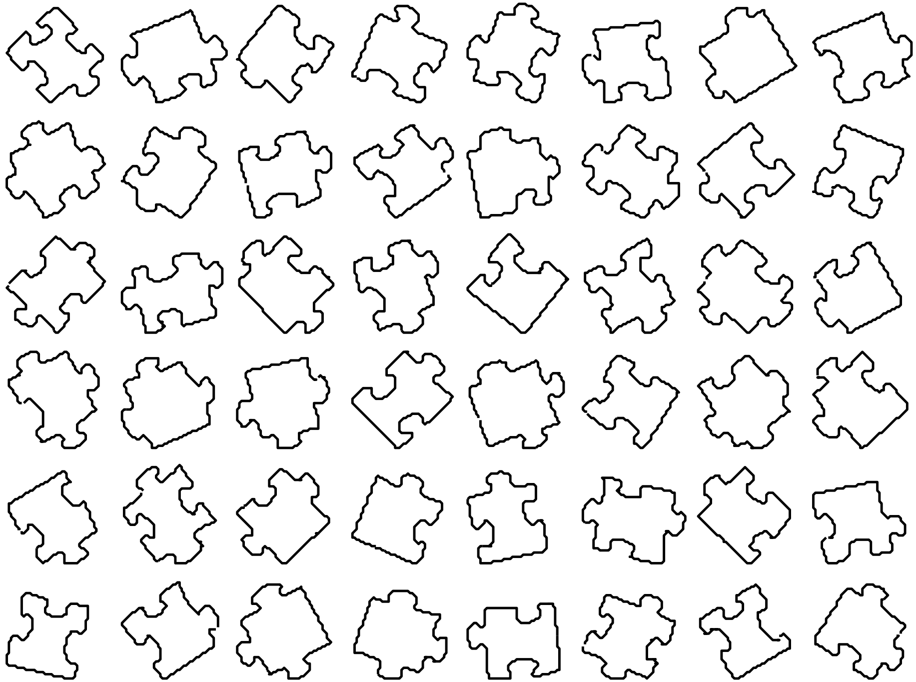


⋆⋆ Minimize force and torque based on gravitational attraction of the two matching edges.

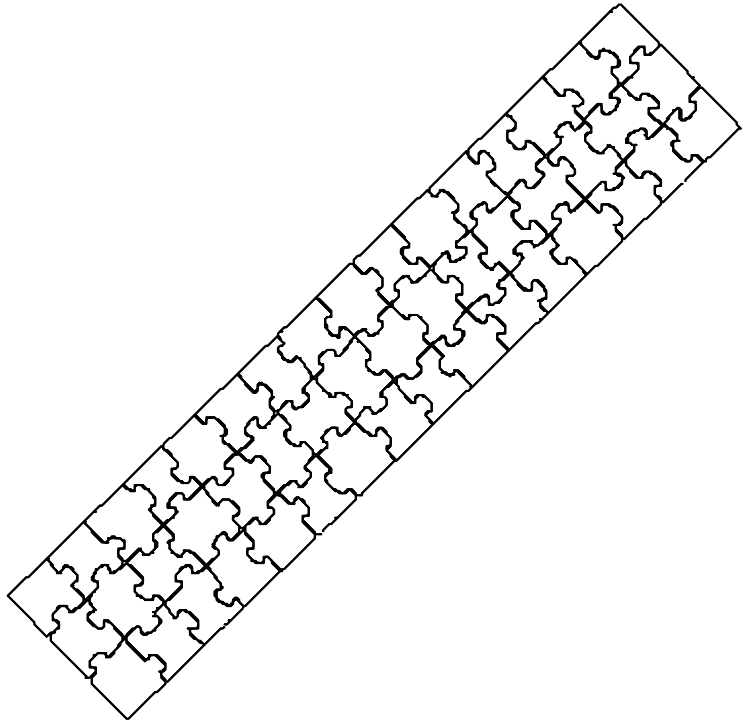
# The Baffler Solved



# The Rain Forest Giant Floor Puzzle

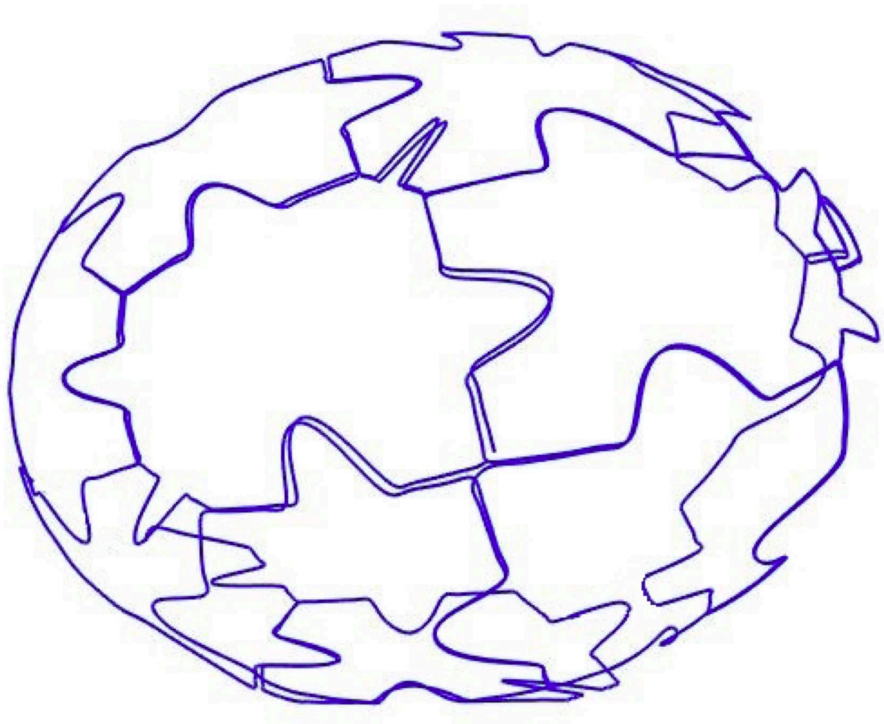


# The Rain Forest Puzzle Solved



⟹ D. Hoff & PJO, Automatic solution of jigsaw puzzles,
*J. Math. Imaging Vision* **49** (2014) 234–250.

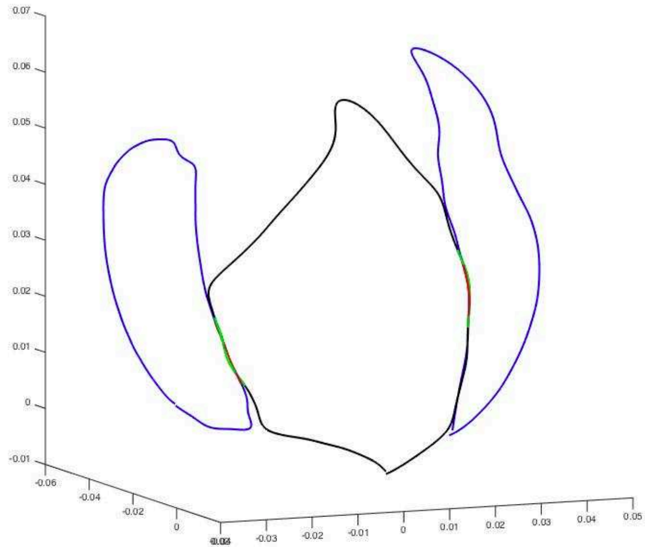
# 3D Jigsaw Puzzles



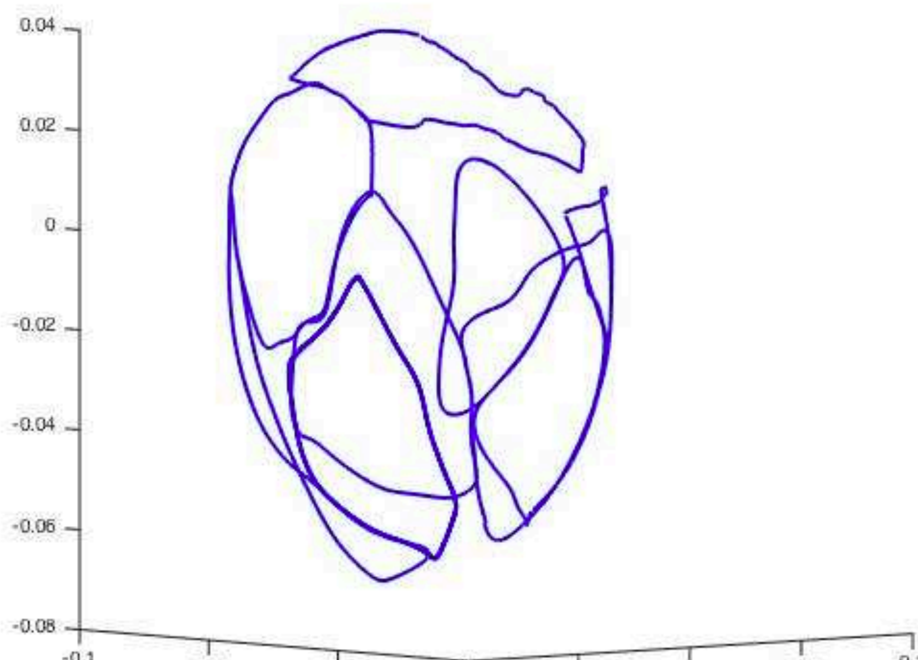
$\Longrightarrow$ Anna Grim, Tim O'Connor, Ryan Schlecta

# Broken Ostrich Egg Shell

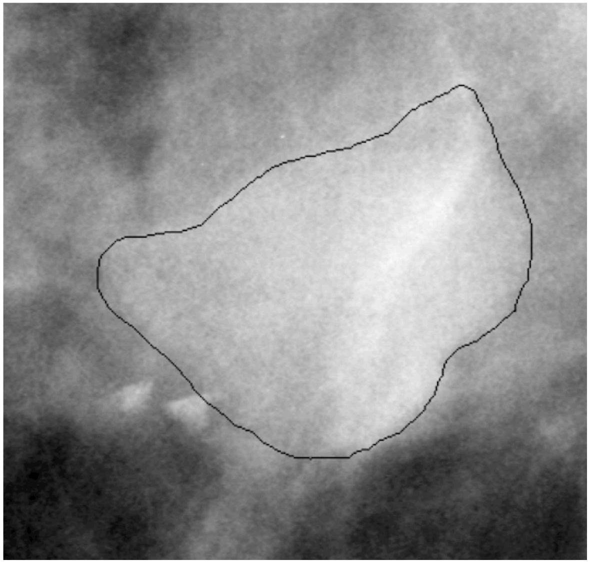
$\Longrightarrow$ Marshall Bern

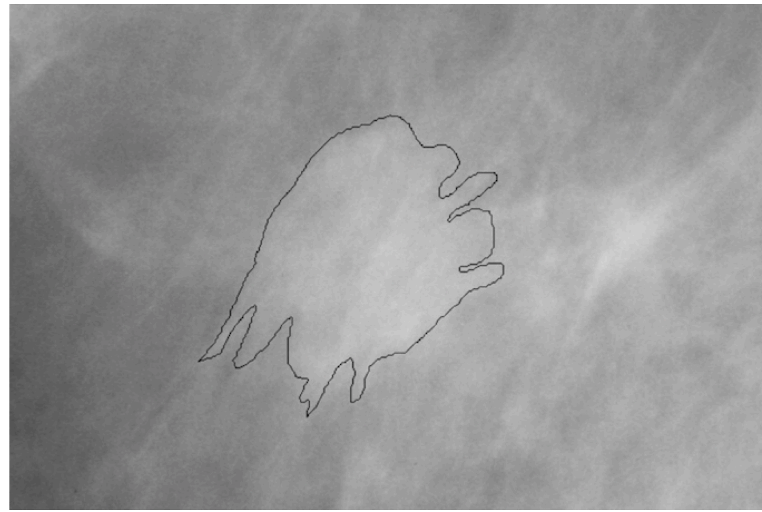


# Reassembling Humpty Dumpty



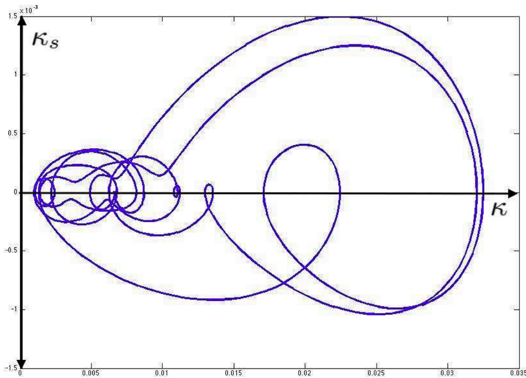
# Benign vs. Malignant Tumors

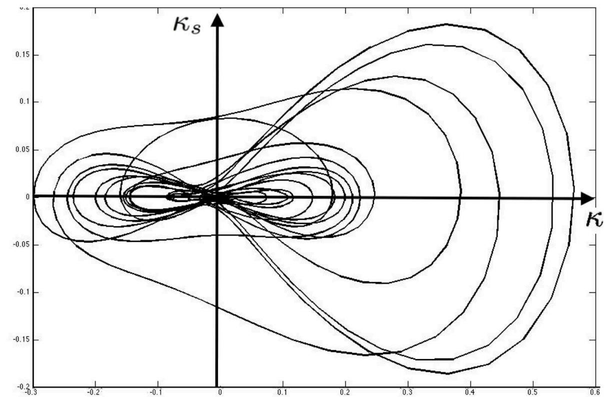




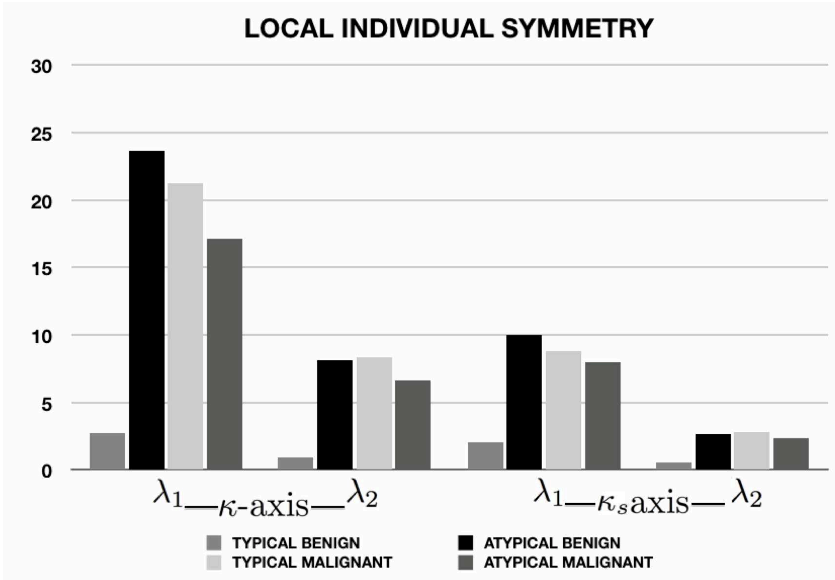
$\Longrightarrow$ A. Grim, C. Shakiban

# Benign vs. Malignant Tumors



# Benign vs. Malignant Tumors



# Joint Invariant Signatures

If the invariants depend on $k$ points on a $p$-dimensional submanifold, then you need at least

$$\ell > k\,p$$

distinct invariants $I_1, \ldots, I_\ell$ in order to construct a syzygy. Typically, the number of joint invariants is

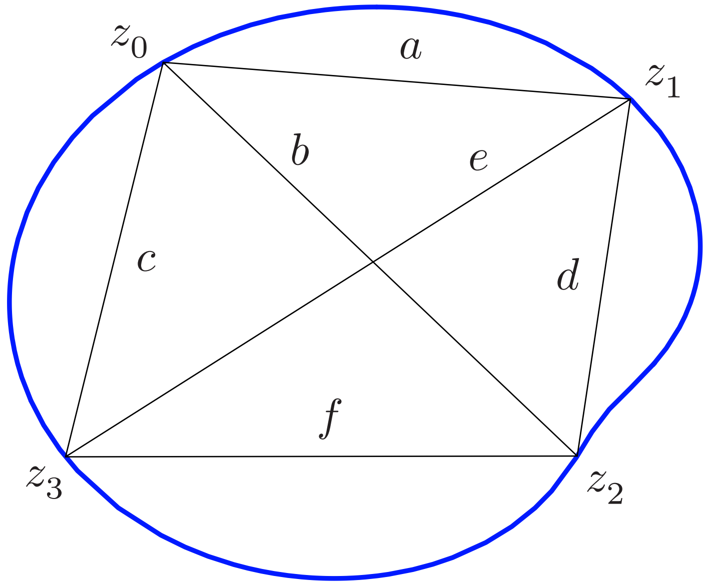
$$\ell = k\,m - r = (\#\text{points})\,(\dim M) - \dim G$$

Therefore, a purely joint invariant signature requires at least

$$k \;\geq\; \frac{r}{m-p} + 1$$

points on our $p$-dimensional submanifold $N \subset M$.

# Joint Euclidean Signature



Joint signature map:

$$\Sigma : \mathcal{C}^{\times 4} \longrightarrow \Sigma \subset \mathbb{R}^6$$

$$a = \| z_0 - z_1 \| \qquad b = \| z_0 - z_2 \| \qquad c = \| z_0 - z_3 \|$$

$$d = \| z_1 - z_2 \| \qquad e = \| z_1 - z_3 \| \qquad f = \| z_2 - z_3 \|$$

$\Longrightarrow$ six functions of four variables

Syzygies:

$$\Phi_1(a, b, c, d, e, f) = 0 \qquad \Phi_2(a, b, c, d, e, f) = 0$$

---

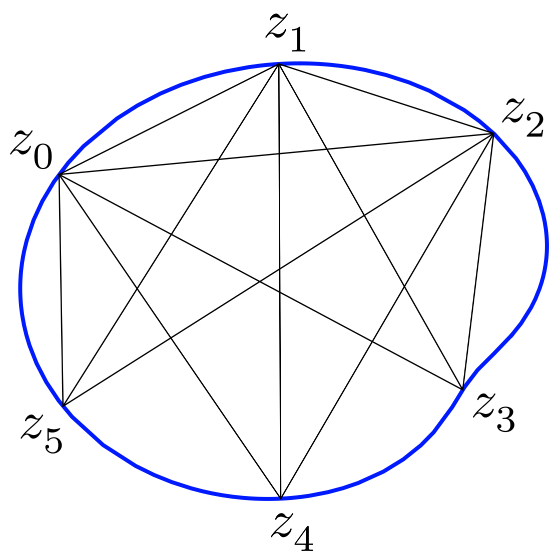
Universal Cayley–Menger syzygy $\iff$ $\mathcal{C} \subset \mathbb{R}^2$

$$\det \begin{vmatrix} 2\,a^2 & a^2 + b^2 - d^2 & a^2 + c^2 - e^2 \\ a^2 + b^2 - d^2 & 2\,b^2 & b^2 + c^2 - f^2 \\ a^2 + c^2 - e^2 & b^2 + c^2 - f^2 & 2\,c^2 \end{vmatrix} = 0$$

# Joint Equi–Affine Signature

Requires 7 triangular areas:

$$[\,0\;1\;2\,],\ [\,0\;1\;3\,],\ [\,0\;1\;4\,],\ [\,0\;1\;5\,],\ [\,0\;2\;3\,],\ [\,0\;2\;4\,],\ [\,0\;2\;5\,]$$



# Joint Invariant Signatures

- The joint invariant signature subsumes other signatures, but resides in a higher dimensional space and contains a lot of redundant information.
- Identification of landmarks can significantly reduce the redundancies (Boutin)
- It includes the differential invariant signature and semi-differential invariant signatures as its "coalescent boundaries".
- Invariant numerical approximations to differential invariants and semi-differential invariants are constructed (using moving frames) near these coalescent boundaries.

# Statistical Sampling

Idea: Replace high dimensional joint invariant signatures by increasingly dense point clouds obtained by multiply sampling the original submanifold.

- The equivalence problem requires direct comparison of signature point clouds.
- Continuous symmetry detection relies on determining the underlying dimension of the signature point clouds.
- Discrete symmetry detection relies on determining densities of the signature point clouds.

# Invariant Histograms

★ To eliminate noise, use histograms based on joint invariants.

**Definition.** The distance histogram of a finite set of points $P = \{z_1, \ldots, z_n\} \subset V$ is the function

$$\eta_P(r) = \#\left\{ (i,j) \;\middle|\; 1 \le i < j \le n,\ d(z_i, z_j) = r \right\}.$$

Brinkman, D., & PJO, Invariant histograms, *Amer. Math. Monthly* **118** (2011) 2–24.

# The Distance Set

The support of the histogram function,

$$\operatorname{supp} \eta_P = \Delta_P \subset \mathbb{R}^+$$

is the distance set of $P$.

---

Erdös' distinct distances conjecture (1946):

If $P \subset \mathbb{R}^m$, then $\# \Delta_P \geq c_{m,\varepsilon} (\# P)^{2/m-\varepsilon}$

# Characterization of Point Sets

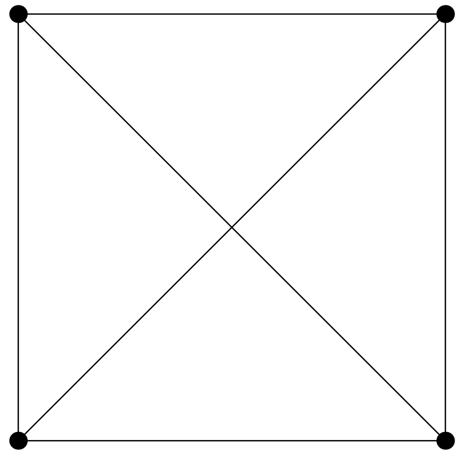
*Note*: If $\widetilde{P} = g \cdot P$ is obtained from $P \subset \mathbb{R}^m$ by a rigid motion $g \in \mathrm{E}(n)$, then they have the same distance histogram: $\eta_P = \eta_{\widetilde{P}}$.

---

Question: Can one uniquely characterize, up to rigid motion, a set of points $P\{z_1, \ldots, z_n\} \subset \mathbb{R}^m$ by its distance histogram?

$\Longrightarrow$ Tinkertoy problem.

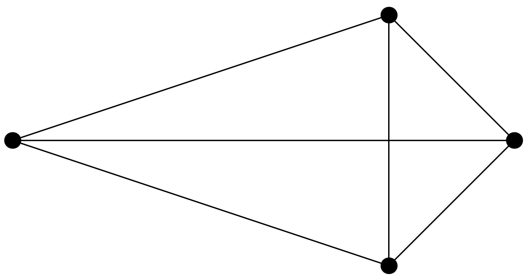
Yes:



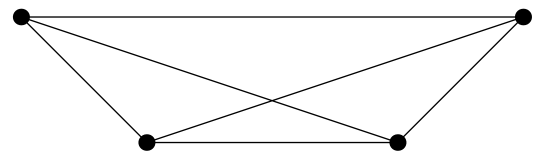
$$\eta = 1, \ 1, \ 1, \ 1, \ \sqrt{2}, \ \sqrt{2}.$$

No:

Kite



Trapezoid



$$\eta = \sqrt{2},\ \ \sqrt{2},\ \ 2,\ \ \sqrt{10},\ \ \sqrt{10},\ \ 4.$$

No:

$$\left.\begin{aligned} P &= \{0, 1, 4, 10, 12, 17\} \\ Q &= \{0, 1, 8, 11, 13, 17\} \end{aligned}\right. \subset \mathbb{R}$$

$$\eta = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17$$

$\implies$ G. Bloom, *J. Comb. Theory, Ser. A* **22** (1977) 378–379

**Theorem.** (*Boutin–Kemper*) Suppose $n \leq 3$ or $n \geq m+2$. Then there is a Zariski dense open subset in the space of $n$ point configurations in $\mathbb{R}^m$ that are uniquely characterized, up to rigid motion, by their distance histograms.

$\Longrightarrow$ M. Boutin, G. Kemper, *Adv. Appl. Math.* **32** (2004) 709–735

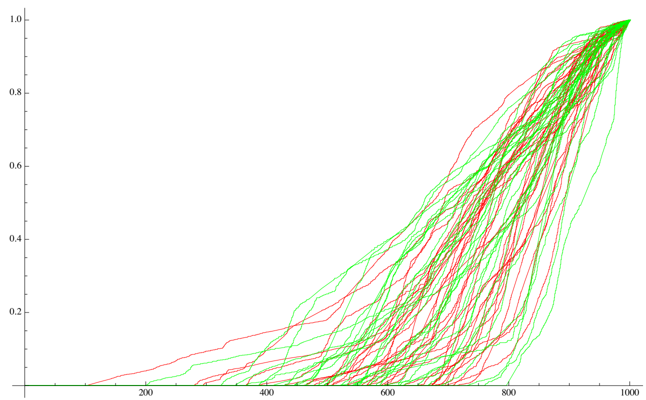
# **Distinguishing Melanomas from Moles**

Melanoma

Mole

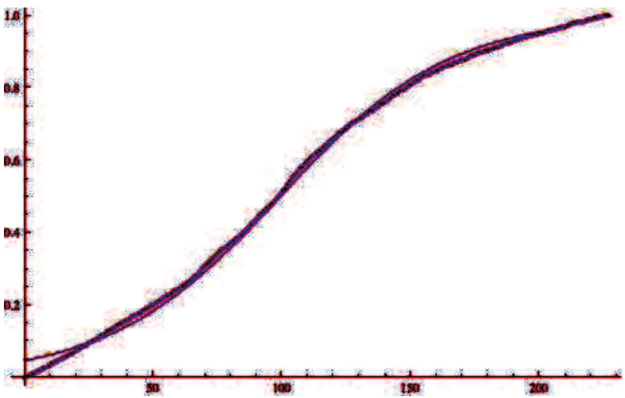
$\Longrightarrow$ A. Rodriguez, J. Stangl, C. Shakiban

# Cumulative Global Histograms

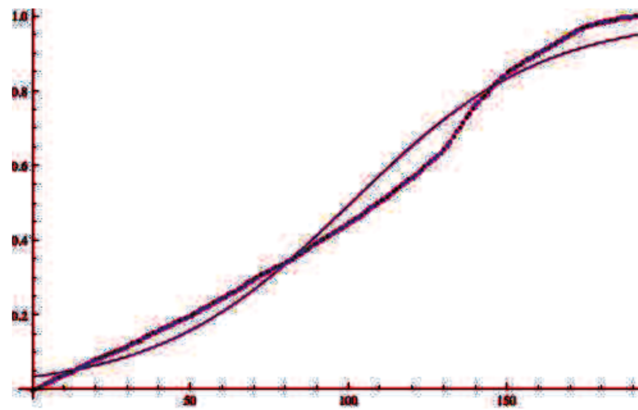



Red: melanoma Green: mole

# Logistic Function Fitting

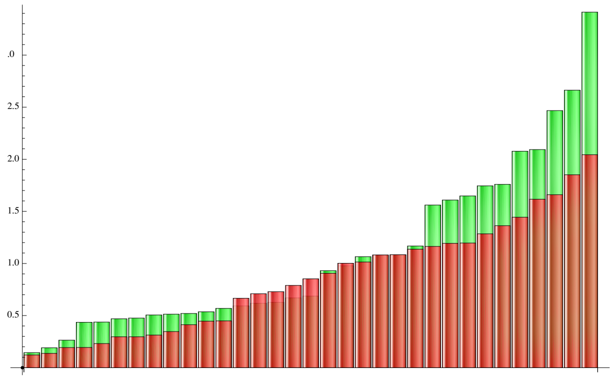



Melanoma




Mole

# Logistic Function Fitting — Residuals

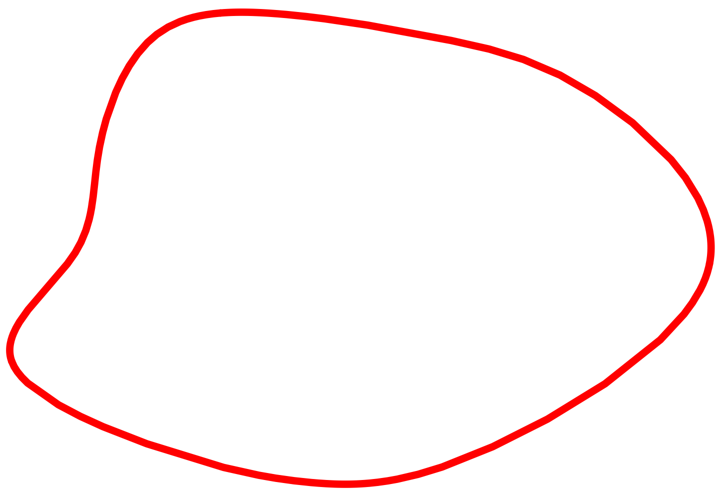



$$\text{Melanoma} = 17.1336 \ \pm \ 1.02253$$

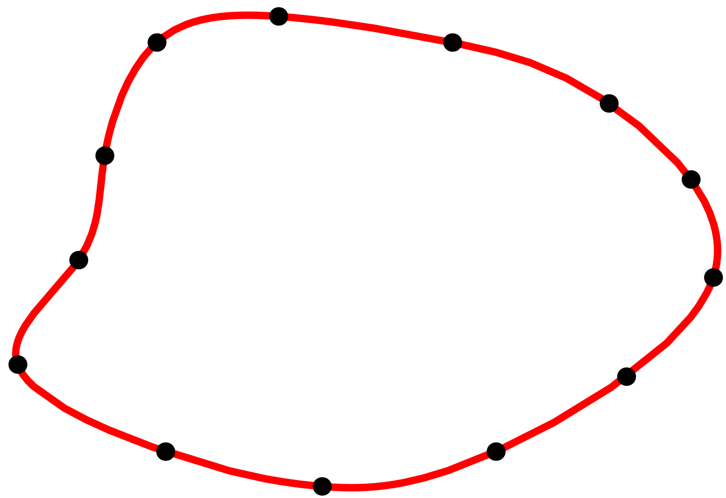
$$\text{Mole} = 19.5819 \ \pm \ 1.42892$$

58.7% Confidence

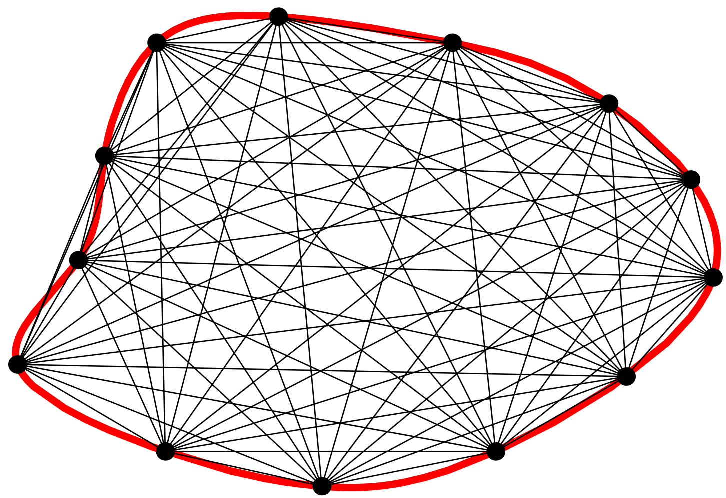
# Limiting Curve Histogram



# Limiting Curve Histogram



# Limiting Curve Histogram



# Sample Point Histograms

Cumulative distance histogram: $n = \#P$:

$$\Lambda_P(r) = \frac{1}{n} + \frac{2}{n^2} \sum_{s \leq r} \eta_P(s) = \frac{1}{n^2} \# \left\{ (i,j) \;\middle|\; d(z_i, z_j) \leq r \right\},$$

Note

$$\eta(r) = \tfrac{1}{2} n^2 [\, \Lambda_P(r) - \Lambda_P(r-\delta) \,] \qquad \delta \ll 1.$$

Local distance histogram:

$$\lambda_P(r,z) = \frac{1}{n} \# \left\{ j \;\middle|\; d(z, z_j) \leq r \right\} = \frac{1}{n} \#(P \cap B_r(z))$$

Ball of radius $r$ centered at $z$:

$$B_r(z) = \{ \, v \in V \mid d(v,z) \leq r \, \}$$

Note:

$$\Lambda_P(r) = \frac{1}{n} \sum_{z \in P} \lambda_P(r,z) = \frac{1}{n^2} \sum_{z \in P} \#(P \cap B_r(z)).$$

# Limiting Curve Histogram Functions

Length of a curve

$$l(C) = \int_C ds < \infty$$

Local curve distance histogram function $\quad z \in V$

$$h_C(r, z) = \frac{l(C \cap B_r(z))}{l(C)}$$

$\Longrightarrow$ The fraction of the curve contained in the ball of radius $r$ centered at $z$.

Global curve distance histogram function:

$$H_C(r) = \frac{1}{l(C)} \int_C h_C(r, z(s))\, ds.$$

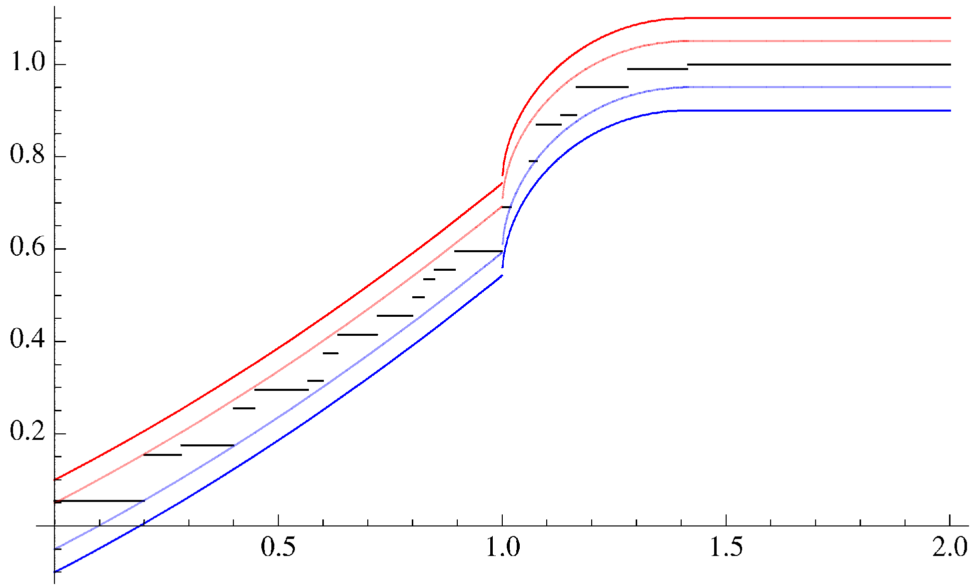
# Convergence

**Theorem.** Let $C$ be a regular plane curve. Then, for both uniformly spaced and randomly chosen sample points $P \subset C$, the cumulative local and global histograms converge to their continuous counterparts:

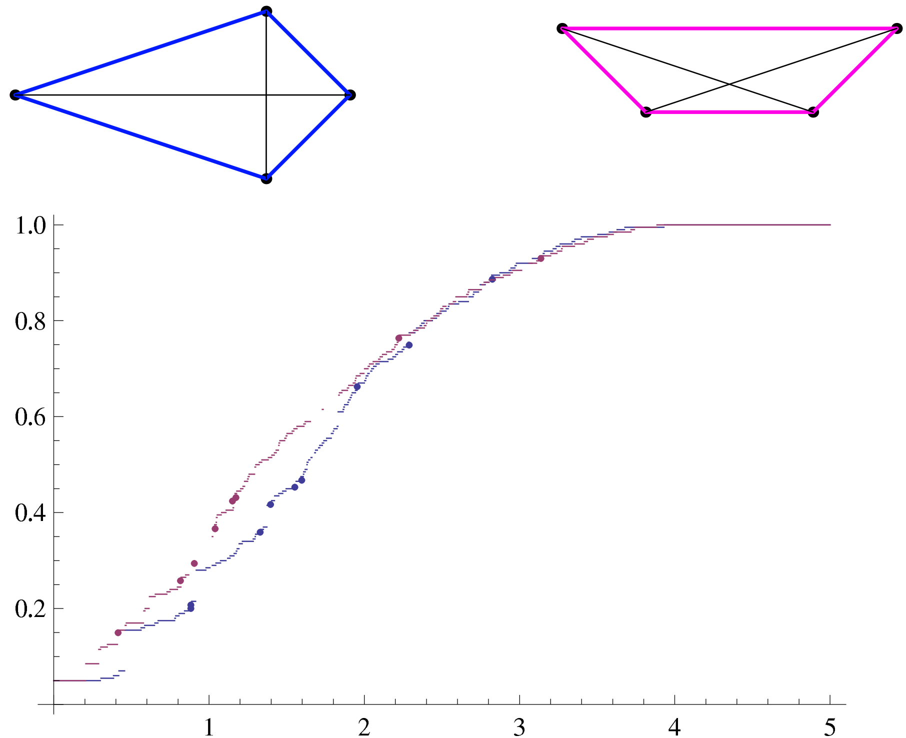
$$\lambda_P(r,z) \longrightarrow h_C(r,z), \qquad \Lambda_P(r) \longrightarrow H_C(r),$$

as the number of sample points goes to infinity.

# Square Curve Histogram with Bounds



# Kite and Trapezoid Curve Histograms



# Histogram–Based Shape Recognition

500 sample points

| Shape | $(a)$ | $(b)$ | $(c)$ | $(d)$ | $(e)$ | $(f)$ |
|---|---|---|---|---|---|---|
| (a) triangle | 2.3 | 20.4 | 66.9 | 81.0 | 28.5 | 76.8 |
| (b) square | 28.2 | .5 | 81.2 | 73.6 | 34.8 | 72.1 |
| (c) circle | 66.9 | 79.6 | .5 | 137.0 | 89.2 | 138.0 |
| (d) $2 \times 3$ rectangle | 85.8 | 75.9 | 141.0 | 2.2 | 53.4 | 9.9 |
| (e) $1 \times 3$ rectangle | 31.8 | 36.7 | 83.7 | 55.7 | 4.0 | 46.5 |
| (f) star | 81.0 | 74.3 | 139.0 | 9.3 | 60.5 | .9 |

# Curve Histogram Conjecture

Two sufficiently regular plane curves $C$ and $\widetilde{C}$ have identical global distance histogram functions, so $H_C(r) = H_{\widetilde{C}}(r)$ for all $r \geq 0$, if and only if they are rigidly equivalent: $C \simeq \widetilde{C}$.

# “Proof Strategies”

- Show that any polygon obtained from (densely) discretizing a curve does not lie in the Boutin–Kemper exceptional set.
- Polygons with obtuse angles: taking $r$ small, one can recover $(i)$ the set of angles and $(ii)$ the shortest side length from $H_C(r)$. Further increasing $r$ leads to further geometric information about the polygon ...
- Expand $H_C(r)$ in a Taylor series at $r = 0$ and show that the corresponding integral invariants characterize the curve.

# Taylor Expansions

Local distance histogram function:

$$L\, h_C(r,z) = 2\,r + \tfrac{1}{12}\,\kappa^2\, r^3 + \left( \tfrac{1}{40}\,\kappa\,\kappa_{ss} + \tfrac{1}{45}\,\kappa_s^2 + \tfrac{3}{320}\,\kappa^4 \right) r^5 + \ \cdots \ .$$

Global distance histogram function:

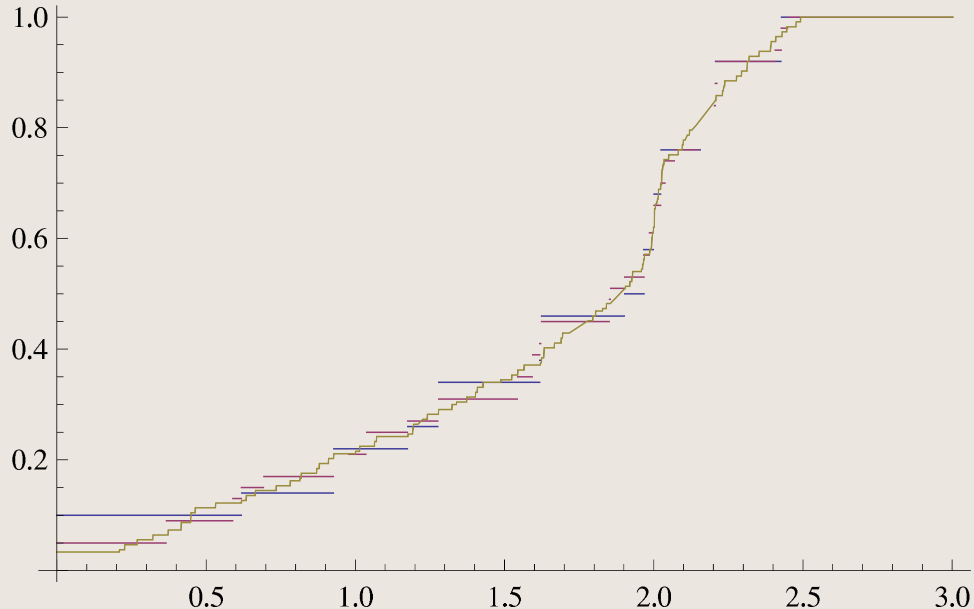
$$H_C(r) = \frac{2\,r}{L} + \frac{r^3}{12\,L^2} \oint_C \kappa^2\, ds + \frac{r^5}{40\,L^2} \oint_C \left( \tfrac{3}{8}\,\kappa^4 - \tfrac{1}{9}\,\kappa_s^2 \right) ds + \ \cdots \ .$$

# Space Curves

Saddle curve:

$$z(t) = (\cos t, \sin t, \cos 2t), \qquad 0 \le t \le 2\pi.$$

Convergence of global curve distance histogram function:

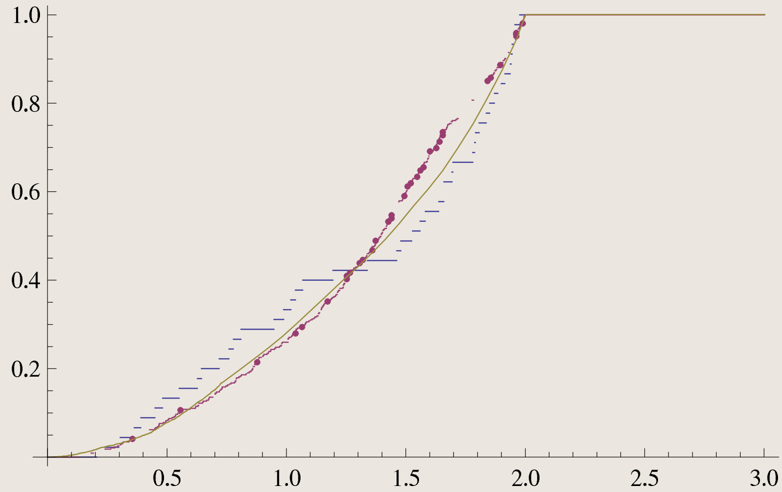


# Surfaces

Local and global surface distance histogram functions:

$$h_S(r,z) = \frac{\text{area}\,(S \cap B_r(z))}{\text{area}\,(S)}, \qquad H_S(r) = \frac{1}{\text{area}\,(S)} \iint_S h_S(r,z)\, dS.$$

Convergence for sphere:



# Area Histograms

Rewrite global curve distance histogram function:

$$H_C(r) = \frac{1}{L}\oint_C h_C(r, z(s))\, ds = \frac{1}{L^2}\oint_C \oint_C \chi_r(d(z(s), z(s'))\, ds\, ds'$$

$$\text{where} \qquad \chi_r(t) = \begin{cases} 1, & t \leq r, \\ 0, & t > r, \end{cases}$$

Global curve area histogram function

$$A_C(r) = \frac{1}{L^3}\oint_C \oint_C \oint_C \chi_r(\text{area}\,(z(\widehat{s}), z(\widehat{s}'), z(\widehat{s}''))\, d\widehat{s}\, d\widehat{s}'\, d\widehat{s}'',$$

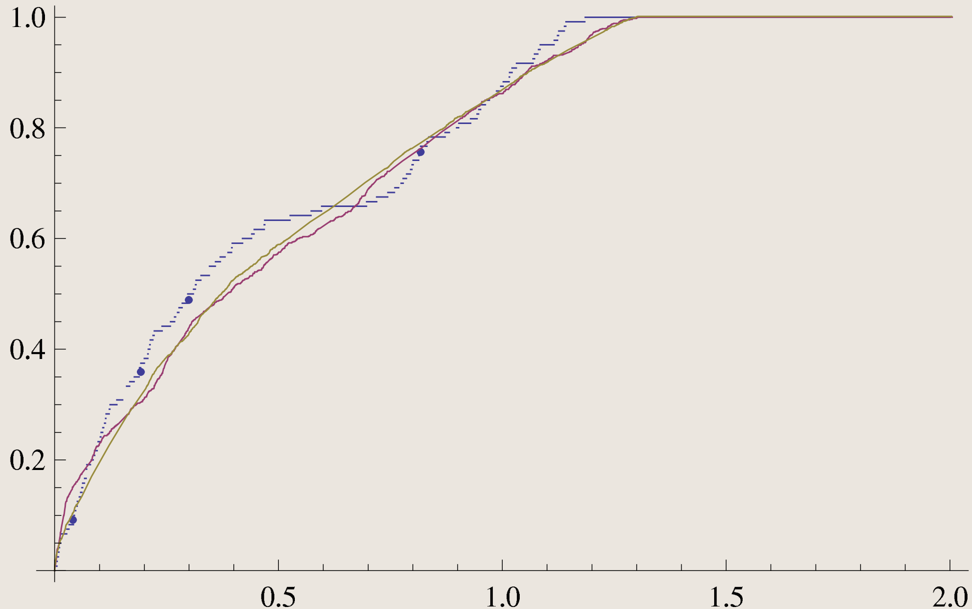
$d\widehat{s}$ — equi-affine arc length element $\quad L = \int_C d\widehat{s}$

Discrete cumulative area histogram

$$A_P(r) = \frac{1}{n(n-1)(n-2)} \sum_{z \neq z' \neq z'' \in P} \chi_r(\text{area}\,(z, z', z'')),$$

Boutin & Kemper: the area histogram uniquely determines generic point sets $P \subset \mathbb{R}^2$ up to equi-affine motion

# Area Histogram for Circle




★ ★ Joint invariant histograms — convergence???

# Triangle Distance Histograms

$Z = (\ldots z_i \ldots) \subset M$ — sample points on a subset $M \subset \mathbb{R}^n$ (curve, surface, etc.)

$T_{i,j,k}$ — triangle with vertices $z_i, z_j, z_k$.

Side lengths:

$$\sigma(T_{i,j,k}) = (\, d(z_i, z_j), d(z_i, z_k), d(z_j, z_k)\,)$$

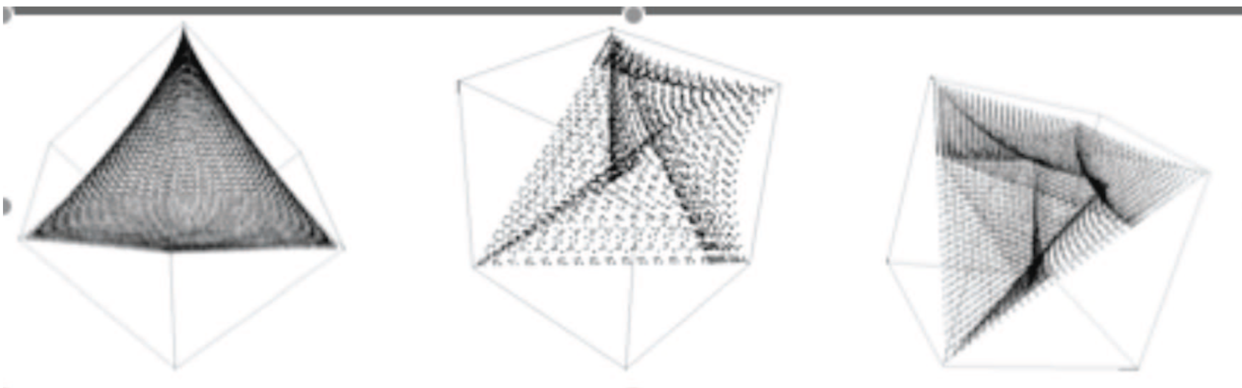
Discrete triangle histogram:

$$\mathcal{S} = \sigma(\mathcal{T}) \subset K$$

Triangle inequality cone

$$K = \{\, (x, y, z) \mid \; x, y, z \geq 0, \; x + y \geq z, \; x + z \geq y, \; y + z \geq x \,\} \subset \mathbb{R}^3.$$

# Triangle Histogram Distributions



Circle Triangle Square

$\Longrightarrow$ Madeleine Kotzagiannidis

# Practical Object Recognition

- Scale-invariant feature transform (SIFT) (Lowe)
- Shape contexts (Belongie–Malik–Puzicha)
- Integral invariants (Krim, Kogan, Yezzi, Pottman, . . . )
- Shape distributions (Osada–Funkhouser–Chazelle–Dobkin)
  Surfaces: distances, angles, areas, volumes, etc.
- Gromov–Hausdorff and Gromov-Wasserstein distances (Mémoli)
  $\Longrightarrow$ lower bounds