Invariant Signatures

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The Basic Equivalence Problem

M — smooth m-dimensional manifold.

G — transformation group acting on M

- finite-dimensional Lie group
- infinite-dimensional Lie pseudo-group

Transformation Groups

- Euclidean — rigid motions
- Similarity — rigid plus scaling
- Equi-affine — volume (area)-preserving
- Conformal — angle-preserving
- Projective
- Video
- Illumination & Color
- Classical Invariant Theory
- Symmetries of differential equations, etc.

- Diffeomorphisms
- Canonical — symplectomorphisms
- Conformal — 2D

Equivalence:

Determine when two n-dimensional submanifolds

$$N \quad \text{and} \quad \overline{N} \subset M$$

are *congruent*:

$$\overline{N} = g \cdot N \qquad \text{for} \qquad g \in G$$

Symmetry:

Find all symmetries,

i.e., self-equivalences or *self-congruences*:

$$N = g \cdot N$$

Tennis, Anyone?

Invariants

Definition. An invariant is a real-valued function $I : M \to \mathbb{R}$ that is unaffected by the group transformations:

$$I(g \cdot z) = I(g)$$

Equivalence & Invariants

- Equivalent submanifolds $N \approx \overline{N}$
  must have the same invariants: $I = \bar{I}$.

Constant invariants provide immediate information:

$$\text{e.g.} \qquad \kappa = 2 \quad \Longleftrightarrow \quad \overline{\kappa} = 2$$

Non-constant invariants are not useful in isolation, because an equivalence map can drastically alter the dependence on the submanifold parameters:

$$\text{e.g.} \qquad \kappa = x^3 \qquad \text{versus} \qquad \overline{\kappa} = \sinh x$$

Syzygies

However, a functional dependency or syzygy among the invariants *is* intrinsic:

$$\tau = \kappa^3 - 1 \qquad \Longleftrightarrow \qquad \overline{\tau} = \overline{\kappa}^{\,3} - 1$$

- Universal syzygies — Gauss–Codazzi
- Distinguishing syzygies.

Signatures

By an invariant signature we mean a set parametrized by a complete system of "distinguishing invariants", that will rigorously resolve the equivalence problem.

Typically, there are not enough ordinary invariants to prescribe a signature. In particular, if G acts transitively on M, there are *no* ordinary invariants.

Constructing enough invariants for a signature requires that we increase the dimension of the underlying space via some kind of natural prolongation procedure.

- Prolonging to derivatives (jet space)

$$G^{(n)} : \mathrm{J}^n(M,p) \quad \longrightarrow \quad \mathrm{J}^n(M,p)$$

$\Longrightarrow$ differential invariants

- Prolonging to Cartesian product actions

$$G^{\times n} : M \times \cdots \times M \quad \longrightarrow \quad M \times \cdots \times M$$

$\Longrightarrow$ joint invariants

- Prolonging to "multi-space"

$$G^{(n)} : M^{(n)} \quad \longrightarrow \quad M^{(n)}$$

$\Longrightarrow$ joint or semi-differential invariants

$\Longrightarrow$ invariant numerical approximations

Basic Framework

M — m-dimensional manifold

$\mathrm{J}^n = \mathrm{J}^n(M,p)$ — n^{th} order jet space for p-dimensional submanifolds $N \subset M$

G — transformation group acting on M

$G^{(n)}$ — prolonged action on the submanifold jet space J^n

Differential Invariants

Differential invariant $\qquad I : \mathrm{J}^n \to \mathbb{R}$

$$I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$$

$\Longrightarrow$ curvature, torsion, ...

Invariant differential operators:

$$\mathcal{D}_1, \ldots, \mathcal{D}_p$$

$\Longrightarrow$ arc length derivative

$\mathcal{I}(G)$ — the algebra of differential invariants

The Basis Theorem

Theorem. The differential invariant algebra $\mathcal{I}(G)$ is generated by a finite number of differential invariants

$$I_1, \ \ldots \ , I_\ell$$

and $p = \dim N$ invariant differential operators

$$\mathcal{D}_1, \ \ldots \ , \mathcal{D}_p$$

meaning that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_\kappa .$$

$\Longrightarrow$ *Lie, Tresse, Ovsiannikov, Kumpera*

Generating Differential Invariants

- Plane curves $C \subset \mathbb{R}^2$:

  curvature κ and arc length derivatives $\kappa_s, \kappa_{ss}, \ldots$

- Space curves $C \subset \mathbb{R}^3$:

  curvature κ, torsion τ, and derivatives $\kappa_s, \tau_s, \kappa_{ss}, \tau_{ss}, \ldots$

- Euclidean surfaces $S \subset \mathbb{R}^3$:

  Gauss curvature K, mean curvature H, and invariant derivatives $\mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1 H, \ldots$.

- Equi–affine surfaces $S \subset \mathbb{R}^3$:

  The Pick invariant P and derivatives $\mathcal{D}_1 P, \mathcal{D}_2 P, \mathcal{D}_1^2 P, \ldots$.

Equivalence & Syzygies

Theorem. (Cartan) Two submanifolds are (locally) equivalent if and only if they have the same syzygies among *all* their (joint) differential invariants.

♠ There are, in general, an infinite number of differential invariants and hence an infinite number of syzygies must be compared to establish equivalence.

♡ But the higher order syzygies are all consequences of a finite number of low order syzygies!

Example — Plane Curves

If non-constant, both κ and κ_s depend on a single parameter, and so, locally, are subject to a syzygy:

$$\kappa_s = H(\kappa) \qquad (*)$$

But then

$$\kappa_{ss} = \frac{d}{ds} H(\kappa) = H'(\kappa)\,\kappa_s = H'(\kappa)\,H(\kappa)$$

and similarly for κ_{sss}, etc.

Consequently, all the higher order syzygies are generated by the fundamental first order syzygy $(*)$.

$\Longrightarrow$ κ and κ_s serve as distinguishing invariants and are used to parametrize the signature in this case.

Definition. The *signature curve* $\mathcal{S} \subset \mathbb{R}^2$ of a curve $\mathcal{C} \subset \mathbb{R}^2$ is parametrized by the two lowest order differential invariants:

$$\mathcal{S} = \left\{ \left(\kappa \, , \, \frac{d\kappa}{ds} \right) \right\} \quad \subset \quad \mathbb{R}^2$$

Theorem. Two curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ are equivalent:

$$\overline{\mathcal{C}} = g \cdot \mathcal{C}$$

if and only if their signature curves are identical:

$$\overline{\mathcal{S}} = \mathcal{S}$$

$\Longrightarrow$ Object recognition:

Calabi–O–Shakiban–Tannenbaum–Haker

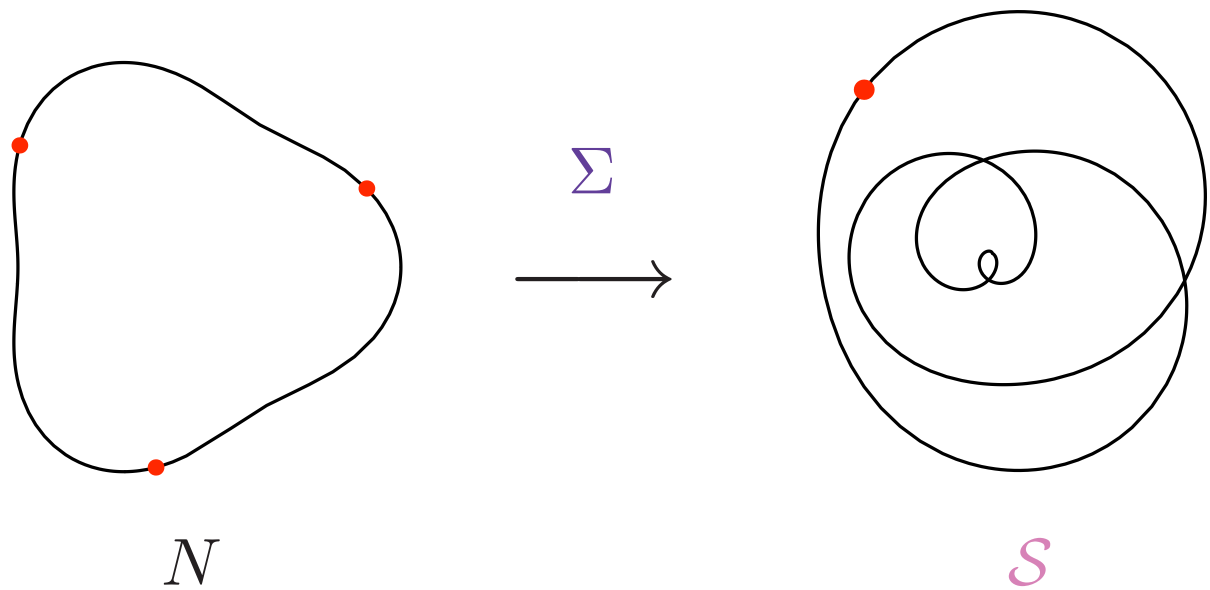
Symmetry and Signature

Theorem. The dimension of the symmetry group of a (nonsingular) submanifold N equals the codimension of its signature:

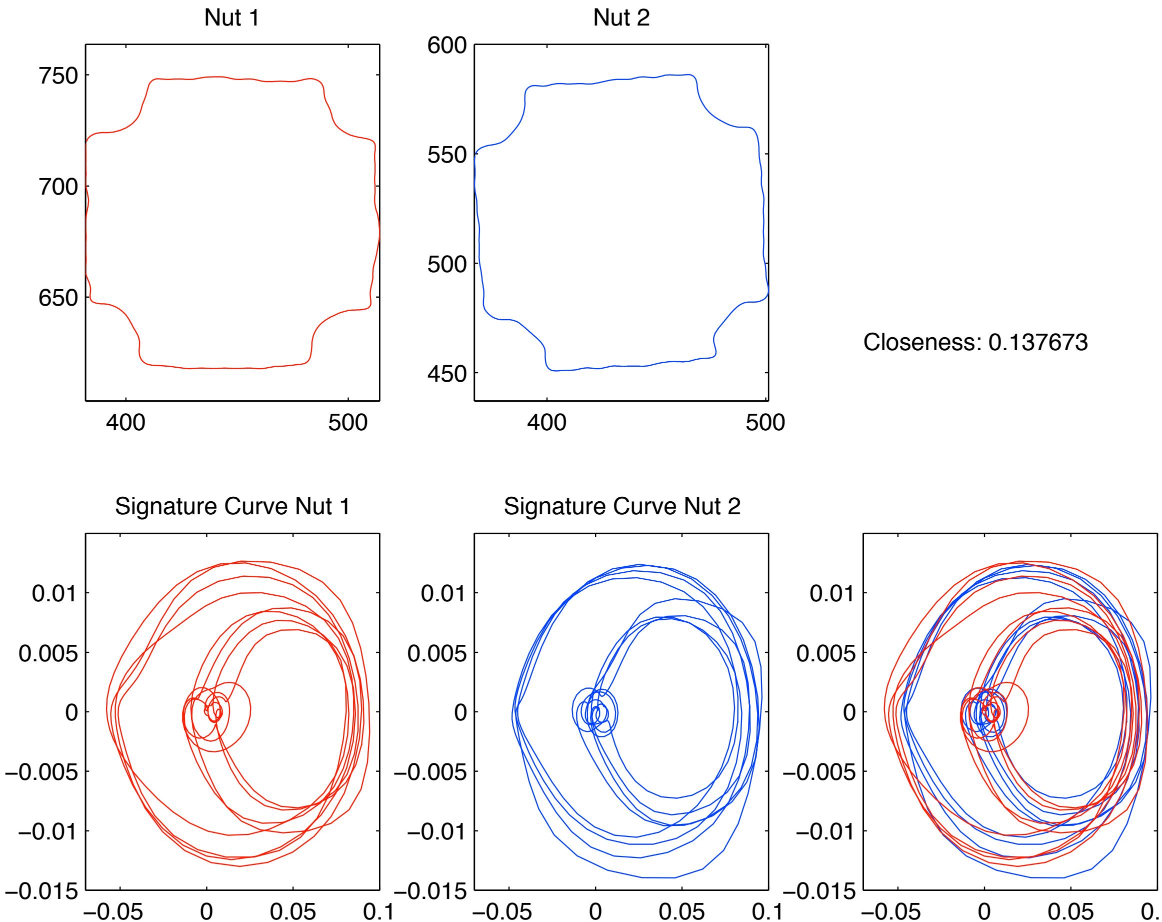
$$\dim G_N = \dim N - \dim \mathcal{S}$$

Theorem. If N has only a discrete symmetry group ($\dim G_N = 0$), the number of its symmetries equals the index of its signature map $\Sigma : N \to \mathcal{S}$.

The Index



Object Recognition


Nut 1
Nut 2
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Closeness: 0.137673
Signature Curve Nut 1
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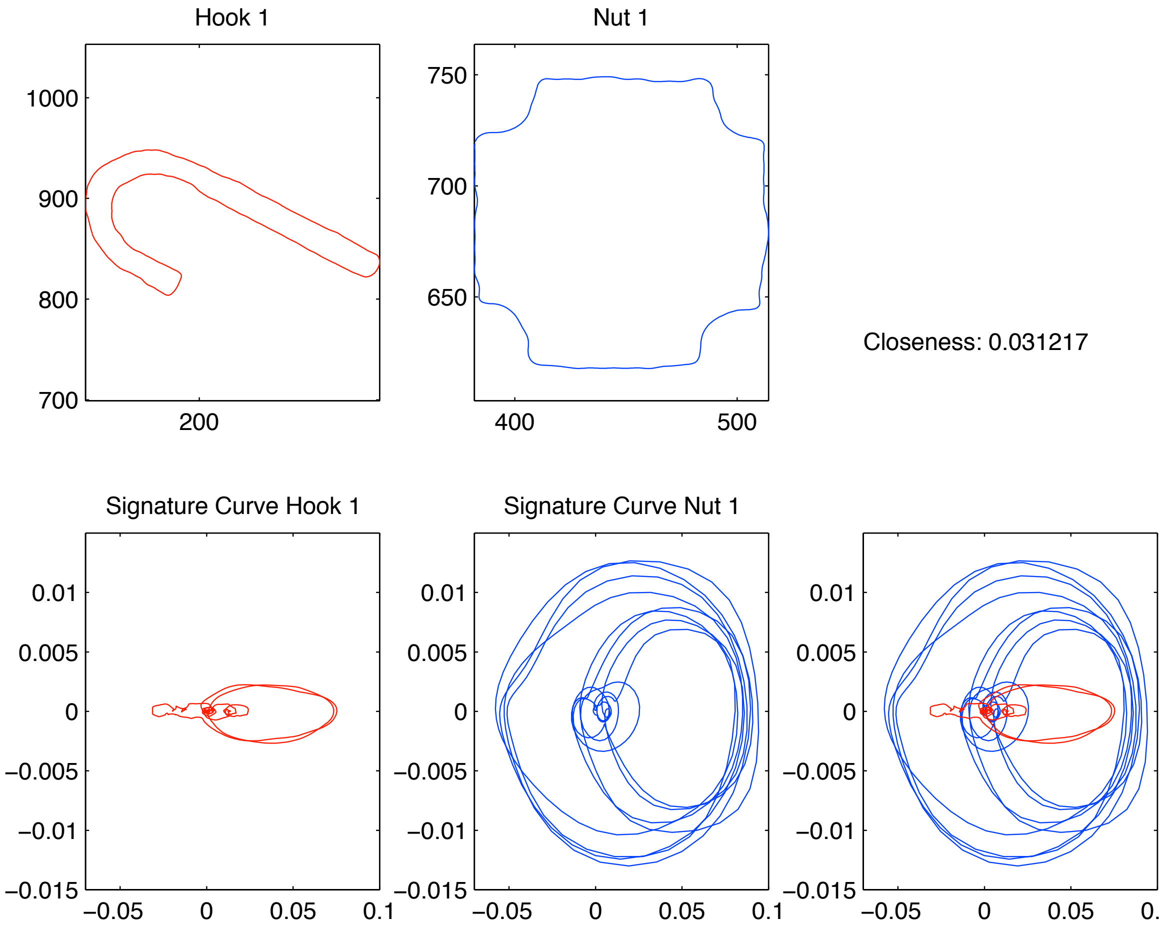
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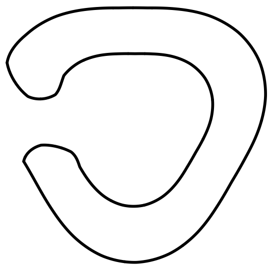
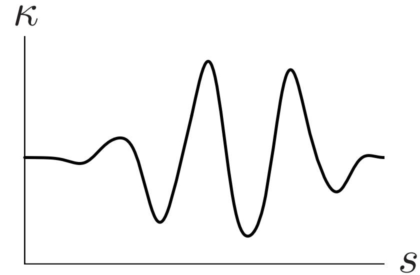
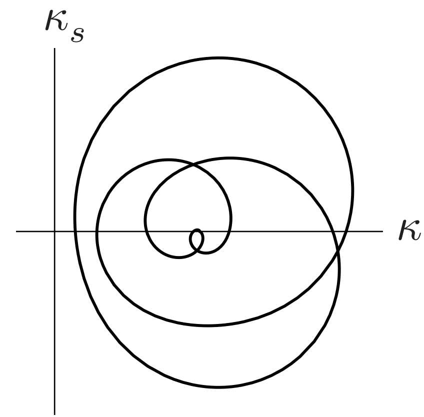
Signatures



Original curve

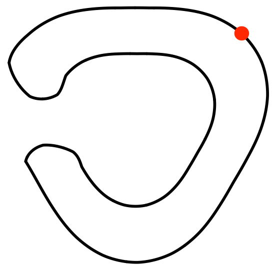


Classical Signature

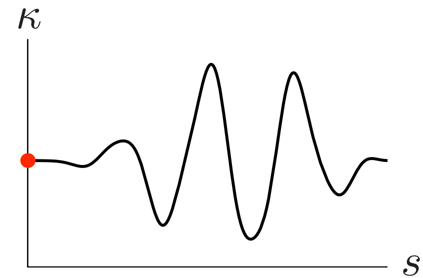


Differential invariant signature

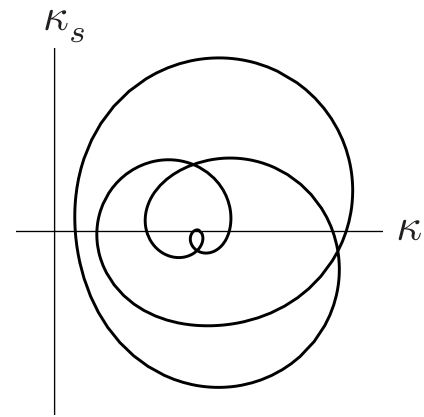
Signatures



Original curve

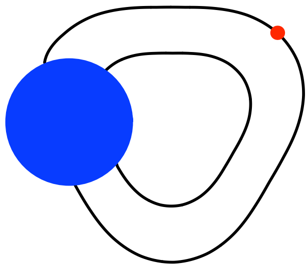


Classical Signature

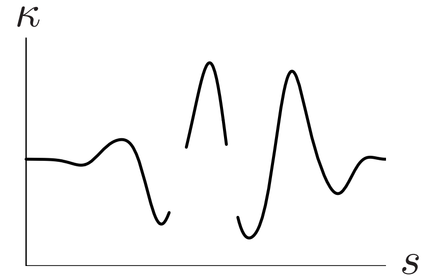


Differential invariant signature

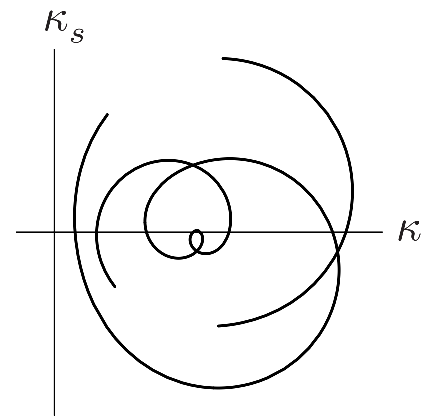
Occlusions



Original curve

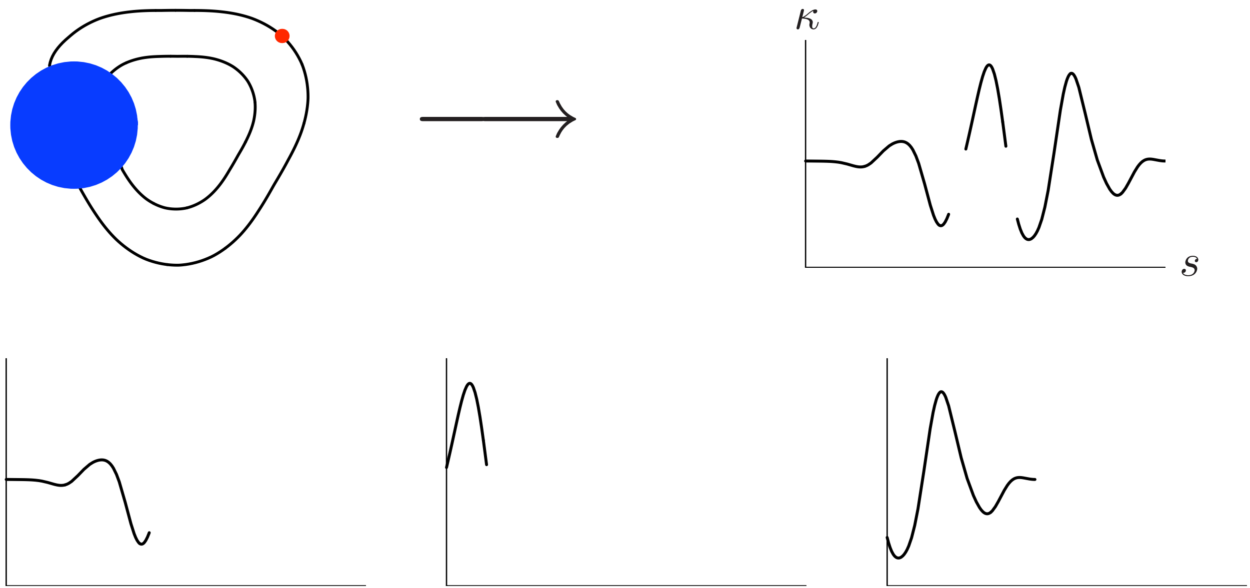


Classical Signature



Differential invariant signature

Classical Occlusions



Possible Signature Metrics

- Hausdorff
- Monge–Kantorovich transport
- Electrostatic repulsion
- Latent semantic analysis
- Histograms
- Geodesic distance
- Diffusion metric
- Gromov–Hausdorff

Advantages of the Signature Curve

- Purely local — no ambiguities
- Symmetries and approximate symmetries
- Readily extends to surfaces and higher dimensional submanifolds
- Occlusions and reconstruction

Main disadvantage: Noise sensitivity due to dependence on high order derivatives.

Noise Reduction

Strategy #1:

Use lower order invariants to construct a signature:

- joint invariants
- joint differential invariants
- integral invariants
- topological invariants
- . . .

Joint Invariants

A joint invariant is an invariant of the k-fold Cartesian product action of G on $M \times \cdots \times M$:

$$\boxed{I(g \cdot z_1, \ldots, g \cdot z_k) \; = \; I(z_1, \ldots, z_k)}$$

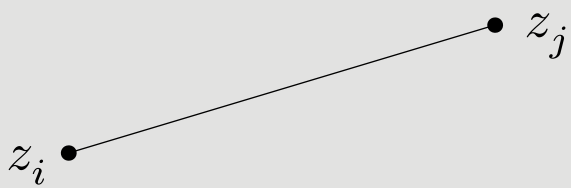
A joint differential invariant or semi-differential invariant is an invariant depending on the derivatives at several points $z_1, \ldots, z_k \in N$ on the submanifold:

$$\boxed{I(g \cdot z_1^{(n)}, \ldots, g \cdot z_k^{(n)}) \; = \; I(z_1^{(n)}, \ldots, z_k^{(n)})}$$

Joint Euclidean Invariants

Theorem. Every joint Euclidean invariant is a function of the interpoint distances

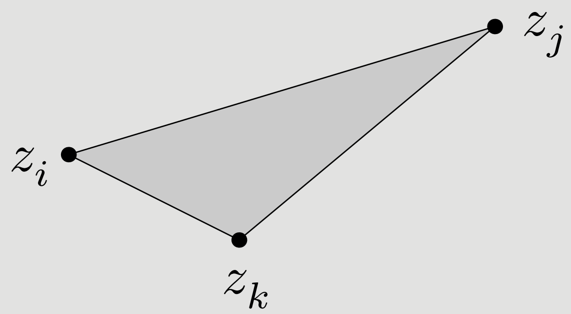
$$d(z_i, z_j) = \| \, z_i - z_j \, \|$$



Joint Equi–Affine Invariants

Theorem. Every planar joint equi–affine invariant is a function of the triangular areas

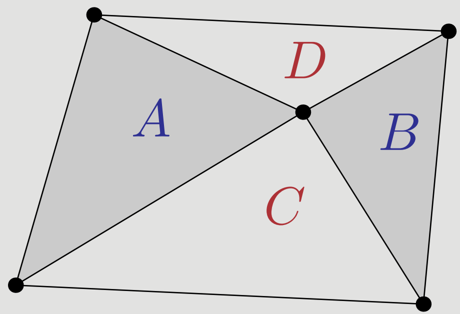
$$[\, i \ j \ k \,] = \tfrac{1}{2}\,(z_i - z_j) \wedge (z_i - z_k)$$



Joint Projective Invariants

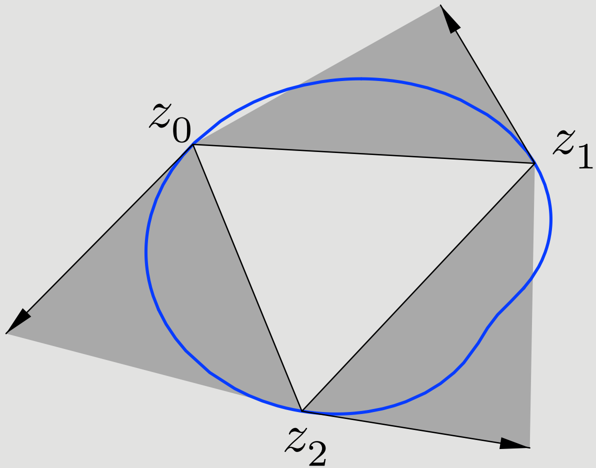
Theorem. Every joint projective invariant is a function of the planar cross-ratios

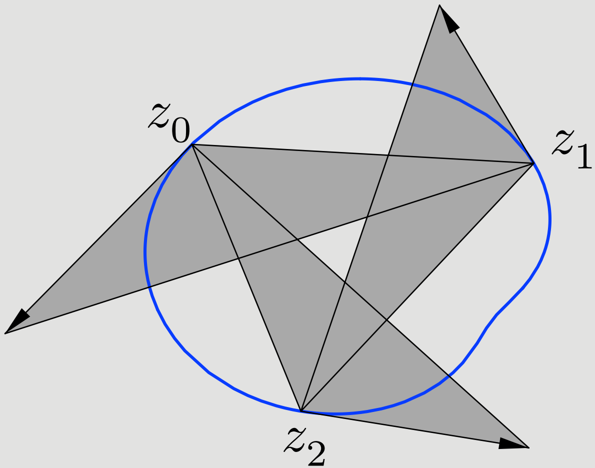
$$[\, z_i, z_j, z_k, z_l, z_m \,] \;=\; \frac{A\,B}{C\,D}$$



- Three–point projective joint differential invariant
  — tangent triangle ratio:

$$\frac{[\,0\;2\;\dot{0}\,]\,[\,0\;1\;\dot{1}\,]\,[\,1\;2\;\dot{2}\,]}{[\,0\;1\;\dot{0}\,]\,[\,1\;2\;\dot{1}\,]\,[\,0\;2\;\dot{2}\,]}\,.$$





Joint Invariant Signatures

If the invariants depend on k points on a p-dimensional submanifold, then you need at least

$$\ell > k\,p$$

distinct invariants $I_1, \ldots, I_\ell$ in order to construct a syzygy. Typically, the number of joint invariants is

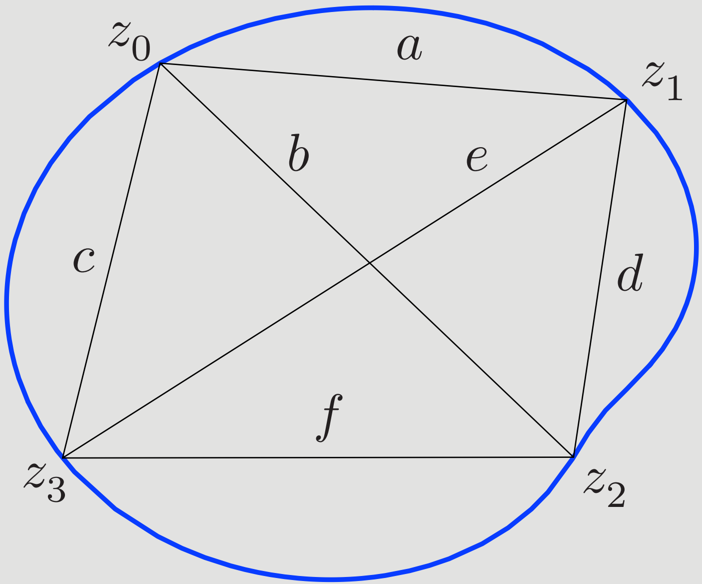
$$\ell = k\,m - r = (\#\text{points})\,(\dim M) - \dim G$$

Therefore, a purely joint invariant signature requires at least

$$k \;\geq\; \frac{r}{m-p} + 1$$

points on our p-dimensional submanifold $N \subset M$.

Joint Euclidean Signature



Joint signature map:

$$\Sigma : \mathcal{C}^{\times 4} \longrightarrow \mathcal{S} \subset \mathbb{R}^6$$

$$a = \| z_0 - z_1 \| \qquad b = \| z_0 - z_2 \| \qquad c = \| z_0 - z_3 \|$$

$$d = \| z_1 - z_2 \| \qquad e = \| z_1 - z_3 \| \qquad f = \| z_2 - z_3 \|$$

$\Longrightarrow$ six functions of four variables

Syzygies:

$$\Phi_1(a, b, c, d, e, f) = 0 \qquad \Phi_2(a, b, c, d, e, f) = 0$$

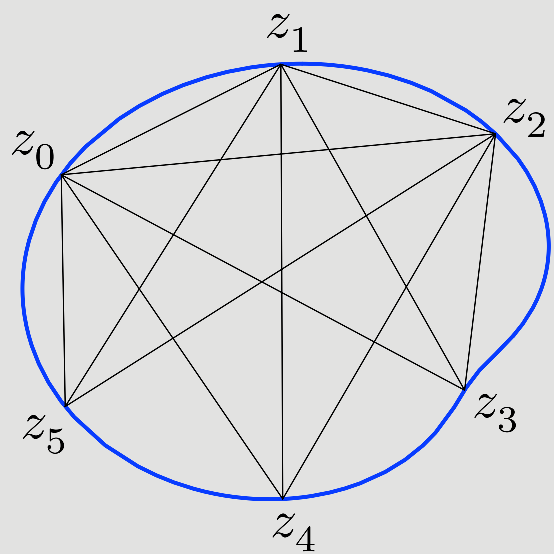
Universal Cayley–Menger syzygy $\quad \Longleftrightarrow \quad \mathcal{C} \subset \mathbb{R}^2$

$$\det \begin{vmatrix} 2\,a^2 & a^2 + b^2 - d^2 & a^2 + c^2 - e^2 \\ a^2 + b^2 - d^2 & 2\,b^2 & b^2 + c^2 - f^2 \\ a^2 + c^2 - e^2 & b^2 + c^2 - f^2 & 2\,c^2 \end{vmatrix} = 0$$

Joint Equi–Affine Signature

Requires 7 triangular areas:

$[\,0\;1\;2\,],\ [\,0\;1\;3\,],\ [\,0\;1\;4\,],\ [\,0\;1\;5\,],\ [\,0\;2\;3\,],\ [\,0\;2\;4\,],\ [\,0\;2\;5\,]$



Joint Invariant Signatures

- The joint invariant signature subsumes other signatures, but resides in a higher dimensional space and contains a lot of redundant information.
- Identification of landmarks can significantly reduce the redundancies (Boutin)
- It includes the differential invariant signature and semi-differential invariant signatures as its "coalescent boundaries". (Discrete $\Longrightarrow$ continuous)
- Invariant numerical approximations to differential invariants and semi-differential invariants are constructed (using moving frames) near these coalescent boundaries.

Statistical Sampling

Idea: Replace high dimensional joint invariant signatures by increasingly dense point clouds obtained by multiply sampling the original submanifold.

- The equivalence problem requires direct comparison of signature point clouds. $\Longrightarrow$ Compressed sampling?

- Continuous symmetry detection relies on determining the underlying dimension of the signature point clouds.

- Discrete symmetry detection relies on determining densities of the signature point clouds.

$\Longrightarrow$ Natural signature statistics

⋆⋆ Moving Frames ⋆⋆

The construction of signatures relies on:

A new, equivariant approach to the classical theory of moving frames developed over the past decade in collaboration with:

M. Fels, G. Marí–Beffa, I. Kogan, M. Boutin, J. Cheh, J. Pohjanpelto, D. Lewis, E. Mansfield, E. Hubert

⋆⋆ Completely constructive ⋆⋆

Additional applications of moving frames:

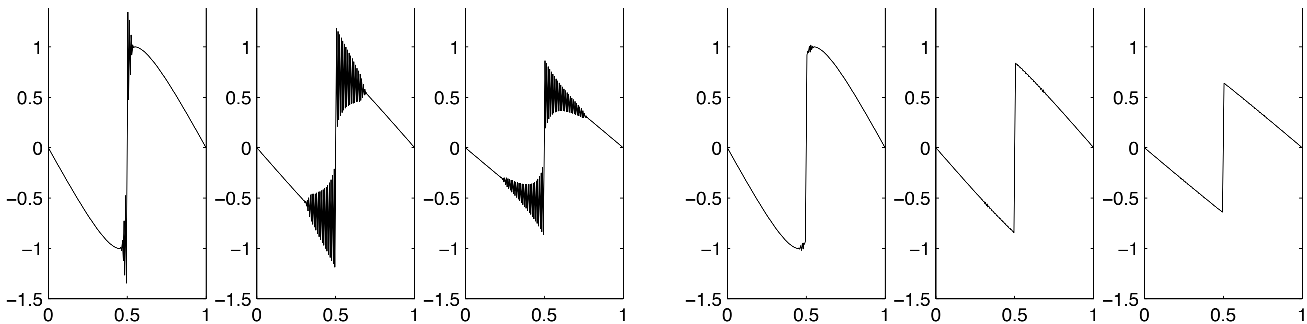
Symmetry–Preserving Numerical Methods

- Invariant numerical approximations to differential invariants.
- Invariantization of numerical integration methods.

$\Longrightarrow$ Structure-preserving algorithms

Invariantization of Crank–Nicolson for Burgers' Equation

$$u_t = \varepsilon\, u_{xx} + u\, u_x$$



$\Longrightarrow$ Pilwon Kim

Evolution of Invariants and Signatures

Theorem. Under the curve shortening flow $C_t = -\kappa\,\mathbf{n}$, the signature curve $\kappa_s = H(t,\kappa)$ evolves according to the parabolic equation

$$\frac{\partial H}{\partial t} = H^2\,H_{\kappa\kappa} - \kappa^3 H_\kappa + 4\,\kappa^2 H$$

$\Longrightarrow$ Signature Noise Reduction Strategy #2

$\Longrightarrow$ Solitons and bi-Hamiltonian systems

Invariant Variational Problems

Problem: Given an invariant variational problem written in terms of the differential invariants, *directly* construct the invariant form of its Euler–Lagrange equations.

$\Longrightarrow$ Willmore, $\int K^2$, etc.

Example. Euclidean plane curves:

Invariant variational problem:

$$\int P(\kappa, \kappa_s, \kappa_{ss}, \ \dots\ )\, ds$$

Invariant Euler-Lagrange formula

$$\mathbf{E}(L) = (\mathcal{D}^2 + \kappa^2)\, \mathcal{E}(P) + \kappa\, \mathcal{H}(P).$$

$\mathcal{E}(P)$ — invariantized Euler–Lagrange expression

$\mathcal{H}(P)$ — invariantized Hamiltonian